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THE DETERMINATION OF SHAPE AND MEAN SHAPE FROM SECTIONS AND PROJECTIONS

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ABSTRACT

Classical and recent uniqueness results for convex bodies by sections or projections are described and the corresponding estimation problems from incomplete information are discussed. The results are applied to planar random sets, spatial fibre processes, and Boolean models. It is shown that convex mean bodies can give information about the shape distribution even for random structures of a non-convex nature.

Key words: sections, projections, convex bodies, random set, fibre process, Boolean model.

1. INTRODUCTION

Modern stereology no longer aims only for numerical estimators of volume, surface area, etc., but also intends to infer additional information about shape from sections and/or projections. Classical convex geometry provides a number of uniqueness results for sections and projections of compact convex sets (convex bodies) in d-space $\mathbf{R}^d$, $d \geq 2$, which can be of some use in stereological problems.

In the following, we first give a short survey of some classical and new reconstruction problems for single convex bodies from section or projection means (projection bodies, mean section bodies). We show in particular that certain integral operators on the sphere play a major role in these uniqueness problems and indicate the use of spherical harmonics in solving the corresponding integral equations. We also discuss the associated approximation problems which correspond to the statistical estimation in practice and show how convex geometry can help to overcome the instability of ill-posed inverse problems.

Then, it will be shown how convex bodies arise in stochastic geometry in connection with stationary random sets and random particle processes (in a direct or indirect way) as mean bodies and how some of the techniques described can be used to estimate mean shapes and related distributions (directional distributions).

In particular, three situations are discussed in more detail where the structure under consideration is non-convex, but convex bodies arise in an indirect manner. The first example of that kind arises in connection with the description of the anisotropy of a stationary random closed set in the plane. The second example is the problem of obtaining the directional distribution of a stationary fibre process X in $\mathbf{R}^3$, which is to be estimated from the intensity of the intersection point process $X \cap E_i$ in finitely many planar sections (with planes $E_1, ..., E_k$). Finally, the third situation concerns stationary, but non-isotropic Boolean models Y with compact grains in $\mathbf{R}^2$ or $\mathbf{R}^3$. Here, in the planar case and for, in general, non-convex grains, a convex mean body can be introduced which gives information about the deviation from isotropy and which can be estimated from finitely many realizations of the union set Y. In three dimensions (and for convex grains) a similarly defined mean body is shown to be uniquely determined by the corresponding mean bodies of (randomly chosen) planar sections of the Boolean model Y.

2. UNIQUENESS THEOREMS FOR CONVEX BODIES

For applications in stereology, convex sets in the plane or in three-dimensional space suffice. Generally, from the theoretical point of view, there is a significant difference between the two-dimensional and the three-dimensional situation. Phenomena of this kind will be described later in this section. It is for some parts of this presentation convenient to work with convex sets in d-dimensional space $\mathbf{R}^d$ and to distinguish the two cases $d = 2$ and $d = 3$ afterwards. Therefore, we consider now *convex bodies* $K \subset \mathbf{R}^d$ (these are compact convex subsets of $\mathbf{R}^d$ with non-empty interior) and denote by $\mathcal{K}^d$ the class of all convex bodies. For background information on convex bodies, Bonnesen and Fenchel (1934) is a classical reference; a modern and up-to-date treatment of the theory can be found in Schneider (1993).

In the following, L always denotes a linear subspace in $\mathbf{R}^d$, whereas E denotes an affine subspace. We let $\mathcal{L}_k^d$ and $\mathcal{E}_k^d$ denote the set of all (linear resp. affine) subspaces in $\mathbf{R}^d$ of dimension k, $k \in \{1, ..., d-1\}$. It is obvious that the orthogonal projection $K|L$ of K onto L and the intersection $K \cap E$ of K with E are convex bodies (of dimension k) in L resp. E. It is also simple to show that the family $\{K|L : L \in \mathcal{L}_k^d\}$ resp. $\{K \cap E : E \in \mathcal{E}_k^d\}$ uniquely determines the original body K. It is therefore more interesting, both from a theoretical point of view as well as for stereological applications, to modify the uniqueness problems in an appropriate way.

2A. PROJECTIONS

For projections we therefore consider the question, how far K is determined by its projection function $v_k(K, L)$, $L \in \mathcal{L}_k^d$. $v_k(K, L)$ is defined as the (k-dimensional) content of $K|L$. We also concentrate on the case $k = d-1$ and replace each $(d-1)$-dimensional subspace L by its normal vector u (more precisely, by its antipodal pair $\{u, -u\}$ of normal vectors). We assume $u \in \Omega$, where Ω is the unit sphere in $\mathbf{R}^d$. In this case $v_{d-1}(K, \cdot)$ can be viewed as an even continuous function on Ω. It is clear that any translate $K + x$ and also the reflection $-K = \{-x : x \in K\}$ of K have the same projection function as K. Therefore we can only expect to determine a convex body K up to a translation and reflection. Consequently, we now concentrate on convex bodies which are centrally symmetric with respect to the origin 0 and denote this class by $\mathcal{K}_0^d$. The following uniqueness result is a famous theorem of Aleksandrov (1937).

Theorem 1 (Aleksandrov). *Given $K, K' \in \mathcal{K}_0^d$ with $v_{d-1}(K, \cdot) = v_{d-1}(K', \cdot)$, then $K = K'$.*

It is worthwhile to give an analytic interpretation of this result, since the latter relates the proof to a well-known integral equation. It follows from the general theory of mixed volumes (but could also be obtained directly for polytopes K and then by approximation for general convex bodies K) that

$$v_{d-1}(K,x) = \frac{1}{2}\int_{\Omega} |\cos\alpha(x,u)|\, dS_{d-1}(K,u), \;\; x \in \Omega. \tag{1}$$

Here, $\alpha(x,u)$ denotes the angle (in $[0,\pi]$) between the unit vectors x and u and $S_{d-1}(K,\cdot)$ is the *surface area measure* of the body K. For a Borel set A in the unit sphere Ω, $S_{d-1}(K,A)$ is defined as the total surface area of those boundary points of K which have an outward normal vector in A; for example, for a polytope P, $S_{d-1}(P,\cdot)$ is a discrete measure which is concentrated on the finitely many facet normals $u_1,...,u_m$ of P, the value $S_{d-1}(P,\{u_i\})$ being the $(d-1)$-content of the facet $P(u_i)$. By a theorem of Minkowski, any (nonnegative) Borel measure μ on Ω which is full-dimensional (i.e. not concentrated on a subsphere) and has center of mass at the origin, is the surface area measure of a convex body K, which is unique up to a translation. Hence, K is centrally symmetric if and only if $S_{d-1}(K,\cdot)$ is even.

The uniqueness problem behind Theorem 1 thus reduces to the question whether the function $v_{d-1}(K,\cdot)$ in (1) determines the measure $S_{d-1}(K,\cdot)$ uniquely. It is convenient here to consider (1) in a slightly more general form as

$$h = \int_{\Omega} |\cos\alpha(\cdot,u)|\, d\mu(u), \tag{2}$$

where μ is a measure and h a continuous function on Ω. It is interesting to note that for the inversion of (2) the symmetry of μ is necessary (which implies that h is even). To understand this phenomenon, a small excursion to harmonic analysis on the sphere Ω is helpful. This will be done in section 2C.

More important for applications is the lack of stability of the integral equation (2) in dimension $d = 3$. The main reason for this is the fact that the function h is of a very special kind; it is the support function $h(Z,\cdot)$ of a centrally symmetric convex body Z which can be approximated by finite (Minkowski) sums of segments. In convex geometry, these bodies Z are called *zonoids* (Matheron (1975) used the term Steiner compacts). Zonoids (with inner points) build a small subclass of $\mathcal{K}_0^d$; e.g. a polytope P is a zonoid if (and only if) all faces (of all dimensions) of P have a center. Such polytopes are called *zonotopes*.

If $h = v_{d-1}(K,\cdot)$, the zonoid Z with $h = h(Z,\cdot)$ is called the *projection body* ΠK of K. In the planar case, ΠK is simply obtained from K by a 90° rotation $\vartheta_{\pi/2}$. Consequently, any $K \in \mathcal{K}_0^2$ is a zonoid and the integral equation (2) is stable. Inversion of (2) or approximation of μ are therefore not problematic in $\mathbf{R}^2$.

We will continue the discussion of these questions in 2C.

2B. SECTIONS

With respect to sections, a corresponding uniqueness problem has to be formulated in a different manner. If, for $E \in \mathcal{E}_k^d$, $w_k(K,E)$ denotes the k-dimensional content of the section $K \cap E$, then the function $w_k(K,\cdot)$ obviously determines K, since K is already specified by the support of this function. Moreover, from a stereological point of view, the evaluation of characteristics of sections $K \cap E$ which depend on the spatial position of E, seems to be a difficult task. Therefore, we consider the *mean section body* $M_k(K)$

of $K \in \mathcal{K}$. The simplest way to define $M_k(K)$ is by its support function (a procedure which shows some similarity with (1)), namely

$$h(M_k(K), x) = \int_{\mathcal{E}_k^d} h(K \cap E, x)\, d\mu_k^d(E), \; x \in \Omega. \tag{3}$$

Here, μ_k^d is the (suitably normalized) motion invariant measure on $\mathcal{E}_k^d$ and, in order to fix the position of $M_k(K)$ among all translates, we assume in addition that the Steiner point of $M_k(K)$ is at the origin, i.e.

$$\int_\Omega x\, h(M_k(K), x)\, d\lambda(x) = 0$$

(here λ denotes the spherical Lebesgue measure). $M_k(K)$ can be interpreted as the limit of Minkowski sums of (independent random) k-dimensional sections of K. The shape of $M_k(K)$ is thus not affected by individual translations of the sectioned sets $K \cap E$, a fact which makes the mean section body interesting for stereological applications. The basis for a reconstruction of K from section means would be a uniqueness theorem for $M_k(K)$. It turns out that here the case $k = 1$ (linear sections) is not interesting, the corresponding mean section body $M_1(K)$ is always a ball (of radius proportional to the volume of K) and therefore does not determine K. In the sequel, we therefore concentrate on planar sections $k = 2$. The following uniqueness result was obtained in Goodey and Weil (1992). It looks similar to Theorem 1, but the main difference is that no symmetry is assumed.

Theorem 2. *Given $K, K' \in \mathcal{K}^d$ with $h(M_2(K), \cdot) = h(M_2(K'), \cdot)$, then $K = K'$, up to a translation.*

The common background of both theorems becomes apparent from the following integral formula for $h(M_2(K), \cdot)$ which results from (3) with the use of more general formulae from translative integral geometry,

$$h(M_2(K), x) = \frac{\kappa_2 \kappa_{d-2}}{\binom{d}{2} \kappa_d} \int_\Omega \alpha(x, u) \sin \alpha(x, u)\, dS_{d-1}(-K, u), \; x \in \Omega, \tag{4}$$

(where κ_j is the j-volume of the j-dimensional unit ball). Again, instead of (4), we consider the integral equation in a slightly more general form,

$$h = \int_\Omega \alpha(\cdot, u) \sin \alpha(\cdot, u)\, d\mu(u), \tag{5}$$

where μ is now a general measure (without the restriction of evenness) on Ω.

The remarks made above in connection with (2) can be made in an analogous way for (5). The instability of (5) is expressed by the fact that the functions h, occurring on the left-hand side of (5) are support functions of special convex bodies. The difference to the previous situation is that the corresponding class $\{M_2(K) : K \in \mathcal{K}^d\}$ (resp. its closure in the Hausdorff metric) is not yet characterized. Only for symmetric K it is known that $M_2(K)$ is an iterated projection body, hence a special zonoid. Due to this open characterization problem, the use of Theorem 2 in practical applications is limited, up to now.

2C. STABILITY AND APPROXIMATION

As remarked the joint background for the uniqueness results in Theorems 1 and 2 as well as the difference in the symmetry assumptions conclude from harmonic analysis on the sphere Ω. Here, we give some further details which also explain the problems with stability and approximation.

The basic result is the Funk-Hecke Theorem for spherical harmonics (restrictions of homogeneous polynomials to the sphere), which we state here in a convenient form which can be found e.g. in Schneider (1970). In the following result, $C_m^{(d-2)/2}$ is the Gegenbauer polynomial of order $\frac{d-2}{2}$ and degree m and S_m denotes a spherical harmonic of degree m.

Theorem 3 (Schneider). *Let f be a continuous function on $[-1,1]$ and*

$$\lambda_m[f] = \int_{-1}^{1} f(t)\, C_m^{(d-2)/2}(t)\,(1-t^2)^{(d-3)/2}\, dt, \quad m = 0,1,2,...$$

Then for a signed Borel measure ρ on Ω the condition

$$\int_{\Omega} f(\cos\alpha(x,v))\; d\rho(v) = 0 \text{ for all } x \in \Omega$$

implies $\rho = 0$ if and only if ρ fulfills

$$\int_{\Omega} S_m(u)\; d\rho(u) = 0 \tag{6}$$

for all S_m and $m \in \{0,1,2,...\}$ with $\lambda_m[f] = 0$.

In the case of equation (1), this result is applied with $f(t) = |t|$ and $\rho = S_{d-1}(K,\cdot) - S_{d-1}(K',\cdot)$. Then, $\lambda_m[f] \neq 0$ precisely for even m, hence the above condition has to be checked for odd m only. In that case S_m is an odd function, hence (6) is fulfilled since ρ is even.

In the case of equation (4), we choose $f(t) = \{\arccos(t)\}\,(1-t^2)^{1/2}$. Here, all coefficients $\lambda_m[f], m = 0,1,2,...$, are non-zero and therefore $\rho = S_{d-1}(K,\cdot) - S_{d-1}(K',\cdot) = 0$.

The spherical harmonics approach not only solves the uniqueness problems, it can also be used for the inversion of (2) and (5) in the smooth case. If the function h has sufficient smoothness (in case of (2), differentiability of order $d+3$ is sufficient), then the measure μ has a continuous density g w.r.t. λ, and the functions h and g have an expansion (as an, in general, infinite series) into spherical harmonics, the coefficients of which are directly related. We do not give the corresponding formulae here, but refer to Schneider (1967), for further details. We use this fact, however, to comment on an important consequence of this expansion.

Obviously, the finite partial sums of such a series can be used for an approximation of g. This suggests the following inversion procedure which can be put into practice: replace the given function h by a smooth approximation $\tilde{h}$, produce an approximate inverse $\tilde{g}$ of $\tilde{h}$ by expansion into a finite series of spherical harmonics, use $\tilde{g}$ as an approximation for g. The already mentioned instability of the integral equations (2) and (5) is expressed by the fact that such an approximation procedure as the one just described does not work in dimension $d = 3$. The reason is the already mentioned special nature of the functions h, which implies that a smooth approximation $\tilde{h}$ is in general not in this class anymore and consequently $\tilde{g}$ is not close to g as a continuous function. Even the measure $\tilde{\mu}$ (which has $\tilde{g}$ as its density) is close to μ only in the topology of Schwartz distributions, which means that it need not be a nonnegative measure at all. This phenomenon has a serious effect if, in the case of stochastic applications, μ represents a probability distribution which is to be estimated.

Such estimation or approximation problems usually occur in stereology, since there only finitely many sections or projections can be evaluated. Consequently, in any such application, only finitely many values of $h(u_1), ..., h(u_k)$ of the function h in (2) or (5) (or even approximations of these values only) will be available, and it is the task to obtain an approximation of μ from these data, e.g. by an interpolation of $h(u_1), ..., h(u_k)$ with a smooth or a piecewise linear function and a subsequent inversion.

In this situation, geometry can help, at least in the situation of (2), since here the functions h are characterized to be support functions of zonoids. For (5) the corresponding theory is still to be developed based on an appropriate characterization of mean section bodies.

With respect to (2), the basic idea is to approximate the values $h(u_1), ..., h(u_k)$ by the support values $h(Z_k, u_1), ..., h(Z_k, u_k)$ of a simple convex body Z_k which belongs to the class of zonoids and for which the inversion of the integral equation (2) is easy, namely a zonotope. The following result was shown in Campi et al. (1993). For the formulation, we denote by $[-x, x]$, $x \in \Omega$, the segment from $-x$ to x and by δ_x the discrete measure concentrated on $\{-x, x\}$ and giving measure 1 to each point. The k planes through the origin and orthogonal to $u_1, ..., u_k$ (respectively) divide $\mathbf{R}^3$ into m closed polyhedral cones $C_1, ..., C_m$ (with interior points), $m \leq 2^k$. For $i = 1, ..., m$ and $j = 1, ..., k$, the sign of $\cos\alpha(u_j, z_i)$ does not change as z_i runs through all (interior) elements of C_i. We put $\epsilon_{ij} = 1$ if this sign is positive and $\epsilon_{ij} = -1$ if it is negative.

Theorem 4. *Let μ be an even finite measure on Ω generating a function h by equation* (2). *Let $u_1, ..., u_k \in \Omega$ be arbitrary. Then there exist k points $x_1, ..., x_k \in \Omega$ and weights $\alpha_1 \geq 0, ..., \alpha_k \geq 0$ such that the zonotope*

$$Z_k = \sum_{i=1}^{k} \alpha_i \, [-x_i, x_i]$$

fulfills

$$h(Z_k, u_1) = h(u_1), \; ... \; , h(Z_k, u_k) = h(u_k).$$

The linear program (LP)

$$\text{Minimize } f(y_1, ..., y_m) = \sum_{j=1}^{k} (h(u_j - \sum_{i=1}^{m} \epsilon_{ij} \cos\alpha(u_j, y_i)))$$

$$\text{subject to } \sum_{i=1}^{m} \epsilon_{ij} \cos\alpha(u_j, y_i) \leq h(u_j), \; j = 1, ..., k,$$

$$\text{and } y_i \in C_i, \; i = 1, ..., m.$$

is solvable and any vertex solution of (LP) *has at most k non-zero entries $y_{i_1}, ..., y_{i_k}$ from which the points $x_1, ..., x_k$ and weights $\alpha_1, ..., \alpha_k$ result due to the equations*

$$y_{i_1} = \alpha_1 x_1, \; ..., \; y_{i_k} = \alpha_k x_k.$$

If $u_1, ..., u_k$ are independent uniform random directions in Ω, then the resulting discrete measures

$$\mu_k = \sum_{i=1}^{k} \alpha_i \delta_{x_i}$$

converge weakly towards μ, as $k \to \infty$.

The two-dimensional situation is much simpler, since in $\mathbf{R}^2$ any centrally symmetric body is a zonoid, hence any centrally symmetric polygon is a zonotope and can be used for the approximation, in analogy to Theorem 4 (but without solving a linear program). In particular, the k lines

$$\{z \in \mathbf{R}^2 : \cos\alpha(z, u_i) = h(u_i)\}, \ i = 1, ..., k,$$

and their reflections build the boundary of such a zonotope. This geometric approach is the basis of the procedure described in Rataj and Saxl (1989).

3. MEAN SHAPES IN STOCHASTIC GEOMETRY

The random models which have been developed and successfully used in stereology in the last decades are the stationary *random closed sets* and the stationary random collections of (possibly overlapping) compact sets (the stationary *particle point processes*). If such random structures X are, in addition, isotropic, any suitably defined mean shape will be a ball. Hence, in the following we are interested in non-isotropic structures X, in particular a mean shape of X should reflect the degree of anisotropy of X.

As is to be expected, there will be a difference between random sets and point processes, but also between the dimensions two and three. We therefore treat some of the resulting cases separately and show how *convex mean bodies* arise.

3A. PLANAR RANDOM SETS

Let $Y \subset \mathbf{R}^2$ be a stationary random closed set, obeying some mild regularity conditions. These are fulfilled, e.g., if Y belongs to the extended convex ring $\mathcal{S}^2$ (i.e. Y is a locally finite union of convex bodies), or if the boundary of Y consists piecewisely of Jordan arcs. Then, in a compact and convex 'sampling window' $K \subset \mathbf{R}^2$ any realization $Y(\omega)$ of Y allows a finite boundary measure $S_1(Y(\omega) \cap K, \cdot)$ which is a nonnegative measure on the unit circle Ω (this measure is the two-dimensional version of the surface area measure considered in section 2). Since

$$\int_\Omega x\, S_1(Y(\omega) \cap K, dx) = 0,$$

by Minkowski's theorem there is a unique convex body $M \subset \mathbf{R}^2$ with $S_1(Y(\omega) \cap K, \cdot) = S_1(M, \cdot)$ (and obeying $\int_\Omega x h(M, x)\, d\omega(x) = 0$), the *convexification* $\mathrm{co}\,(Y(\omega) \cap K)$ of $Y(\omega) \cap K$. This convexification has been studied by Pach (1978) and Fáry and Makai (1982) (for geometric reasons) and in Weil (1993a) (for random sets). From $\mathrm{co}\,(Y(\omega) \cap K)$, a mean shape of Y can be obtained by a limiting procedure, in analogy to the definition of the boundary length density L_A. We need some integrability conditions on Y, which are omitted here (see Weil (1993a), for details).

Theorem 5. *Let Y be a stationary random closed set in $\mathbf{R}^2$ and $K \subset \mathbf{R}^2$ a convex body. Then the mapping $\omega \mapsto h(\mathrm{co}\,(Y(\omega) \cap K), \cdot)$ is measurable, the expectation*

$$\mathbf{E}\, h(\mathrm{co}\ (Y \cap K), \cdot)$$

exists, and (under some integrability condition) the limit

$$h^{\star}_{A}(Y, \cdot) = \lim_{r \to \infty} \frac{\mathbf{E}\, h(\mathrm{co}\ (Y \cap K), \cdot)}{A(rK)} \tag{7}$$

is independent of K and is the support function of a convex body $K(Y)$.

We call $K(Y)$ the *mean body* of Y.

The star in $h^{\star}_{A}(Y, \cdot)$ refers to the choice of the convexification as a 'centred' set. If Y and the reflection $-Y$ have the same distribution, then $K(Y)$ is centrally symmetric with center 0. We may also call the function $h^{\star}_{A}(Y, \cdot)$ the *support density* of Y. The boundary measure $S_1(K(Y), \cdot)$ fulfills

$$S_1(K(Y), \cdot) = \lim_{r \to \infty} \frac{\mathbf{E}\, S_1(\mathrm{co}\ (Y \cap K), \cdot)}{A(rK)}, \tag{8}$$

independently of K. We have

$$S_1(K(Y), \Omega) = L_A,$$

hence $S_1(K(Y), \cdot)$ is a local version of the boundary length density. If Y is isotropic, $h^{\star}_{A}(Y, \cdot)$ is a constant (a multiple of L_A), hence $K(Y)$ is a circle and $S_1(K(Y), .)$ is a multiple of λ, but the reverse implications are not true. (8) follows immediately from (7) with the well-known relation between the support function $h(K, \cdot)$ of a convex body K and its surface area measure. In particular, in the plane we have

$$S_1(K, A) = \int_A (h''(K, x) + h(K, x))\, d\lambda(x), A \subset \Omega, \tag{9}$$

if K is smooth enough, the circle Ω is parametrized by the angle $x, x \in [0, 2\pi)$, and h'' denotes the second derivative with repect to this parametrization. For arbitrary K, a corresponding relation holds in the sense of Schwartz distributions.

The boundary ∂Y of Y is (under the regularity conditions we mentioned) a 1-dimensional random set (a fibre process in the sense of Stoyan et al. (1987)). For ∂Y, the distribution $P_{\partial Y}$ of the tangential direction in a 'typical' point of ∂Y is usually called the *rose of directions*, $P_{\partial Y}$ is an even probability measure on Ω. The zonoid $Z(\partial Y)$ generated by $L_A P_{\partial Y}$ is the Steiner compact associated with ∂Y. It follows easily from the above considerations that the symmetrized boundary measure of $K(Y)$ fulfills

$$S_1(K(Y), .) + S_1(-K(Y), .) = \vartheta_{\pi/2}\, L_A P_{\partial Y},$$

or equivalently,

$$K(Y) + (-K(Y)) = \vartheta_{\pi/2}\, Z(\partial Y),$$

where $\vartheta_{\pi/2}$ is again the rotation by 90°. Hence, in those cases where $K(Y)$ is not centrally symmetric, $K(Y)$ carries more information about the random set Y than the rose of directions (or the Steiner compact) of the boundary set ∂Y. The difference is that, in $K(Y)$, the boundary points are taken into account together with their outer normals.

In Weil (1993a) a number of further properties of the mean body $K(Y)$ are studied. We mention only the following observation which also gives a practical procedure for estimating $K(Y)$. Assume that a realization $Y(\omega)$ of Y is observed in the unit square W and, for practical purposes, the boundary of $Y(\omega) \cap W$ is given (or approximated) as a sequence of segments. If we order these segments clockwisely according to their outer normals, they fit together to form a closed convex polygon, namely $K_1 = \text{co}\,(Y(\omega) \cap W)$. Due to the edge effects, K_1 is not yet an unbiased estimate for $K(Y)$. In order to obtain such an estimate, we have to consider in addition the rectangle K_2 which is the sum of those segments of $\partial(Y(\omega) \cap W)$ which lie in the upper or the right boundary of W. Then the Minkowski difference $K_1 - K_2$ is an unbiased estimate of $K(Y)$. In general, $K_1 - K_2$ need not be a convex body, since K_2 is not automatically a summand of K_1, but if we repeat this sampling procedure independently k times and take as K_1^k and K_2^k the corresponding sampling means (in the sense of Minkowski addition), the difference $K_1^k - K_2^k$ tends to $K(Y)$ almost surely (a precise statement is possible using the support function).

3B. SPATIAL FIBRE PROCESSES

A stationary spatial fibre process X can be described either as a point process of curves or a random 1-dimensional set. For the analysis of X which we have in mind, both representations are equivalent. The main quantities of Y which are of stereological interest, are the *length density* L_V and the *directional distribution* P_0 (the distribution of the tangential direction in a 'typical' point of X, the rose of directions). If $E_1, ..., E_k$ are planes with (randomly chosen) normal directions $u_1, ..., u_k$, respectively, the intersection $X \cap E_i$ is (almost surely) an ordinary stationary point process, the intensity of which depends only on u_i. We denote it by $\gamma(u_i)$. As is well known (see e.g. Stoyan et al (1987) or Weil (1987)),

$$\gamma(u_i) = L_V \int_\Omega |\cos \alpha(u_i, x)|\; dP_0(x).$$

Since $\gamma(u_i)$ can simply be estimated by counting the number of intersection points of $Y(\omega) \cap E_i$, the basic stereological problem is to estimate L_V and P_0 from $\gamma(u_1), ..., \gamma(u_k)$. This situation fits now easily into our previous considerations if we define the *mean body* $K(X)$ associated with K as the zonoid in $\mathbf{R}^3$ which is generated by the measure $\mu = L_V P_0$. Then

$$\gamma(u_i) = h(K(X), u_i), \; i = 1, ..., k,$$

and the optimization method described in 2C is applicable.

3C. BOOLEAN MODELS

A *Boolean model* Y is a random closed set which results as the union of a stationary Poisson process X of compact sets, the grains. X is uniquely determined by two quantities, the *intensity* γ and the *shape distribution* P_0 (also called the *distribution of the primary grain*). We do not assume isotropy and, moreover, in the planar case the grains can be non-convex (but with similar regularity conditions as in 2A; i.e. either they are finite unions of convex bodies or they are bounded by finitely many Jordan arcs). For the intensity γ, we assume $0 < \gamma < \infty$. P_0 is a probability measure on the 'space of shapes' $\mathcal{C}_0$, i.e. the collection of compact sets (of the described manner) centred at the origin (but not necessarily centrally symmetric). A familiar choice of a center is the mid-point of the circumsphere.

We consider the two-dimensional situation first. Then, the union set Y allows a mean body $K(Y)$ as in 2A. On the other hand, for each $C \in \mathcal{C}_0$, the convexification $\operatorname{co} C$ is defined as in 2A. $\operatorname{co} C$ is a planar convex body which contains (a translate of) the convex hull $\operatorname{conv} C$ of C. Both sets (the convexification and the convex hull) coincide (up to a translation), if and only if C is convex. For the Poisson process X, a *mean body* $K(X)$ can simply be defined by

$$h(K(X),\cdot) = \gamma \int_{\mathcal{C}_0} h(\operatorname{co} C,\cdot)\, dP_0(C). \tag{10}$$

For non-convex grains, the body $\gamma^{-1}K(X)$ is larger than the set-valued expectation of P_0, as it is discussed e.g. in Vitale (1988) (see also Weil (1993a)). The latter mean body would result if, in the integral in (10), $\operatorname{co} C$ is replaced by $\operatorname{conv} C$. Of course, for convex grains the two notions of mean bodies (or expectations) coincide. Our definition of $K(X)$ has the advantage, that we get a simple connection between $K(Y)$ and $K(X)$ (in the planar case) which allows to estimate $K(X)$ from observations of Y. The following result is mentioned in Weil (1990) (more details will appear in Weil (1993b)). We denote by $\bar{A}$ and $\bar{L}$ the mean area and mean boundary length of the particles of X, i.e. the expectation of the area A and the boundary length L w.r.t P_0.

Theorem 6. *For a Boolean model Y in $\mathbf{R}^2$ with underlying Poisson particle process X, we have*

$$h(K(Y),\cdot) = e^{-\gamma\bar{A}}\, h(K(X),\cdot). \tag{11}$$

If $K \in \mathcal{K}_0^2$, then

$$\mathbf{E}\, h(\operatorname{co}(Y \cap K),\cdot) = A(K)\, e^{-\gamma\bar{A}}\, h(K(X),\cdot) + h(K,\cdot)\,(1 - e^{-\gamma\bar{A}}). \tag{12}$$

As a direct consequence of (9), (11) can be transformed into a relation for boundary length measures, namely

$$S_1(K(Y),\cdot) = e^{-\gamma\bar{A}}\, S_1(K(X),\cdot). \tag{13}$$

(13) is the local version of the familiar stereological equation

$$L_A = e^{-\gamma\bar{A}}\gamma\bar{L} \tag{14}$$

between the length density L_A of Y and the characteristics of X (see Stoyan et al. (1987) or Weil (1988)). As we discussed in 3A, these local versions are in general non-symmetric and, therefore, carry more information than the corresponding relations between the roses of directions of the boundaries ∂Y and ∂X. Such a relation follows from (13), if we replace X and Y by $-X$ and $-Y$, add up the corresponding equations, and apply the rotation $\vartheta_{\pi/2}$. If we normalize the resulting measures with the help of (14), we get the information that the roses of directions of ∂Y and ∂X coincide, a result which can be obtained directly from the independence properties of Poisson processes. The rose of direction was used recently in Molchanov and Stoyan (1993) and Molchanov et al. (1993) for a statistical analysis of certain Boolean models.

As was described in 3A in a more general situation, (12) can be used for estimation in practice and therefore, $K(X)$ can be estimated. In Weil (1993b) this procedure is described in more detail and used to give an estimator of the intensity (particle number) γ, which turns out to be quite effective for non-isotropic Boolean models.

The three-dimensional situation is more involved, at the present stage. We have to assume convex grains here and consider the situation, where planar sections $Y \cap E$ (with randomly chosen direction) of the Boolean model $Y \subset \mathbf{R}^3$ are investigated. As described above, Theorem 6 can be used to estimate the mean body $K(X \cap E)$ of the intersection process $X \cap E$. The following result from Weil (1993c) describes the connection with the mean body $K(X)$ (since we assume convex grains here, the mean bodies are defined by their support functions as Minkowski integrals of the grains as in (10)).

Theorem 7. *Let X be a stationary Poisson process of convex bodies in $\mathbf{R}^3$ and $E \subset \mathbf{R}^3$ a random plane with uniformly distributed direction. Then,*

$$\mathbf{E}\, h(K(X \cap E), \cdot) = \frac{2}{3}\, h(M_2(K(X)), \cdot). \tag{15}$$

Here, on the left-hand side of (15), we consider the set $K(X \cap E)$ as a two-dimensional body in $\mathbf{R}^3$ and average the corresponding support functions. The resulting convex body is thus the mean section body of the mean body $K(X)$. Theorems 6 and 7 (together with Theorem 2) imply that the mean body of the spatial Poisson particle process X is uniquely determined by the planar sections $Y \cap E$ of the union set Y. However, a practical estimation procedure of that kind is at the moment not apparent due to the open characterization (and approximation) problem in the class of mean section bodies.

However, this final result emphasizes once more the importance of recent developments in convex and integral geometry for advanced estimation problems in stochastic geometry (and their stereological applications).

REFERENCES

Aleksandrov AD. On the theory of mixed volumes of convex bodies, I: Extension of certain concepts in the theory of convex bodies, II: New inequalities between mixed volumes and their applications (Russian). Mat Sbornik N S 1937; 2: 947–72, 1205–38.

Bonnesen T, Fenchel W. Theorie der konvexen Körper. Berlin: Springer, 1934: 1–164.

Campi S, Haas D, Weil W. Approximation of zonoids by zonotopes in fixed directions. Discrete Comput Geom 1993; to appear.

Fáry I, Makai E Jr. Isoperimetry in variable metric. Stud Sci Math Hungar 1982; 17: 143–58.

Goodey P, Weil W. The determination of convex bodies from the mean of random sections. Math Proc Camb Phil Soc 1992; 112: 419–30.

Matheron G. Random sets and integral geometry. New York: Wiley, 1975: 1–261.

Molchanov IS, Stoyan D. Directional analysis of fibre processes related to Boolean models. Metrika 1993; to appear.

Molchanov IS, Stoyan D, Fyodorov KM. Directional analysis of planar fibre networks: application to cardboard microstructure. J Microsc 1993; to appear.

Pach J. On an isoperimetric problem. Stud Sci Math Hungar 1978; 13: 43–5.

Rataj J, Saxl I. Analysis of planar anisotropy by means of the Steiner compact. J Appl Prob 1989; 26: 490–502

Schneider R. Zu einem Problem von Shephard über die Projektionen konvexer Körper. Math Z 1967; 101: 71–82.

Schneider R. Über eine Integralgleichung in der Theorie der konvexen Körper. Math Nachr 1970; 44: 55–75.

Schneider R. Convex bodies: the Brunn-Minkowski theory. Cambridge: Cambridge Univ Press, 1993: 1–490.

Stoyan D, Kendall WS, Mecke J. Stochastic geometry and its applications. Berlin: Akademie-Verlag, 1987: 1–345.

Vitale RA. An alternate formulation of the mean value for random geometric figures. J Microsc 1988; 151: 197–204.

Weil W. Point processes of cylinders, particles and flats. Acta Applicandae Math 1987; 9: 103–36.

Weil W. Expectation formulas and isoperimetric properties for non-isotropic Boolean models. J Microsc 1988; 151: 235–45.

Weil W. Lectures on translative integral geometry and stochastic geometry of anisotropic random geometric structures. Atti del Primo Convegno Italiano di Geometria Integrale. Rend Sem Mat Messina, Ser II. 1990; 13: 79–97.

Weil W. Support functions on the convex ring in the plane and support densities for random sets and point processes. Suppl Rend Circ Mat Palermo 1993a; to appear.

Weil W. The estimation of mean shape and mean particle number in overlapping particle systems in the plane. 1993b; in preparation.

Weil W. On the mean shape of particle processes. 1993c; in preparation.