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ABOUT THE COHOMOLOGY OF THE LIE SUPERALGEBRA OF VECTOR FIELDS ON $\mathbb{R}^{n|n}$

AMMAR FAOUZI AND KAMOUN KAOUTHAR

ABSTRACT. In this paper, we compute the first space of cohomology of $Vect(\mathbb{R}^{n|n})$, the Lie superalgebra of vector fields on the supermanifold $\mathbb{R}^{n|n}$ with coefficients in $\mathcal{F}(\mathbb{R}^{n|n})$, the space of smooth functions on $\mathbb{R}^{n|n}$. We give a super analog of the cohomologies of vector fields that where studied for instance by D.B. Fuchs [2]. This work allows us to classify the deformations of the action of $Vect(\mathbb{R}^{n|n})$ on $\mathcal{F}(\mathbb{R}^{n|n})$.

1. INTRODUCTION

Let $Vect(\mathbb{R}^{n|n})$ be the Lie superalgebra of vector fields on the super manifold $\mathbb{R}^{n|n}$ and $\mathcal{F}(\mathbb{R}^{n|n})$ be the space of smooth functions on the manifold $(\mathbb{R}^{n|n})$. As $\mathcal{F}(\mathbb{R}^{n|n})$ can be identified with the supercommutative superalgebra $\Omega(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \Omega^k(\mathbb{R}^n)$ of differential forms on $\mathbb{R}^n$, then $Vect(\mathbb{R}^{n|n})$ is identified with the superalgebra of superderivations of $\Omega(\mathbb{R}^n)$. So, $Vect(\mathbb{R}^{n|n})$ is identified to a sum of two copies of the space of tensor valued differential forms on $\mathbb{R}^n$, $\Omega = \bigoplus_k \Omega^k(\mathbb{R}^n, T\mathbb{R}^n)$, one with the Frölicher-Nijenhuis bracket $[[\ ,\ ]]$, the other one with the Richardson Nijenhuis bracket $[\ ,\ ]^\wedge$. We shall set $\mathfrak{F} = (\Omega, [[\ ,\ ]])$, and $\mathcal{R} = (\Omega, [\ ,\ ]^\wedge)$. For this identification, as well as relationship between the two brackets, see the book by Michor, Kolar and Slovac [3]. Here we compute $H^1(Vect(\mathbb{R}^{n|n}), \mathcal{F}(\mathbb{R}^{n|n}))$.

1.1. Notations and definitions.

1.1.1. *Identification of* $Vect(\mathbb{R}^{n|n})$. We shall first precise the structure of $Vect(\mathbb{R}^{n|n})$. The space $\mathcal{F}(\mathbb{R}^{n|n})$ of smooth functions on $\mathbb{R}^{n|n}$ can be identified with the graded commutative algebra

$$\Omega(\mathbb{R}^n) = \bigoplus_{s=0}^{n} \Omega^s(\mathbb{R}^n)$$

of differential forms on $\mathbb{R}^n$. We denote by $Der_s(\Omega(\mathbb{R}^n))$ the space of all graded derivations of degree s, i.e all linear mappings

$$D : \Omega(\mathbb{R}^n) \longrightarrow \Omega(\mathbb{R}^n)$$

Key words and phrases. cohomology space, the Lie superalgebra of vector fields on the super manifold $\mathbb{R}^{n|n}$, Frölicher-Nijenhuis bracket, Richardson-Nijenhuis bracket.

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with $D(\Omega^l(\mathbb{R}^n)) \subset \Omega^{s+l}(\mathbb{R}^n)$ and

$$D(\varphi \wedge \psi) = D\varphi \wedge \psi + (-1)^{kl}\varphi \wedge D(\psi)$$

for $\varphi \in \Omega^l(\mathbb{R}^n)$ and $\psi \in \Omega^k(\mathbb{R}^n)$. The space

$$Der(\Omega(\mathbb{R}^n)) = \bigoplus_s Der_s(\Omega(\mathbb{R}^n))$$

is a graded Lie superalgebra with the graded commutator:

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{s_1 s_2} D_2 \circ D_1$$

for $D_i \in Der_{s_i}(\Omega(\mathbb{R}^n))$, for $i \in \{1, 2\}$. Then the space

$$Vect(\mathbb{R}^{n|n}) := Der(\Omega(\mathbb{R}^n)).$$

We call $\Omega(\mathbb{R}^n, T\mathbb{R}^n) = \bigoplus_{s=0}^n \Omega^s(\mathbb{R}^n, T\mathbb{R}^n)$ the space of all vector valued differential forms. We shall frequently use the identification between $\Omega^*(\mathbb{R}^n, T\mathbb{R}^n)$ and the completed tensor product over functions $\Omega^*(\mathbb{R}^n) \otimes T\mathbb{R}^n$. So, by a slight abuse notations, we shall identify $\omega \otimes X$ where $\omega \in \Omega^*(\mathbb{R}^n)$ and $X \in T\mathbb{R}^n$, with the corresponding tensor valued differential form.

A derivation $D \in Der_s(\Omega(\mathbb{R}^n))$ is algebraic if its restriction to $\Omega^0(\mathbb{R}^n)$ vanishes identically. Then $D(f\omega) = fD(\omega)$ for $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$. So, from C. Roger ([6] p 68), D is given by a tensor field. So, D induces a derivation $D_x \in Der_s \wedge T_x^*\mathbb{R}^n$ for each $x \in \mathbb{R}^n$. It is uniquely determined by its restriction to 1-forms:

$$D_{x|T_x^*\mathbb{R}^n} : T_x^*\mathbb{R}^n \longrightarrow \wedge^{s+1} T^*\mathbb{R}^n$$

which we may view as an element $K_x \in \wedge^{k+1} T_x^*\mathbb{R}^n \otimes T_x\mathbb{R}^n$ depending smoothly on $x \in \mathbb{R}^n$. We write $D = i_K$, where

$$K \in C^\infty(\wedge^{s+1} T^*\mathbb{R}^n \otimes T\mathbb{R}^n) =: \Omega^{s+1}(\mathbb{R}^n, T\mathbb{R}^n).$$

Note the defining equation: $i_K(w) = w \circ K$ for $w \in \Omega^1(\mathbb{R}^n)$.

The exterior derivative d is an element of $Der_1(\Omega(\mathbb{R}^n))$. In view of the formula

$$\mathcal{L}_X = [i_X, d] = i_X \circ d + d \circ i_X$$

for vector fields $X \in Vect(\mathbb{R}^n)$, we define for $K \in \Omega^s(\mathbb{R}^n, T\mathbb{R}^n)$ the Lie derivation $\mathcal{L}_K \in Der_s(\Omega(\mathbb{R}^n))$ by

$$\mathcal{L}_K := [i_K, d] = i_K \circ d + (-1)^s d \circ i_K,$$

then the mapping $\mathcal{L} : \Omega(\mathbb{R}^n, T\mathbb{R}^n) \longrightarrow Der(\Omega(\mathbb{R}^n))$ is injective, since $\mathcal{L}_K f = i_K df = df \circ K$ for $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$.

Lemma 1. [6] *For any graded derivation $D \in Der_k(\Omega(\mathbb{R}^n))$, there exists an unique $K \in \Omega^k(\mathbb{R}^n, T\mathbb{R}^n)$ and $L \in \Omega^{k+1}(\mathbb{R}^n, T\mathbb{R}^n)$ such that $D = \mathcal{L}_K + i_L$, where*

(1) $\quad i_L(\omega \otimes X) := i_L(\omega) \otimes X$ *and* $i_L(\omega) = \eta \wedge i_Y(\omega)$ *for* $L = \eta \otimes Y$.

The degree of D is denoted $|D|$ and is equal to k.

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1.1.2. *Richardson-Nijenhuis algebra.* The injection

$$i : \Omega^{*+1}(\mathbb{R}^n, T\mathbb{R}^n) \longrightarrow Der^*(\Omega(\mathbb{R}^n)); \quad i([K, L]^\wedge) := [i_K, i_L]$$

is a graded Lie bracket on $\Omega^{*+1}(\mathbb{R}^n, T\mathbb{R}^n)$. So, we get a bracket on $\Omega^{*+1}(\mathbb{R}^n, T\mathbb{R}^n)$ which defines a graded Lie algebra structure with the grading as indicated. For $K \in \Omega^{k+1}(\mathbb{R}^n, T\mathbb{R}^n)$ and $L \in \Omega^{\ell+1}(\mathbb{R}^n, T\mathbb{R}^n)$ we have

$$[K, L]^\wedge = i_K L - (-1)^{k\ell} i_L K.$$

The space $\mathcal{R} = (\bigoplus \Omega^{*+1}(\mathbb{R}^n, T\mathbb{R}^n), [\ ,\ ]^\wedge)$ is called the Richardson-Nijenhuis algebra. It is a subalgebra of $Vect(\mathbb{R}^{n|n})$

Remark 2. This Lie superalgebra is linked with $\mathbb{R}^{0|n}$ the Lie superalgebra of vector fields on a purely odd space. More precisely, if one identifies as a space

$$\Omega^{*+1}(\mathbb{R}^n, T\mathbb{R}^n) = Vect(\mathbb{R}^{0|n}) \otimes C^\infty(\mathbb{R}^n)$$

with completed tensor product, then the Richardson-Nijenhuis bracket reads as follows: for $K = a \otimes \xi$ and $L = b \otimes \lambda$ with $\xi,\ \lambda$ in $Vect(\mathbb{R}^{0|n})$ and $a,\ b$ in $C^\infty(\mathbb{R}^n)$, one has $[K, L]^\wedge = ab \otimes [\xi, \lambda]$, where $[\xi, \lambda]$ is the bracket of vector fields on the supermanifold $\mathbb{R}^{0|n}$. So, it can be identified with the super Lie algebra of currents with value in $Vect(\mathbb{R}^{0|n})$.

1.1.3. *Frölicher-Nijenhuisalgebra.* The bracket of $\mathcal{L}_\theta$ and $\mathcal{L}_\eta$ is still a derivation, which gives the Frölicher-Nijenhuis bracket by the following formula:

$$\mathcal{L}_{[[\theta, \eta]]} = [\mathcal{L}_\theta, \mathcal{L}_\eta].$$

For $\theta = \alpha \otimes X$ and $\eta = \beta \otimes Y$ with $\alpha \in \Omega^k(\mathbb{R}^n)$, $\beta \in \Omega^l(\mathbb{R}^n)$, X and Y in $Vect(\mathbb{R}^n)$ one has:

$$\begin{aligned}[][\alpha \otimes X, \beta \otimes Y]] &= \alpha \wedge \beta \otimes [X, Y] + \alpha \wedge L_X \beta \otimes Y - L_Y \alpha \wedge \beta \otimes X \\ &\quad + (-1)^k (d\alpha \wedge i_X \beta \otimes Y + i_Y \alpha \wedge d\beta \otimes X).\end{aligned}$$

The space $\mathfrak{F} = (\bigoplus \Omega^*(\mathbb{R}^n, T\mathbb{R}^n), [[\ ,\ ]])$ is called Frölicher-Nijenhuis algebra. It is a subalgebra of $Vect(\mathbb{R}^{n|n})$ (see [5]). The above formula has been obtained by Michor in [3].

We get, so, for $Vect(\mathbb{R}^{n|n}) = \mathcal{R} + \mathfrak{F}$ the following bracket:

Lemma 3. [6] *For $K_i \in \Omega^{k_i}(\mathbb{R}^n, T\mathbb{R}^n)$ and $L_i \in \Omega^{k_i+1}(\mathbb{R}^n, T\mathbb{R}^n)$ where $i \in \{1, 2\}$, we have:*

$$\begin{aligned}[][\mathcal{L}_{K_1} + i_{L_1}, \mathcal{L}_{K_2} + i_{L_2}] &= \mathcal{L}\big([[K_1, K_2]] + i_{L_1}(K_2) - (-1)^{k_1 k_2} i_{L_2}(K_1)\big) \\ &\quad + i\big([L_1, L_2]^\wedge + [[K_1, L_2]] - (-1)^{k_1 k_2}[[K_2, L_1]]\big).\end{aligned}$$

Remark 4. As a consequence of this lemma, for $K \in \Omega^k(\mathbb{R}^n, T\mathbb{R}^n)$ and $L \in \Omega^{l+1}(\mathbb{R}^n, T\mathbb{R}^n)$, one has

$$[\mathcal{L}_K, i_L] = i([[K, L]]) - (-1)^{kl} \mathcal{L}(i_L K)$$

and

$$[i_L, \mathcal{L}_K] = \mathcal{L}(i_L K) - (-1)^k i([[L, K]]).$$

2. Mains results

With the notations of previous subsection, one has:

Proposition 5. *If $n > 2$, the space of cohomology $\mathrm{H}^1(\mathfrak{F}, \Omega(\mathbb{R}^n))$ is one dimensional and is generated by the 1-cocycle given by:*

$$\begin{array}{rccc} c_1 : & \mathfrak{F} & \longrightarrow & \Omega(\mathbb{R}^n) \\ & \omega \otimes X & \longrightarrow & d(i_X\omega) \end{array}$$

Proposition 6. *The space of cohomology $\mathrm{H}^1(\mathcal{R}, \Omega(\mathbb{R}^n))$ is one dimensional and is generated by the 1-cocycle given by:*

$$\begin{array}{rccc} c_2 : & \mathcal{R} & \longrightarrow & \Omega(\mathbb{R}^n) \\ & \omega \otimes X & \longrightarrow & (-1)^{|\omega|-1} i_X\omega \end{array}$$

where $|\omega|$ denotes the degree of ω.□

This result can be deduced from C. Roger and P. Lecomte in [7]. Here we take an other proof and rectify their result.

The cohomology of $Vect(\mathbb{R}^{n|n}) = \mathfrak{F} + \mathcal{R}$ is given by:

Theorem 7. *If $n > 2$, the space of cohomology $\mathrm{H}^1(\mathrm{Vect}(\mathbb{R}^{n|n}), \mathcal{F}(\mathbb{R}^{n|n}))$ is generated by the 1-cocycles*

$$c : Vect(\mathbb{R}^{n|n}) \longrightarrow \mathcal{F}(\mathbb{R}^{n|n})$$

defined by

$$c(\mathcal{L}_K + i_L) = -c_1(K) + \partial\omega_1(K) + c_2(L)$$

where $\partial\omega_1(K) = \mathcal{L}_K(\omega_1)$, a coboundary on $\mathfrak{F}$ with $\omega_1 \in \Omega^0(\mathbb{R}^n)$.

3. Proof of propositions 5 and 6 and theorem 7

Before proving the propositions and the theorem, we shall give some definitions and preliminary results.

3.1. **Polynomial notation.** (see [1] and [5])

Polynomial notation is very useful to handle computation with differential operators. It allows to apply polynomial computations for operators.

We suppose that $E \to M$ and $F \to M$ are vector bundles, with typical fibers E_0 and F_0, that $\Gamma(E)$ and $\Gamma(F)$ denote their spaces of smooth sections. Then fixing a local chart $(U, x_1, ..., x_n)$, we can identify $\Gamma(E)$ and $\Gamma(F)$ to $C^\infty(U, E_0)$ and $C^\infty(U, F_0)$ respectively. Then a differential operator of order k can be written in following form:

$$f \longmapsto \sum_{|\alpha| \leq k} A_\alpha(x) D^\alpha f(x)$$

where $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ denotes partial derivatives with respect to $(x_1, ..., x_n)$, furthermore the mappings A_α is in $C^\infty(U, \mathcal{L}(E_0, F_0))$. Then

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the symbolic polynomial associated to A is defined by

$$P(\xi;X)(x)=\sum_{|\alpha|\leq k}A_{\alpha,x}(X)\xi^{\alpha}.$$

For example if $X=\sum_i X^i\partial_{x_i}\in Vect(\mathbb{R}^n)$, where $\partial_{x_i}=\frac{\partial}{\partial x_i}$, acting on a function $f\in C^{\infty}(\mathbb{R}^n)$ through the operator of Lie derivative:

$$f\longrightarrow L_X(f)=\sum_i X^i\frac{\partial f}{\partial x_i}$$

is represented by the polynomial function

$$\sum_i X^i\xi_i f=<X,\xi>f.$$

3.2. **Preliminary results.** Let

$$c_1:\mathfrak{F}\longrightarrow\Omega(\mathbb{R}^n)$$

be a cochain, the condition of 1-cocycle applied to c_1 reads:

$c_1([[\alpha\otimes X,\beta\otimes Y]])-\mathcal{L}_{(\alpha\otimes X)}c_1(\beta\otimes Y)=(-1)^{|\alpha||\beta|+1}\mathcal{L}_{(\beta\otimes Y)}c_1(\alpha\otimes Y).$

Remark that for every $\alpha\otimes X\in\mathfrak{F}$ and $\gamma\in\Omega^q(\mathbb{R}^n)$ one has

(2) $$\mathcal{L}_{(\alpha\otimes X)}(\gamma)=i_{(\alpha\otimes X)}d\gamma+\ (-1)^q\ d(i_{(\alpha\otimes X)}\gamma)$$

where $i_{(\alpha\otimes X)}\gamma=\alpha\wedge i_X\gamma$.

Lemma 8. *If*

$$c:\Omega(\mathbb{R}^n)\otimes Vect(\mathbb{R}^n)\longrightarrow\Omega(\mathbb{R}^n)$$

is a 1-cocycle, then c is a differential operator.

Proof. This is a simple adaptation of the result of [8]. □

Lemma 9. *Each cohomology class $[c]$ in $\mathrm{H}^1(\mathfrak{F},\Omega^{\mathrm{k}}(\mathbb{R}^{\mathrm{n}}))$ contains a 1-cocycle with constants coefficients.*

Proof. We suppose that c is a 1-cocycle, then its restriction to the Lie subalgebra $Vect(\mathbb{R}^n)\subset\mathfrak{F}$ of vector fields is also a 1-cocycle. The first cohomology space of the Lie algebra of vector fields is generated by "*div*" and "*ddiv*" so there exist $a,b\in\mathbb{R}$ and $\omega\in\Omega^k(\mathbb{R}^n)$ such that

$$c(X)=a\ \ div(X)+b\ \ ddiv(X)+\partial_X\omega\ \ \forall X\in Vect(\mathbb{R}^n).$$

Now, the 1-cocycle $c-\partial\omega$ vanishes on constant vector fields. Here we use the identification of the algebra of vector fields $Vect(\mathbb{R}^n)$ as a subalgebra of $\mathfrak{F}$ and the fact that the restriction of the action $\mathcal{L}$ of $\mathfrak{F}$ on forms to Lie algebra $Vect(\mathbb{R}^n)$ coincides with the classical Lie derivative L. It follows from the relation of 1-cocycle:

(3) $$L_X(c(K))=c([[X,K]])-\mathcal{L}_K(c(X))$$

for $K\in\mathfrak{F}$, that c commutes with the Lie derivative in the direction of constant vector fields:

(4) $$L_X(c(K))=c([[X,K]]).$$

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A direct computation finishes the proof. □

3.3. **Proof of proposition 5.** Since $\mathfrak{F}$ is a graded Lie algebra and $\Omega(\mathbb{R}^n)$ is a graded module by the degree of forms, the space of cohomology $\mathrm{H}^1(\mathfrak{F}, \Omega(\mathbb{R}^n))$ is graded, then we have

$$\mathrm{H}^1(\mathfrak{F}, \Omega(\mathbb{R}^n)) = \bigoplus_{\mathrm{q}} \mathrm{H}^1(\mathfrak{F}, \Omega(\mathbb{R}^n))_{\mathrm{q}}$$

where $\mathrm{H}^1(\mathfrak{F}, \Omega(\mathbb{R}^n))_{\mathrm{q}}$ is the space of class of homogeneous cocycle $c_{1,q}$ of degree q i.e transforms an argument of degree p on an argument of degree $p+q$. The restriction of $c_{1,q}$ to $\Omega^p(\mathbb{R}^n) \otimes Vect(\mathbb{R}^n)$ is noted

$$c_{1,p,q} : \Omega^p(\mathbb{R}^n) \otimes Vect(\mathbb{R}^n) \longrightarrow \Omega^{p+q}(\mathbb{R}^n).$$

The condition of 1-cocycle applied to $c_{1,p,q}$ can be written

(5) $c_{1,p,q}([[K_1, K_2]]) - \mathcal{L}_{K_1}(c_{1,p,q}(K_2)) = (-1)^{|K_1||K_2|+1}\mathcal{L}_{K_2}(c_{1,p,q}(K_1))$

for K_1 and K_2 in $\mathfrak{F}$. Up to a coboundary, we may suppose that the restriction of c to $Vect(\mathbb{R}^n) \cong \Omega^0(\mathbb{R}^n) \otimes Vect(\mathbb{R}^n) \subset \mathfrak{F}$ is a combination of "*div*" and "*ddiv*". Hence, in equation (5), if we set K_1 to be a linear vector field X, since $c_{1,p,q}(K_1)$ is constant, we directly obtain the relation

$$c_{1,p,q}(L_X(K_2)) - L_X(c_{1,p,q}(K_2)) = 0,$$

where L_X is the classical Lie derivative in the direction of X. If η denote the derivative affecting K, then one may write the symbolic form $c_{1,p,q}(\eta, K)$ associated to $c_{1,p,q}$ (see[5]). For $X_1, ..., X_{n+q}$ in $T\mathbb{R}^n$, the polynomial $c_{1,p,q}(\eta, K)(X_1, ..., X_{n+q})$ is invariant with respect to the action of the algebra $gl(n, \mathbb{R})$. The classical result of Weyl (see [1]) states that such invariant polynomials are generated by contractions. Hence one gets that the degree in η (say r) is equal to $q+1$. Now, the polynomial $c_{1,p,q}(\eta, K)(X_1, ..., X_{n+q})$ must be symmetric in η and antisymmetric in $X_1, ..., X_{n+q}$, so, $r \in \{0, 1, 2\}$. Hence, as a result of the invariance property, we obtain (where a_p, b_p, c_p and e_p are reals numbers):

$c_{1,p,-1}(\eta, K) = a_p\tau(\eta, K)$ where $K = \alpha \otimes X$ and $\tau(\eta, K) = i_X\alpha$;

$c_{1,p,0}(\eta, K) = b_p\ \tau_1(\eta, K) + c_p\ \tau_2(\eta, K)$ where $\tau_1(\eta, K) = \eta \wedge \tau(\eta, K)$ and $\tau_2(\eta, K) =< K, \eta >$;

$c_{1,p,1}(\eta, K) = d_p\ \tau_3(\eta, K)$ where $\tau_3(\eta, K) = e_p\eta\wedge < K, \eta >$.

Thus, we compute the coefficients a_p, b_p, c_p and e_p in accordance with the degree q.

- Case $q = -1$

  In this case, the condition for $c_{1,p,-1}$ to be a 1-cocycle forces a_p to be zero for all p, if $n > 1$.

- Case $q = 0$

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Take $c_{1,p,0}(\eta, K) = b_p\ \tau_1(\eta, K) + c_p\ \tau_2(\eta, K)$ and plug it in equation (5) one can show that b_p is equal to $b_{p'}$ for all p and p' and c_p must be zero for all p, if $n > 2$.

- Case $q = 1$

In this case we have

$$\delta\tau_3(\eta, K)(\alpha \wedge \mathbf{1}, X) = e_p \quad ddivX \wedge d\alpha$$

where $\mathbf{1} = \sum_{i=1}^n dx^i \otimes \partial_{x_i}$. A straightforward computation shows that if $n > 2$, e_p must be zero for all p.

3.4. **Proof of proposition 6.** Consider the mapping

$$\begin{array}{rccc} c_2: & \mathcal{R} & \longrightarrow & \Omega(\mathbb{R}^n) \\ & \omega \otimes X & \longrightarrow & (-1)^{|\omega|-1} i_X \omega \end{array}$$

where $|\omega|$ denotes the degree of ω.

We shall prove that c_2 is a 1-cocycle. Then for $L_1 = \alpha \otimes X$, where $|\alpha| = |L_1| = l_1 + 1$ and $|i_{L_1}| = l_1$, and for $L_2 = \beta \otimes Y$ where $|\beta| = |L_2| = l_2 + 1$ and $|i_{L_2}| = l_2$, one has:

$$c_2([\alpha \otimes X, \beta \otimes Y]^\wedge) - i_{L_1} c_2(\beta \otimes Y) - (-1)^{l_1 l_2 + 1} i_{L_2} c_2(\alpha \otimes X)$$
$$= c_2(\alpha \wedge i_X\beta \otimes Y \ + \ (-1)^{l_1 l_2+1} \beta \wedge i_Y \alpha \otimes X) \ - \ \alpha \wedge i_X((-1)^{l_2} i_Y \beta)$$
$$- (-1)^{l_1 l_2+1} \beta \wedge i_Y((-1)^{l_1} i_X \alpha)$$
$$= (-1)^{l_1+l_2} i_Y(\alpha \wedge i_X \beta) + (-1)^{l_1 l_2 + 1 + l_1 + l_2} i_X(\beta \wedge i_Y \alpha) - (-1)^{l_2} \alpha \wedge i_X i_Y \beta$$
$$- (-1)^{l_1 l_2 + 1 + l_1} \beta \wedge i_Y i_X \alpha$$
$$= (-1)^{l_1+l_2} i_Y \alpha \wedge i_X \beta + (-1)^{l_2+1} \alpha \wedge i_Y i X \beta - (-1)^{l_1 l_2 + l_1 + l_2} i_X \beta \wedge i_Y \alpha$$
$$- (-1)^{l_1 l_2 + l_1 + 1} \beta \wedge i_X i_Y \alpha - (-1)^{l_2} \alpha \wedge i_X i_Y \beta + (-1)^{l_1 l_2 + l_1} \beta \wedge i_Y i_X \alpha$$
$$= 0.$$

To prove that this 1-cocycle is unique we use the same method as in the Proposition 5.

3.5. **Proof of theorem 7.** Let

$$c : Vect(\mathbb{R}^{n|n}) \longrightarrow \mathcal{F}(\mathbb{R}^{n|n})$$

be a 1-cocycle. The restriction of c to the subalgebra $\mathfrak{F}$ (respectively $\mathcal{R}$) is a 1-cocycle over $\mathfrak{F}$ (respectively $\mathcal{R}$). According to propositions 5 and 6 the 1-cocycle c reads

$$c(\mathcal{L}_K + i_L) = c(\mathcal{L}_K) + c(i_L)$$

and

$$\left\{ \begin{array}{ll} c(\mathcal{L}_K) = a\ c_1(\mathcal{L}_K) + \partial\omega_1(K), & (6) \\ c(i_L) \ = b\ c_2(i_L) + \overline{\partial}\omega_2(L) & (7), \end{array} \right.$$

where a, b are real constants and $\partial\omega_1$ and $\overline{\partial}\omega_2$ are coboundaries of $\mathfrak{F}$ and $\mathcal{R}$ respectively given by $\partial\omega_1(K) = L_K(\omega_1)$ and $\overline{\partial}\omega_2(L) = i_L(\omega_2)$. Besides, since the superalgebras $\mathfrak{F}$ and $\mathcal{R}$ are graded and $\mathcal{F}(\mathbb{R}^{n|n}) \cong \Omega(\mathbb{R}^n)$ is a graded module, too, the terms in the right hand of the equation (6) (respectively (7)) must have the same degrees. Then we must have in equation (6):

$$|c_1(\mathcal{L}_K)| = |\partial\omega_1(K)|,$$

but $|c_1(\mathcal{L}_K)| = |c_1(\alpha\otimes X)| = |\alpha|$ where $K = \alpha\otimes X \in \mathfrak{F}$ and $|\partial\omega_1(K)| = |\mathcal{L}_K(\omega_1)| = |\alpha| + |\omega_1|$ (see equation (2)) then $|\omega_1| = 0$ besides $\omega_1 \in \Omega^0(\mathbb{R}^n)$, moreover we must have in equation (7):

$$|c_2(i_L)| = |\overline{\partial}\omega_2(L)|$$

but $|c_2(i_L)| = |c_2(\beta \otimes Y)| = |i_Y(\beta)| = |\beta| - 1$ where $L = \beta \otimes Y \in \mathcal{R}$, then $|\overline{\partial}\omega_2(L)| = |i_{\beta\otimes Y}(\omega_2)| = |\beta| + |\omega_2| - 1$ (see equation (1)), one deduces that $\omega_2 \in \Omega^0(\mathbb{R}^n)$. Since, $i_L(\omega_2) = i_{\beta\otimes Y}(\omega_2) = \beta \wedge i_Y(\omega_2) = 0$, one has $\overline{\partial}\omega_2(L) = 0$.

Now, the condition of 1-cocycle applied to c reads:

(6)
$$bc_2([[K,L]]) - (-1)^{kl}ac_1(i_L(K)) - b\mathcal{L}_K(c_2(i_L)) + (-1)^{lk}ai_L(c_1(\mathcal{L}_K)) = -(-1)^{kl}\partial\omega_1(i_L(K)) + (-1)^{lk}i_L(\partial\omega_1(\mathcal{L}_K)).$$

We use (2), we obtain that the right hand of equation (6) vanish and it becomes

$$bc_2([[K,L]]) - (-1)^{kl}ac_1(i_L(K)) - b\mathcal{L}_K(c_2(i_L)) + (-1)^{lk}ai_L(c_1(\mathcal{L}_K)) = 0. \tag{7}$$

Now, if we substitute the expressions of the 1-cocycles c_1 and c_2 in equation (7), we show that we must have $a + b = 0$. The result follows immediately.

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Ammar Faouzi: Faculté des Sciences, Université de Sfax, B.P.802, Sfax, Tunisie. E-mail: faammar@yahoo.com

Kamoun Kaouthar: Faculté des Sciences, Université de Sfax, B.P.802, Sfax, Tunisie. E-mail:lkkaouthar@Yahoo.com