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STEREOLOGICAL MODEL TESTS FOR THE SPATIAL POISSON-VORONOI TESSELLATION II

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ABSTRACT

This paper discusses three stereological model tests for the spatial Poisson-Voronoi tessellation. The tests aim to discriminate the Poisson-Voronoi tessellation from more regular or more irregular tessellations. For some chosen sample sizes the quantiles of the distribution of the test variables are estimated by simulation. The power of the model tests is investigated under some special parametric alternative hypotheses.

Key words: model test, spatial Poisson-Voronoi tessellation, random plane section.

INTRODUCTION

The random Voronoi tessellation is an important model of stochastic geometry which seems to be suitable for describing space-filling mosaic-like structures resulting from growth processes. Voronoi tessellations have successfully been used as models in many fields of science, e.g. in materials science, geography, astrophysics, cell biology and geology (see e.g. Stoyan et al., 1987 or Okabe et al., 1992).

A random spatial Voronoi tessellation is a random division of space into convex polyhedra (cells) defined with respect to a generating point process of so-called germs. Each cell consists of those points of the space which are closer to a given germ than to all other germs. If the germs constitute a homogeneous Poisson point process, then the tessellation is called Poisson-Voronoi tessellation (PVT). The only parameter of this model is the intensity λ, the mean number of points of the generating Poisson point process per unit volume.

The proposed model tests shall help to decide whether the Poisson-Voronoi tessellation is an appropriate model to describe a given mosaic structure, when the only available information can be taken from random planar sections.

In a foregoing paper (Krawietz and Lorz, 1991), various model tests based on the number of vertices of the section cells were presented. Here, three completely different procedures are proposed.

A preceding study comparing different types of Voronoi tessellations with respect to geometric characteristics of spatial cells and of the planar section cells revealed only small differences between the means of the characteristics, but appreciable differences between their variances (Lorz and Hahn, 1993). In the case of the section cells, the variances of the cell areas showed particularly significant differences, see also Lorz (1990). Therefore, the first two tests proposed here are based on the variability of the area of the section cells. While the first test requires a direct measurement of these areas, the second test consists in a simple point counting. The third test is motivated by a well-known relation between specific edge length L_A and point process intensities λ and P_A, cf. Eq. 3 and 4 (Stoyan et al., 1987).

Both the first and the second test allow to test the Poisson-Voronoi tessellation versus more regular or more irregular tessellations, respectively. They can therefore be designed as one-sided and as two-sided tests.

After a description of the tests, quantiles of the distributions are given, which are estimated by computer simulation. Four sample sizes are taken into account. A further computer simulation reveals the power of the tests with respect to several parametric alternative hypotheses. The discussion validates the tests and deals with their possible fields of application.

STEREOLOGICAL MODEL TESTS

The first test variable is the coefficient of variation of the section cell areas. Being dimensionless, it is independent of the model parameter λ.

For each cell, a 'sampling point' is determined, namely the centre of gravity of the cell. The section cells are sampled by determining whether their sampling point is inside the observation window or not. The observation window has to be adequately small, so that all the cells intersecting the observation window can be entirely seen. Let n denote the number of selected section cells, and a_i be the area of the i-th section cell, then

$$C = \frac{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(a_i - \bar{a})^2}}{\bar{a}} \qquad (1)$$

is the first test variable, where

$$\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i.$$

The second test is based on the fact that the section cells are not uniformly arranged in the section plane. The more regular (irregular) the generating point process, the more regular (irregular) will be the distribution of section cells over the plane. The test is similar to the well-known index-of-dispersion test, cf. Stoyan et al. (1987), p. 59. To check the dispersion of the cells within the observation window, each cell is represented by its 'sampling point', here the rightmost vertex of the cell. Then, the observation window is partitioned into m subwindows of equal size, see Fig. 1.

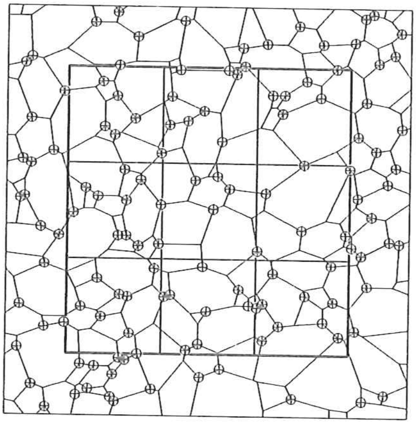

Figure 1. The square observation window is divided into $m = 9$ subwindows. The sampling points are marked by a $\oplus$.

The numbers $n_1, n_2, \dots, n_m$ of sampling points in subwindow $1, 2, \dots, m$ are counted. The test variable

$$X_m = \frac{\sum_{i=1}^{m} (n_i - \bar{n})^2}{\bar{n}} \tag{2}$$

with

$$\bar{n} = \frac{1}{m} \sum_{i=1}^{m} n_i$$

would be asymptotically χ^2-distributed (with $m-1$ degrees of freedom) if the sampling points of the section cells were Poisson distributed in the plane. X_m reflects the variability of the section cell areas, because the number of cells per unit area is inversely proportional to their area. In the following, only square observation windows (and square subwindows) are considered.

The third proposed model test is motivated by a relation between P_A, the mean number of section cell vertices per unit area, and L_A, the mean length of the section cell edges per unit area on the one hand, and the intensity λ of the generating spatial Poisson point process on the other hand (see Stoyan et al., 1987):

$$P_A = \frac{L_V}{2} = \frac{8}{15} \cdot \left(\frac{3}{4}\right)^{1/3} \cdot \pi^{5/3}\, \Gamma\left(\frac{4}{3}\right) \cdot \lambda^{2/3} \approx 2.9159\ \lambda^{2/3}, \tag{3}$$

$$L_A = \frac{\pi}{4} S_V = \pi \cdot \left(\frac{\pi}{2}\right)^{1/3} \cdot \Gamma\left(\frac{5}{3}\right) \cdot \lambda^{1/3} \approx 2.2859\ \lambda^{1/3}. \tag{4}$$

Both P_A and L_A can be unbiasedly estimated in the same observation window without edge correction. A test variable which eliminates the dependency on the model parameter λ is

$$T = \frac{\hat{L}_A^2}{\hat{P}_A} = \frac{L^2(W)}{N(W)\ A(W)}, \tag{5}$$

where $L(W)$ is the total edge length and $N(W)$ the total number of vertices inside an observation window W of area $A(W)$.

For planar sections of the spatial Poisson-Voronoi tessellation, T takes larger values than for sections of more regular or more irregular tessellations. The corresponding test is therefore designed as one-sided test.

QUANTILES OF THE TEST VARIABLES

Until now, the theoretical distribution of the three test variables C, X_m and T is unknown. Therefore, a computer simulation was carried out in order to estimate the quantiles for the proposed model tests. To this end, aggregates of cells of spatial PVT with $\lambda = 1$ were generated. For details concerning the simulation procedure see (Lorz and Hahn, 1993) and (Møller et al., 1989). Isotropic random planar sections of the aggregates were taken and square observation windows were drawn in the section planes with an expected number of 50, 100, 150, and 200 cells, respectively. The aggregates were made large enough to ensure that the observation windows were completely filled with cells. The expected number of cells with sampling point inside the observation window follows from the relation, cf. (Stoyan et al., 1987)

$$N_A = \frac{1}{4} L_V \approx 1.4580\ \lambda^{2/3}. \tag{6}$$

About 7000 samples were generated for each sample size. Empirical α-quantiles for the levels of significance $\alpha = 0.01$, 0.025, 0.05, and 0.1 for the test variables C, X_{36} and T are given in Tbl. 1, 2, and 3, respectively.

Table 1. Estimated quantiles c_α of test variable C.

sample size	c_α											
	0.005	0.01	0.0125	0.025	0.05	0.1	0.9	0.95	0.975	0.9875	0.99	0.995
50	0.511	0.530	0.533	0.552	0.574	0.599	0.782	0.811	0.832	0.857	0.863	0.885
100	0.562	0.573	0.578	0.594	0.609	0.627	0.756	0.775	0.794	0.810	0.812	0.823
150	0.587	0.598	0.601	0.612	0.624	0.640	0.745	0.759	0.773	0.785	0.788	0.798
200	0.602	0.610	0.614	0.623	0.634	0.648	0.739	0.753	0.764	0.773	0.776	0.785

Table 2. Estimated quantiles $x_{36,\alpha}$ of test variable X_{36}.

sample size	$x_{36,\alpha}$											
	0.005	0.01	0.0125	0.025	0.05	0.1	0.9	0.95	0.975	0.9875	0.99	0.995
50	13.73	15.00	15.24	16.57	18.16	20.00	37.24	40.46	43.33	46.31	47.12	49.94
100	10.89	11.74	11.99	13.04	14.25	15.77	29.81	32.26	34.50	36.68	37.34	39.53
150	10.19	10.96	11.18	12.15	13.20	14.64	27.12	29.34	31.41	33.48	34.01	35.99
200	9.82	10.48	10.77	11.62	12.74	14.04	26.12	28.20	30.22	32.02	32.66	34.58

Table 3. Estimated quantiles t_α of test variable T.

sample size	t_α					
	0.005	0.01	0.0125	0.025	0.05	0.1
50	1.59	1.61	1.61	1.64	1.66	1.69
100	1.65	1.66	1.67	1.69	1.70	1.72
150	1.67	1.69	1.69	1.70	1.72	1.74
200	1.69	1.70	1.71	1.72	1.73	1.74

POWER OF THE MODEL TESTS

In order to assess the power of the model tests, values of their power functions were estimated. As special parametric alternative hypotheses Voronoi tessellations were chosen, which are constructed with respect to a Matern hard-core point process (HVT), to a simple sequential inhibition point process (SVT), and to a Matern cluster point process (CVT). SVT and HVT are more regular than PVT, CVT is more irregular. For a mathematical definition of these point processes see (Diggle, 1983) and (Stoyan et al., 1987).

The HVT as well as the SVT model can be characterized by the scale parameter λ_{hc}, the mean number of points of the generating point process per unit volume, and the shape parameter $p_{hc} = \lambda_{hc} \frac{4}{3}\pi R_{hc}^3$, the mean volume fraction of the hard cores with radius R_{hc} (Lorz and Hahn, 1993). For the HVT the parameter p_{hc} has to be taken from the interval $[0, \frac{1}{8})$ whereas for the SVT p_{hc} can be chosen between 0 and approximately 0.4. Consequently, with the SVT model a higher degree of regularity in the tessellation can be reached. Since in both models the limiting case $p_{hc} = 0$ corresponds to the PVT, the parametric test problem can be formulated as the simple null hypothesis $p_{hc} = 0$ versus the compound alternative hypothesis $p_{hc} > 0$.

The model parameters of the CVT are the scale parameter λ_{cl}, the mean number of points of the generating cluster point process per unit volume, and the shape parameters N_{cl}, the mean number of points per cluster, and R_{cl}, the cluster radius (Stoyan et al., 1987). Instead of R_{cl},

$$p_{cl} = 1 - \exp\left\{-\frac{\lambda_{cl}}{N_{cl}} \frac{32}{3} \pi R_{cl}^3\right\} \tag{7}$$

is used as third model parameter. It is scale invariant and can be interpreted as (approximately) the probability that neighbouring clusters 'overlap'.

As in the foregoing paper of Krawietz and Lorz (1991), the parameters $p_{hc} = 0.1$ (HVT), $p_{hc} = 0.2$ (SVT), and $N_{cl} = 10$ and $p_{cl} = 0.7$ (CVT) were chosen for the investigation of the power of the model tests. The intensities λ_{hc} and λ_{cl} were set to unity. About 7000 samples were generated for each model and sample size. The results for sample sizes of 50, 100, 150, and 200 section cells are presented in Fig. 2.

DISCUSSION

As the simulations have shown, the most powerful test is the first test, based on the coefficient of variation of the section cell areas, see Eq. 1. Its power is greater than 95% for CVT, already at a sample size of about 100 section cells. In contrast to the other tests, a considerable power is also reached for the more regular tessellations SVT and HVT. It requires, however, the measuring of the section cell areas, what is not difficult, if automatic image analysers are used. This is also the case for the second test, using the index-of-dispersion, see Eq. 2. It reaches a considerable power for CVT, and is better for CVT and HVT than the third test. It provides therefore a useful method for a preliminary evaluation of the sections, particularly if the cell areas cannot be determined automatically.

The third test variable T, see Eq. 5, is based on unbiased estimators which allow full exploitation of the entire section. However, only in case of SVT satisfactory results can be expected here. Thus, the use of T can not be recommended.

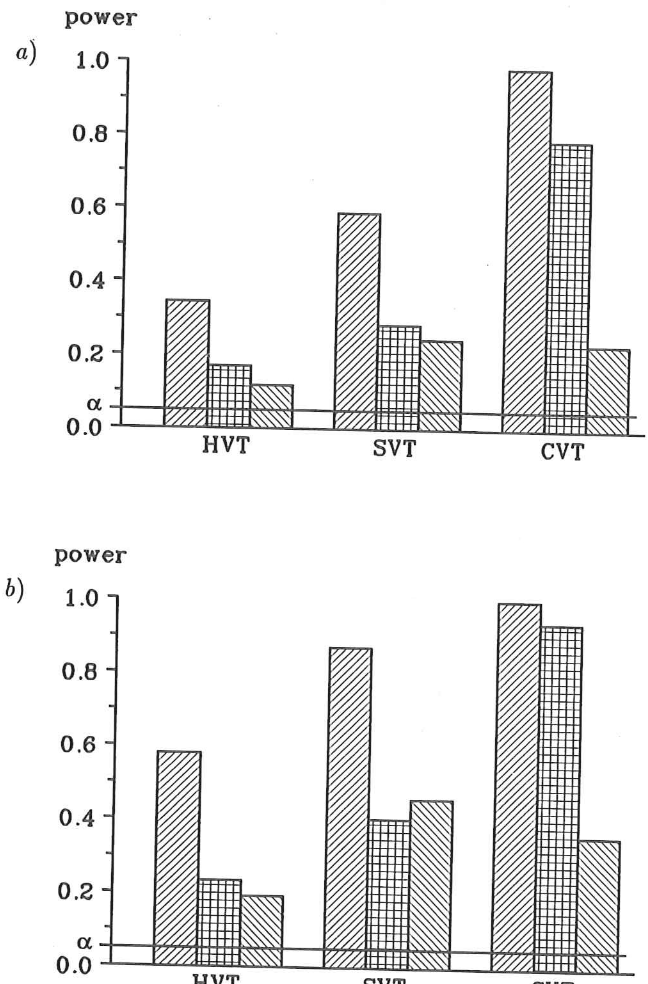

Figure 2. Estimated values of the power functions for square observation windows and sample sizes of 100 (a) and 200 (b) section cells.

one-sided coefficient-of-variation test based on test variable C, Eq. 1

one-sided index-of-dispersion test based on test variable X_{36}, Eq. 2

test based on test variable T, Eq. 5

The first and the second test reach a higher power for more irregular tessellations than for tessellation, which are more regular than the PVT. A similar effect was already described by Krawietz and Lorz (1991) for tests based on the number of vertices of the section cells.

APPLICATION TO SINGLE-PHASE ALUMINA CERAMICS

As discussed in Krawietz and Lorz (1991), the spatial Poisson-Voronoi tessellation is an appropriate model for certain single-phase microstructures, e.g. alumina ceramics, and mean values of cell characteristics can easily be calculated for the PVT, if the intensity λ is known. In that paper, three specimen of single-phase alumina ceramics were investigated, using model tests based on the number of vertices of the section cells. The same specimen shall be checked here using the new tests described above.

In Fig. 3, parts of plane sections of three specimen of single-phase alumina ceramics are shown together with the corresponding schemes as planar tessellations consisting of polygons. These preprocessings were obtained by neglecting pores and inclusions, and connecting the cell vertices by straight line segments. Specimen I is rather regular whereas specimen III is rather irregular. Specimen II takes an intermediate position.

In order to check the applicability of the PVT model for each specimen, the three model tests were carried out with level of significance $\alpha = 0.05$.

Whereas for specimen III the null hypothesis (PVT) is rejected by all model tests, for specimen II the second test (X_{36}), and for specimen I the first test (C) and and the second test (X_{36}) accept the null hypothesis. In the foregoing paper (Krawietz and Lorz, 1991), none of tests considered there rejected the null hypothesis for specimen II, but one of the tests accepted the null hypotheses for specimen III. Obviously, the new tests proposed in the present paper are more sensitive than the test based on the number of vertices of section cells, otherwise the null hypothesis would not have been rejected for specimen II. It should be noted that the sample size had to be reduced for the first and the second test, so that it was possible to observe cells intersecting the observation window as whole cells. This might be the reason why these tests do not lead to a rejection of the null hypothesis in case of specimen II.

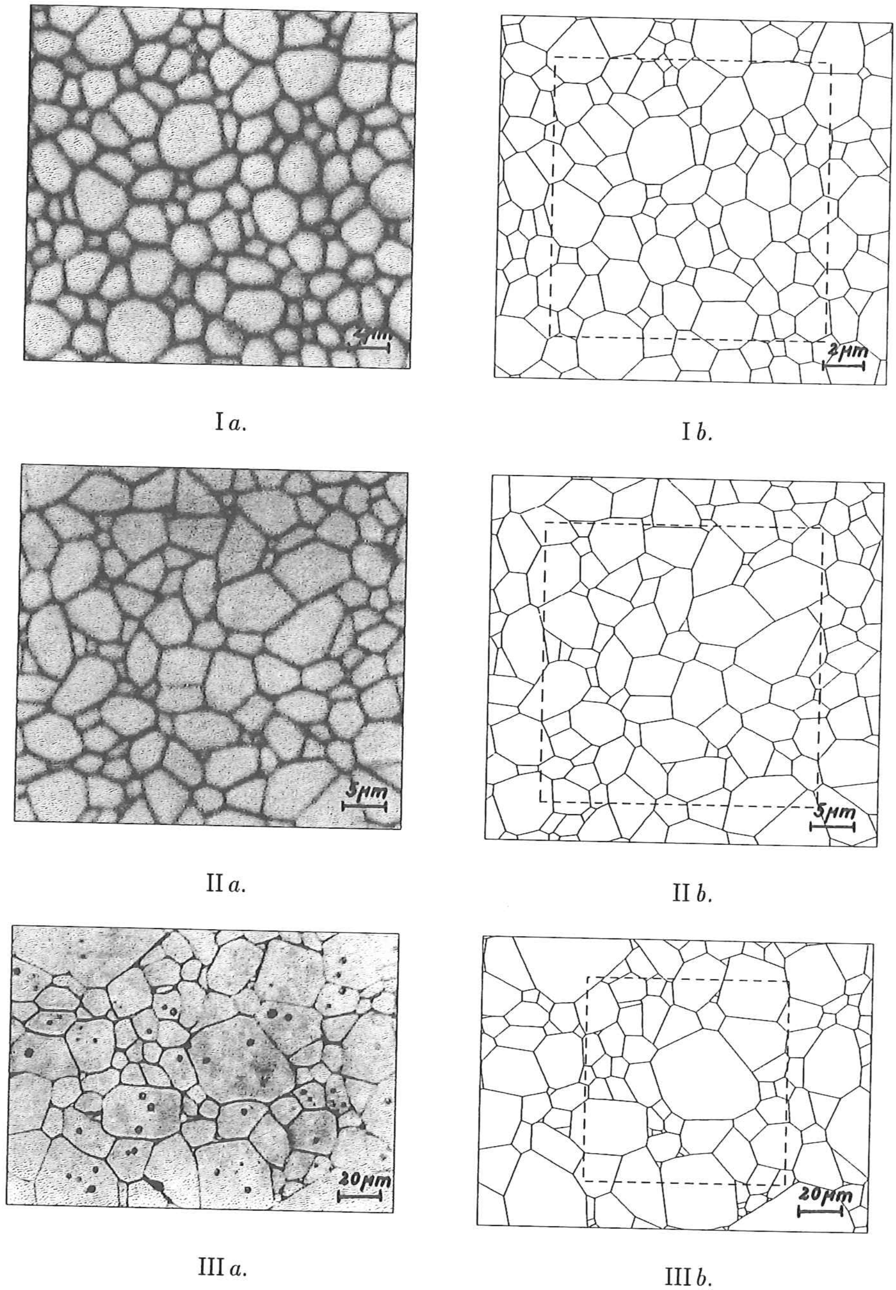

Figure 3. Photographs (a) of plane sections of alumina ceramics and their schemes as planar tessellations (b) (preprocessings). The observation windows are indicated by dashed lines in I (b)–III (b).

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