ACTA STEREOL 1993; 12/2: 89-94
PROC 6ECS PRAGUE, 1993
ORIGINAL SCIENTIFIC PAPER

A DIFFERENT APPROACH OF DISTANCES IN THE SET OF PLANAR BODIES

Isabelle Fillère*, Marie-José Labouré, Michel Jourlin*, Jean-Marie Becker*****

*Lab. LETIA, ***Lab. LIDI, Institut de Chimie et Physique Industrielles, 31 Place Bellecour 69288 Lyon cedex 02 ; France
**Lab. TSI, U. A. CNRS 842, 23 rue du Docteur Michelon 42023 St. Etienne cedex 02 ; France

ABSTRACT

New metrics in the set of planar bodies are proposed : one is a generalization of the Hausdorff distance, using any convex body as structuring element, the other ones are based on radial functions.

Keywords : Gauge distance, Hausdorff metric, Planar Bodies, Radial function.

INTRODUCTION

In shape recognition, one needs a notion of distance between two bodies, to evaluate their relative position or overlapping, to compare their geometric shapes...

One should think that the more interesting distance to use is the euclidean distance, but, in the space of compact sets, it does not yield a distance. Thus, specific metrics have been defined adapted to different kinds of problems : The Hausdorff distance which can estimate, for example, the relative position of two shapes ; or the Asplund distance which can gauge one shape with regard to the other.

Based on these ideas, new metrics are defined here, one being based on the Hausdorff distance defined with other structuring elements than the usual ones in order to get an anisotropic operation ; the others more adapted to the space of classes of shapes equivalent up to a translation.

PRELIMINARIES

Let $\mathcal{K}$ be the set of planar bodies (compact sets with a non empty interior) in the euclidean plane $\mathbb{R}^2$, $\mathcal{K}_s$ be the set of bodies star-shaped with respect to interior points. We define $\mathcal{K}_s^t$ as being the spaces of classes of elements of $\mathcal{K}_s$ consisting of bodies equivalent under translations.

The Hausdorff distance (d_H) in $\mathcal{K}$ can be defined in two ways :

1) $\forall\ (K, L) \in \mathcal{K}^2 \qquad d_H(K,L) = \mathrm{Max}\{\sup_{x\in K} d(x, L), \sup_{y\in L} d(y, K)\}$ (1)

where d is the euclidean distance in $\mathbb{R}^2$.

2) morphological definition

$\forall\ (K, L) \in \mathcal{K}^2 \qquad d_H(K,L) = \mathrm{Inf}\{r \in \mathbb{R}^{+*} ; K \oplus r B \supset L \text{ and } L \oplus r B \supset K\}$ (2)

where B is the unit disk of IR^2 and $\oplus$ the Minkowski addition defined, for A and C two compact sets of IR^2, by $A \oplus C = \{a + c ; a \in A, c \in C\}$

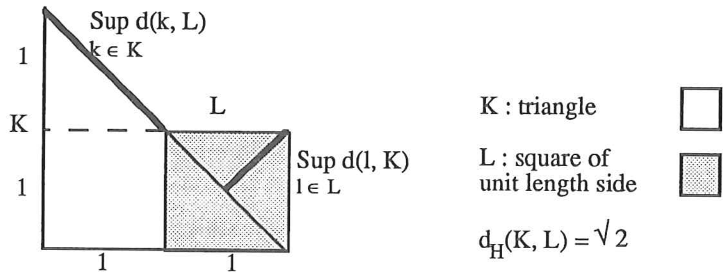


Figure 1 . Example of the Hausdorff distance between a square of unit length side and a triangle.

The space ($\mathcal{K}$, d_H) is known to be a complete metric space ; with this topology, $\oplus$ is a continuous mapping from $\mathcal{K}^2$ to $\mathcal{K}$ (Matheron, 1975).

By using a disk as structuring element, we obtain an isotropic operation. In order to take into account prominent directions, we need to generalize the Hausdorff distance by using non circular and not necessarily symmetric structuring elements.

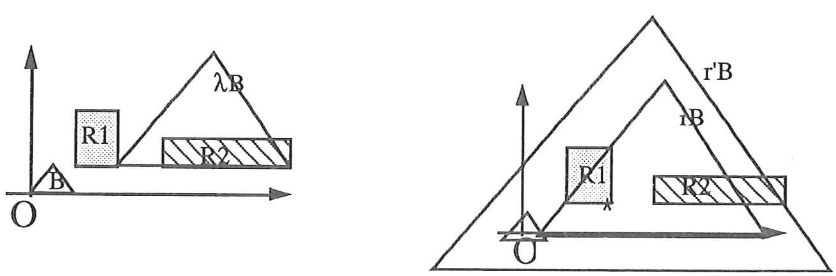
THE GENERALIZED HAUSDORFF DISTANCE

Let B be a fixed convex body in IR^2. Let us define for K and L two bodies in IR^2 :

$$d_{GH}(K,L) = \text{Inf } \{ r \in IR^{+*} ; K \oplus r B \supset L \text{ and } L \oplus r B \supset K \} \qquad (3)$$

Notes: $d_{GH}(K, L) \in IR^+ \cup \{+\infty\} = \overline{IR}^+$

We can notice that the distance between two bodies will be different if the position of the origine in the structuring element is moved (cf fig. 2).



a) b)

Figure 2 . Example of the distance between two bodies R1 and R2 with a triangular structuring element. If its origine is choosen at a vertex (a), the generalized Hausdorff distance will be infinite (no dilation of R2 will contain R1 : $d_{GH}(R_1, R_2) = \text{Max}(\lambda, +\infty)$). When the origin is taken in the triangular structuring element (b), the distance is then finite ($d_{GH}(R_1, R_2) = \text{Max}(r, r')$).

PROPOSITION 1 : The mapping d_{GH} is a distance in $\mathcal{K}$.

Proof :

a) By definition $d_{GH}(K, L) \geq 0$ and $d_{GH}(K, L) = d_{GH}(L, K)$

b) $d_{GH}(K, L) = 0 \Leftrightarrow \forall r > 0 \quad K \oplus r B \supset L \quad$ and $\quad L \oplus r B \supset K$

By continuity of the operation $\oplus$ and of the function $B \rightarrow rB$ for the d_H topology, it implies :

$$\lim_{r \to 0} K \oplus r B = K \text{ and } \lim_{r \to 0} L \oplus r B = L$$

So $\quad d_{GH}(K, L) = 0 \Leftrightarrow \quad K \supset L$ and $L \supset K \quad \Leftrightarrow \quad K = L$

c) Let us prove now the triangular inequality :

Let K, L, M be three bodies in $\mathcal{K}$.

We set $d_{GH}(K, L) = r_1 \quad d_{GH}(L, M) = r_2 \quad d_{GH}(K, M) = r_3 \quad$ where $r_i \in \overline{IR}^+$

We must prove that $r_3 \leq r_1 + r_2 \quad (*)$

- this inequality is obvious if r_1 or r_2 is infinite
- if r_1 and r_2 are finite :

From $(K \oplus r_1 B \supset L$ and $L \oplus r_1 B \supset K)$ and $(L \oplus r_2 B \supset M$ and $M \oplus r_2 B \supset L)$ we get that:

$$(K \oplus r_1 B) \oplus r_2 B \supset M \quad \text{and} \quad (M \oplus r_2 B) \oplus r_1 B \supset K$$

Associativity of $\oplus$ and convexity of B (Matheron 1975) give :

$K \oplus (r_1 + r_2) B \supset M \quad$ and $\quad M \oplus (r_1 + r_2) B \supset K$ which proves (*). ♦

Without the convexity hypothesis about the structuring element B, d_{GH} is no more a distance.

The generalized Hausdorff distance have been used for a long time : effectively, until now the computation of the Hausdorff distance on a grid needs to approximate the ball by a square or an hexagonal structuring element. We have shown here that the use of such structuring elements also leads to a distance.

THE RADIAL DISTANCE

Another extension of the Hausdorff distance can be defined without a structuring element, the dilation being replaced by another kind of shape transformation. This transformation, called "radial dilation", will be defined in the space of star-shaped bodies using radial functions.

We will denote by $\mathcal{St}(K)$ the set of interior points with respect to which K is star-shaped. For $K \in \mathcal{K}$ and $\omega \in \mathcal{St}(K)$, the distance of ω to the edge of K in a direction θ is defined with no ambiguity. This distance $\rho_K(\omega, \theta)$ is the value of the radial function of K measured with respect to ω in the direction θ ($\theta \in [0, 2\pi]$). It will simply be denoted $\rho_K(\theta)$ or $\rho_\theta(K)$ when there is no ambiguity.

K is fully characterized, up to a translation, by the knowledge of $\rho_K(\omega,\theta)$, when θ describes $[0, 2\pi]$. For $h \in IR^{+*}$, and for $\omega \in \mathcal{St}(K)$, we define the "h-radial dilation of K with respect to ω" by its radial function :

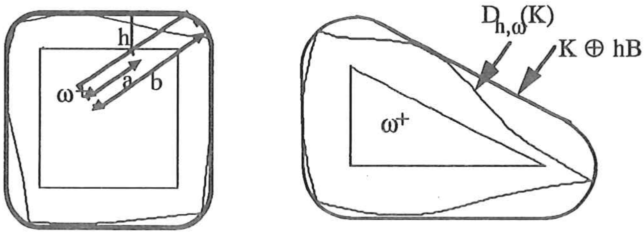
$$\rho_{D_{\omega,h}(K)}(\omega, \theta) \quad = \quad \rho_K(\omega, \theta) + h \qquad \theta \in [0, 2\pi] \qquad (4)$$

This shape clearly belongs to $\mathcal{K}_S$, and will be denoted $D_{\omega,h}(K)$, or $D_h(K)$ when there is no ambiguity.

PROPOSITION 2 : The radial dilated set of K of length h ,with respect to any point ω is contained in the "classical" dilation of K by a disk of radius h.

$$\forall K \in \mathcal{K}_s \qquad \forall \omega \in \mathcal{St}(K) \qquad K \oplus hB \supset D_{\omega,h}(K) \quad (5)$$

proof : (Labouré, 1987)
It suffices to remark that : $\forall \omega \in \mathcal{St}(K), \forall \theta \in [0, 2\pi] \qquad \rho_\theta(K \oplus hB, \omega) \geq \rho_\theta(K, \omega) + h$ ♦



$a = \rho_\theta(K) \quad b = \rho_\theta(K \oplus hB)$

Figure 3 . Radial dilation of a square and a triangle, and illustration of (5).

The radial distance between K and L is defined, by analogy with the Hausdorff distance.

If $\omega \in \mathcal{St}(K) \cap \mathcal{St}(L)$ let d_R be defined by :

$$d_R(K, L) \quad = \quad \text{Inf}\{h>0, D_h(K) \supset L \text{ and } D_h(L) \supset K\} \qquad (6)$$

PROPOSITION 3 : The mapping d_R is a distance on $\mathcal{K}_s$.

proof: It can be expressed by $d_R(K, L) = \|\rho_K - \rho_L\|_\infty = \text{Sup}|\rho_\theta(K) - \rho_\theta(L)|, \theta \in [0, 2\pi]\}$ ♦

PROPOSITION 4 : The mappings d_H and d_R satisfy the following inequality:

$$\forall (K, L) \in \mathcal{K}_s^2 \qquad d_H(K, L) \leq d_R(K, L) \qquad (7)$$

These distances are not equal in general.
proof : (Labouré, 1987) ♦

In order to get a definition which would be independent of the relative position of K and L, let us generalize this distance to the case where $\mathcal{St}(K) \cap \mathcal{St}(L) = \emptyset$.
The points $\omega \in \mathcal{St}(K)$ and $\omega' \in \mathcal{St}(L)$ being given, let u be the vector $\omega'\omega$ and let L_u be the translated set L + u. Thus $\omega \in \mathcal{St}(K) \cap \mathcal{St}(L_u)$. We then define ∂_r the radial distance between K and L by :

$$\partial_r(K, L) \quad = \quad \text{Inf}_{\substack{\omega \in \mathcal{St}(K) \\ \omega' \in \mathcal{St}(L)}} \|\rho_K - \rho_{L_u}\|_\infty \qquad (8)$$

This expression can be generalized to all star-shaped bodies. It is independent of the relative positions of K, L, ω and ω'.

PROPOSITION 5 : The mapping ∂_r is a distance on $\mathcal{K}_s^t$

proof :

a) $\partial_r(K, L)$ is clearly symmetric and positive.

b) We have shown (proposition 2) that $\forall\ \omega \in \mathcal{St}(K) \cap \mathcal{St}(L_u)$ $\quad d_H(K, L_u) \leq \|\rho_K - \rho_{Lu}\|_\infty$

Thus $\operatorname{Inf}_{\substack{\omega \in \mathcal{St}(K)\\ \omega' \in \mathcal{St}(L)}} d_H(K, L_u) \leq \operatorname{Inf}_{\substack{\omega \in \mathcal{St}(K)\\ \omega' \in \mathcal{St}(L)}} \|\rho_K - \rho_{Lu}\|_\infty = \partial_r(K, L)$

We deduce that $\partial_r(K, L) = 0 \Rightarrow \quad \operatorname{Inf}_{\omega, \omega'} d_H(K, L_u) = 0$

Since the set of vectors u is bounded and since K and L are compacts sets, we deduce that K = L up to a translation.

c) We just need now to verify the triangular inequality :

Let $\omega \in \mathcal{St}(K)$, $\omega' \in \mathcal{St}(L)$ and $\omega'' \in \mathcal{St}(M)$

$\|\rho(K, \omega) - \rho(L, \omega')\|_\infty \quad \leq \quad \|\rho(K, \omega) - \rho(M, \omega'')\|_\infty + \|\rho(M, \omega'') - \rho(L, \omega')\|_\infty$

Taking lower bounds for $\omega \in \mathcal{St}(K)$, $\omega' \in \mathcal{St}(L)$ and finally $\omega'' \in \mathcal{St}(M)$, yields the result.

The radial dilation can be used in industrial applications:

-On an assembly line, for example, a way to throw away objects recognized as "bad" objects is to use a piston whose movement is given by the rotation of an ellipse around the origine. The position of the extremity of the piston from O is defined to be the radial function of the radial dilated of the ellipse (Fig. 4 a).

-In a control of circularity, the use of the Hausdorff metric allows to quantify the width of the hole and the radial distance allows to quantify its depth (Fig. 4 b).

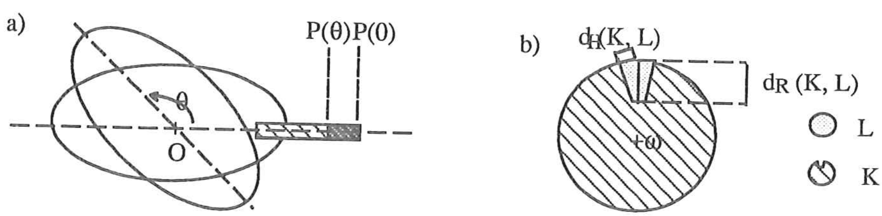


Figure 4 . 4a : movement of a piston given by the rotation of an ellipse.
4b : measures realized by the Hausdorff and the radial distance.

THE "RADIAL GAUGE"

In connexion with gauge notion, we define the "radial gauge" for two sets K_1, K_2 assumed to be star-shaped with respect to an interior point O and lying in the set $\mathcal{K}_S$, by :

$$r(K_1, K_2) = \operatorname{Max}\left\{ \operatorname{Sup}\{\frac{\rho_\theta(K_1)}{\rho_\theta(K_2)}, \theta \in [0, 2\pi]\}, \operatorname{Sup}\{\frac{\rho_\theta(K_2)}{\rho_\theta(K_1)}, \theta \in [0, 2\pi]\} \right\} \tag{9}$$

PROPOSITION 6 : The function $d_G : \mathcal{K}_S \times \mathcal{K}_S \to IR^+$

$(K_1, K_2) \to \ln r(K_1, K_2)$

is a distance on $\mathcal{K}_S$.

Note : Despite appearances, this "gauge distance" is different from Asplund distance (Asplund, 1960). $d_A(K, L) = d_A(K, L) = \text{Log Inf } \{\frac{\alpha}{\beta} ; K \subset \alpha L \text{ and } \beta L \subset K\}$

CONCLUSION

In this paper,we have generalized the Hausdorff distance to non symmetrical structuring elements and we have created new distances based on the radial function. The problems of the discretization and also the problems directly linked to the expressions of these metrics, lead to the fact that the implementation is not easy, and it has to be improved. Moreover, these are specific metrics, and we still have to define their field of applications.

REFERENCES

Asplund E. Comparison between plane symmetric convex bodies and parallelograms. Math Scand 1960; 8: 171-180.

Labouré MJ. Faisabilité d'une carte électronique d'opérateurs de seuillage ; déformation d'objets plans lors de transformations de type morphologiques. Thèse université de Saint Etienne 1987.

Matheron G. Random sets and integral geometry. Wiley J, 1975.

Valentine FA. Convex Set. Mac Graw Hill ; New York, 1964.