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ON ANISOTROPIC SAMPLING IN STEREOLOGY

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ABSTRACT

The paper is devoted to the ten years anniversary of Baddeley's vertical sections, the first anisotropic sampling design in stereology. A general theory is presented which includes vertical sections and vertical projections as special cases. The model based approach is applied, variances of intensity estimators are studied.

KEYWORDS: anisotropy, Buffon transform, vertical projections (sections).

INTRODUCTION

Classical stereological formulas concerning the estimation of length intensity L_V and surface intensity S_V require one of the following isotropy assumptions: either the probes are generated with isotropic orientation (IUR) or the geometrical structure can be modelled by an isotropic random set. If neither the probe nor the structure is known to be isotropic, special formulas of anisotropic stereology have to be used.

Baddeley (1984) developed an ingenious anisotropic sampling design of uniform vertical sections (VUR), which enables an unbiased and efficient S_V estimation. More recently, **Gokhale(1990)** and **Cruz-Orive and Howard(1991)** worked out the design of vertical random projections for the estimation of L_V. Their approach is essentially design-based, evaluation of estimation variances is in most cases an open problem.

In the presented paper the model-based approach is applied, stationary random fibre and surface processes are studied. The notion of a projection measure was developed for anisotropic random measures in **Beneš et al.(1993)** using systematically the Buffon transform. Special case of this measure evaluated in planar section (projection) of the surface (fibre) process, respectively, is in fact the model-based analogue of the quantity W in **Baddeley(1985)** and **Cruz-Orive and Howard(1991)**. Second-order analysis enables to express variance of intensity estimator based on this measure.

In the following text first the theoretical background is rewieved in general d-dimensional case. Further we proceed in three dimensions, first the relation to recent intensity estimators is explained and finally some remarks to practical estimation and estimation variances are presented.

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THE BUFFON TRANSFORM

Let $(R, \mathcal{B}, \nu)^d$ be the d-dimensional Euclidean space with Borel σ-algebra and Lebesgue measure ν and $(M, \mathcal{M})^d$ the measurable space of axial orientations, i.e. a hemisphere in R^d. The general index d is omitted if possible in the following, e.g. $R^d = R$, from the point of view of applications the most important cases are $d = 2$ and $d = 3$.

For any probability measure $\mathcal{P}$ on $\mathcal{M}$ the Buffon transform $\mathcal{F}_\mathcal{P}$ of $\mathcal{P}$ is a nonnegative function on M defined by

$$\mathcal{F}_\mathcal{P}(l) = \int_M \mid cos \sphericalangle(l, m) \mid \mathcal{P}(dm) \ , \ l \in M \tag{1}$$

and for two probability measures $\mathcal{P}$, $\mathcal{Q}$ on $\mathcal{M}$ the Buffon constant

$$\mathcal{F}_{\mathcal{P}\mathcal{Q}} = \int_M \mathcal{F}_\mathcal{P}(l)\mathcal{Q}(dl) = \int_M \mathcal{F}_\mathcal{Q}(l)\mathcal{P}(dl) = \mathcal{F}_{\mathcal{Q}\mathcal{P}} \tag{2}$$

The Buffon transform of a uniform probability measure $\mathcal{U}$ on $\mathcal{M}$ is a constant function denoted

$$\mathcal{F}_\mathcal{U}(l) = \kappa \ , \text{ for all } l \in M \tag{3}$$

if either $\mathcal{P}$=$\mathcal{U}$ or $\mathcal{Q}$=$\mathcal{U}$ then again $\mathcal{F}_{\mathcal{P}\mathcal{Q}} = \kappa$. Elementary integration yields $\kappa_2 = \frac{2}{\pi}$ for $d = 2$ and $\kappa_3 = \frac{1}{2}$ for $d = 3$.

The function $\mathcal{F}_\mathcal{P}(l)$ in (1) may be interpreted as the mean projected length of a unit segment in R of orientation l projected onto a random line $m \in M$ with orientation distribution $\mathcal{P}$. A dual interpretation is possible, as the mean projected length of a unit random segment with orientation distribution $\mathcal{P}$, projected onto a line l. If both the segment and projection direction are random with orientation distributions $\mathcal{P}$ and $\mathcal{Q}$, respectively, duality relation (2) shows that the interchange of both distributions does not influence the resulting mean projected length $\mathcal{F}_{\mathcal{P}\mathcal{Q}}$.

Let Φ be a stationary random fibre process in R^d, see **Stoyan** *et al.*(**1987**). The weighted fibre process Ψ is derived from Φ by joining to each point x of Φ an element of M corresponding to the tangent orientation of the fibre at x. The intensity measure Λ of Ψ can be written as (**Stoyan** *et al.*, **1987**)

$$\Lambda(B \times D) = E\Psi(B \times D) = L\nu(B)\mathcal{R}(D) \ , \ B \in \mathcal{B}, \ D \in \mathcal{M}$$

where the real constant L is the mean fibre length per unit d-dimensional volume in R and the probability measure $\mathcal{R}$ on $\mathcal{M}$, called the rose of directions, is the distribution of tangent orientations of fibres.

Consider a $(d-1)$-dimensional testing hyperplane $H(l)$ with orientation l of its normal and investigate the intersection $\Phi \cap H(l)$, which is a random point process. Let $P(l)$ be the mean number of intersection points per unit $(d-1)$-dimensional area of the hyperplane. Then (**Hilliard, 1967**)

$$P(l) = L\mathcal{F}_\mathcal{R}(l) \tag{4}$$

holds. In particular, if Φ is isotropic, i.e. $\mathcal{R}$=$\mathcal{U}$, then $P(l) = P$ is constant and equal to $P = \kappa L$ Specially this formula is expressed as $P_L = \frac{2}{\pi}L_A$ for $d = 2$ and $P_A = \frac{1}{2}L_V$ for $d = 3$.

For the second order anisotropic stereology Buffon transform of second order is a useful tool. Let $\mathcal{W}_h$ be the two point weight distribution of Ψ, which is interpreted as the joint distribution of weights (fibre tangent orientations) in points x, y such that $y - x = h$, under the condition that x and y are the points of the weighted fibre process. The special type of the second order Buffon transform which will be used here we denote for fixed Q

$$I_Q(h) = \int_M \int_M \mathcal{F}_Q(m_1)\mathcal{F}_Q(m_2)\mathcal{W}_h(d(m_1, m_2)), \ \ h \in R \tag{5}$$

THE PROJECTION MEASURE

For given $l \in M$ and $B \in \mathcal{B}$ with $\nu(B) > 0$, we denote $\mathrm{Pr}_l(B)$ the projection of B onto l (i.e. onto the one-dimensional subspace R_l of R with orientation l). For $y \in \mathrm{Pr}_l(B)$, let N_y be the number of intersection points of $\Phi \cap B$ with the hyperplane $H(l, y)$ with normal orientation l including y. N_y is a random function; we assume further finite second moments for all $y \in R_l$. For a fixed realization ϕ of Φ N_y is measurable on R_l. Then a random measure Φ_l on $\mathcal{B}$ is defined by

$$\Phi_l(B) = \int_{\mathrm{Pr}_l(B)} N_y \, dy \tag{6}$$

that means $\Phi_l(B)$ is the sum of projection lengths of all fibres from $\Phi \cap B$ onto l. Generally, for a given probability measure $\mathcal{Q}$ on $\mathcal{M}$ a projection measure on $\mathcal{M}$ is defined as

$$\Phi_{\mathcal{Q}}(B) = \int_M \Phi_l(B)\, \mathcal{Q}(dl) \tag{7}$$

Theorem 1 *The intensity measures Λ_l, $\Lambda_{\mathcal{Q}}$ corresponding to Φ_l, $\Phi_{\mathcal{Q}}$ respectively, are equal to*

$$\Lambda_l(B) = E\Phi_l(B) = L\nu(B)\mathcal{F}_{\mathcal{R}}(l) \tag{8}$$

$$\Lambda_{\mathcal{Q}}(B) = E\Phi_{\mathcal{Q}}(B) = L\nu(B)\mathcal{F}_{\mathcal{R}\mathcal{Q}}, \quad B \in \mathcal{B} \tag{9}$$

where $\mathcal{R}$ is the rose of directions of Φ.

Proof: We use the Campbell theorem $E\int f(x, m(x))\Phi(dx) = L\int\int f(x,m)\mathcal{R}(dm)dx$ (**Stoyan** *et al.*,**1987**), which holds for any nonnegative measurable function f on $R \times M$. Clearly

$$\Phi_l(dx) = |\cos \sphericalangle(m(x), l)|\, \Phi(dx) \tag{10}$$

and therefore $\Phi_{\mathcal{Q}}(dx) = \mathcal{F}_{\mathcal{Q}}(m)\Phi(dx)$. (8), (9) is then obtained directly from the Campbell theorem by putting $f(x,m) = 1_B(x)\,|\cos \sphericalangle(m(x), l)|$ and $f(x,m) = 1_B(x)\mathcal{F}_{\mathcal{Q}}(m)$, respectively, where 1_B is the indicator function of the set $B \in \mathcal{B}$.

Concerning the second order analysis the following general result will be used, see **Stoyan et al.(1987)**. Let Ψ be a stationary random measure on $(R, \mathcal{B})$ with intensity constant λ and reduced second moment measure $\mathcal{K}$ defined by

$$E\Psi(B)\Psi(C) = \lambda^2 \int_R \int_R 1_B(x) 1_C(x+h)\mathcal{K}(dh)dx \tag{11}$$

for any $B, C \in \mathcal{B}$, 1_B denoting the indicator function of the set B. Let $g_B(x) = \nu(B \cap B_{-x})$, assume that the pair correlation $p(x)$ of Ψ defined by

$$p(x)dx = \mathcal{K}(dx), \quad x \in R$$

exists. Then

$$\mathrm{var}\Psi(B) = \lambda^2 \left[\int_R g_B(x)(p(x) - 1)dx\right]. \tag{12}$$

For the exceptions of this formula see **Ohser(1991)**, but for fibre processes it is valid. In the following we study specially random measures corresponding to the stationary fibre process Φ.

The relation between pair correlation functions $p(x)$ and $p_{\mathcal{Q}}(x)$ of measures Φ, $\Phi_{\mathcal{Q}}$, respectively, can be expressed by means of the second order Campbell theorem (**Schwandtke, 1988**):

$$\begin{aligned} E\int_R\int_R f(x, y, m_1(x), m_2(y))\Phi(dx)\Phi(dy) = \\ L^2 \int_R\int_R\int_{M\times M} f(x, x+h, m_1, m_2)\mathcal{W}_h(d(m_1, m_2))p(h)dhdx \end{aligned} \tag{13}$$

holds for any nonnegative measurable real function f on $R \times R \times M^2$.

Theorem 2 *For the pair correlation functions p, p_Q of random measures Φ, Φ_Q, respectively, holds for almost all $h \in R$*

$$p_Q(h) = \frac{I_Q(h)}{\mathcal{F}^2_{\mathcal{R}Q}} p(h). \tag{14}$$

Proof: (see **Beneš** *et al.*, **1993**). Putting $f(x, y, m(x), m(y)) = 1_B(x)1_C(y)\mathcal{F}_Q(m(x))\mathcal{F}_Q(m(y))$ in (13), $B, C \in \mathcal{B}$, one obtaines using (10)

$$E\Phi_Q(B)\Phi_Q(C) = L^2 \int_R \int_R 1_B(x)1_C(x+h)I_Q(h)p(h)dhdx$$

On the other hand formula (11) applied to Φ_Q yields

$$E\Phi_Q(B)\Phi_Q(C) = L^2\mathcal{F}^2_{\mathcal{R}Q} \int_R \int_R 1_B(x)1_C(x+h)p_Q(h)dhdx$$

As these both equalities are valid for any Borel B and C, (14) follows.

Using (9), (12) and (14) we obtain the corrollary

$$var\Phi_Q(B) = L^2 \int_R g_B(x)(I_Q(x)p(x) - \mathcal{F}^2_{\mathcal{R}Q})dx \tag{15}$$

Example 1: An analytical evaluation of the above formulas will be demonstrated on an anisotropic Boolean segment process Φ in R, which is defined as a union of line segments S the centres of which form a stationary Poisson point process with intensity constant λ. Assuming that the orientation distribution $\mathcal{R}$ of segments is independent of the distribution H of segment lengths, $\mathcal{R}$ is in fact the rose of directions of Φ. Denoting $\bar{H} = \int_0^\infty y dH(y)$ the expected segment length, the intensity L of Φ is $L = \lambda\bar{H}$. Further let $f(r) = \frac{1}{\bar{H}}\left(\int_0^r x^2 dH(x) + \int_r^\infty (2xr - r^2)dH(x)\right)$ be the mean length of $S \cap D_r$ under the condition that a random segment S hits the origin 0, D_r being the sphere centered in origin with radius r. Then for the Boolean segment process Φ it holds (**Beneš** *et al.*, **1993**) $\mathrm{var}\Phi(B) = L\int_0^\infty \int_M g_B(r,l)df(r)\mathcal{R}(dl)$ and

$$\mathrm{var}\Phi_Q(B) = L\int_0^\infty \int_M g_B(r,l)\mathcal{F}^2_Q(l)df(r)\mathcal{R}(dl) \tag{16}$$

in polar coordinates $x = (r, l), dx = r^{d-1}drdl, x \in R, r$ real magnitude, $l \in M$.

ANISOTROPIC SAMPLING DESIGNS

There is a pair of dual formulas in stereology: $S_V = \frac{4}{\pi}L_A$ by means of IUR planar sections and $L_V = \frac{4}{t\pi}L_A$ by means of IUR projections of a slab of thickness t. Our aim is to express these quantities by means of VUR sections and projections, respectively, using the model-based approach. Consider a stationary random surface (fibre) process Ψ (Φ), in R^3, respectively. Define z axis to be vertical axis and any plane $V_\xi = \{(x, y, z) : x\cos\xi + y\sin\xi = 0\}, \xi \in <0, \pi)$ as vertical plane.

Let $D = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$ be the unit sphere and $D_\xi = D \cap V_\xi$ circular subsets of vertical planes. Let B_ξ be a unit cube centered in origin with one vertical edge and orientation ξ of an edge in horizontal plane. Put $B^p_\xi = B_\xi \cap V_\xi, \xi \in <0, \pi)$. Let Ψ_ξ, Φ_ξ be random fibre processes, where $\Psi^\xi = \Psi \cap D_\xi$ and Φ^ξ be the orthogonal projection of $\Phi \cap B_\xi$ onto B^p_ξ. Let Q_1, Q_2 be fixed probability measures on $\mathcal{M}^2$ defined by

$$Q_1(dl) = \frac{1}{2}\sin l dl, \tag{17}$$

$$Q_2(dl) = \frac{1}{2} \mid \cos l \mid dl, \quad l \in <0, \pi)$$

where l is interpreted as the angle from horizontal axis in any vertical plane V_ξ. For these measures it holds

$$\begin{aligned}\mathcal{F}_{Q_1}(m) &= \tfrac{1}{2}[m\sin m + \cos m], & m \in < 0, \tfrac{\pi}{2})\\ &= \tfrac{1}{2}[(\pi - m)\sin m - \cos m], & m \in < \tfrac{\pi}{2}, \pi)\\ \mathcal{F}_{Q_2}(m) &= \tfrac{1}{2}[(\tfrac{\pi}{2} - m)\cos m + \sin m] & m \in < 0, \pi)\end{aligned} \tag{18}$$

and given the orientation distributions $\mathcal{R}, \mathcal{W}_h$ of the process, formulas for $\mathcal{F}_{Q\mathcal{R}}, I_Q(h)$ follow from (2) and (5).

Theorem 3 *For the intensities S_V, L_V of Ψ, Φ, respectively, it holds*

$$S_V = \frac{2}{\pi\nu^2(D_\xi)} E\int_0^\pi \Psi^\xi_{Q_1}(D_\xi)d\xi \tag{19}$$

$$L_V = \frac{2}{\pi\nu^3(B_\xi)} E\int_0^\pi \Phi^\xi_{Q_2}(B^p_\xi)d\xi \tag{20}$$

where $\Psi^\xi_{Q_1}, \Phi^\xi_{Q_2}$ are projection measures corresponding to Ψ^ξ, Φ^ξ. (It is $\nu^2(D_\xi) = \pi$ and $\nu^3(B_\xi) = 1$ here.)

Proof: Let $P_L(l, \xi)$ be the intensity of intersection number of Ψ on test line in M^3 with spherical coordinates (l, ξ), then using (4) and (9)

$$S_V = 2P_L = \frac{1}{\pi}\int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} P_L(l,\xi)\cos l\, dl d\xi = \frac{1}{\pi}\int_0^\pi \int_0^\pi L_A(\xi)\mathcal{F}_{\mathcal{R}_\xi}(l)\sin l\, dl d\xi =$$

$$= \frac{2}{\pi}\int_0^\pi L_A(\xi)\mathcal{F}_{\mathcal{R}_\xi Q_1} d\xi = \frac{2}{\pi\nu^2(D_\xi)}\int_0^\pi E\Psi^\xi_{Q_1}(D_\xi)d\xi$$

where $L_A(\xi), \mathcal{R}_\xi$ is the intensity, rose of directions of Ψ^ξ, respectively. To prove (20) denote the edge length of the cube t, $P_A(l,\xi), P^\xi_L(l)$ are intensities of intersection number on the plane with normal orientation (l, ξ), linear projection of this plane onto V_ξ, respectively. Then

$$L_V = 2P_A = \frac{1}{\pi}\int_0^\pi \int_0^\pi P_A(l,\xi) \mid \cos l \mid dl d\xi =$$

$$= \frac{1}{t\pi}\int_0^\pi \int_0^\pi P^\xi_L(l) \mid \cos l \mid dl d\xi = \frac{2}{\pi\nu^3(B_\xi)}\int_0^\pi E\Psi^\xi_{Q_2}(B^p_\xi)d\xi.$$

Omitting expectations in (19), (20) one obtains unbiased estimators of intensities. The theory of the previous section enables to evaluate variances of these estimators using (15) and (18).

Example 2(cf. **Baddeley, 1985**): Consider a surface process Ψ of horizontal circular plates of diameters d, centres of which form a stationary Poisson process of intensity λ in R^3. Then Ψ^ξ is a Boolean process of horizontal segments of mean length $\frac{\pi d}{4}$ and intensity of centres $d\lambda$ for each $\xi \in < 0, \pi)$. Then it holds $var2\Psi^\xi_Q(D_\xi) = \frac{\lambda\pi d^2}{4}J$, where $J = \int_0^\infty g_{D_\xi}(r)df(r)$ using (16), where $\mathcal{F}_Q(0) = \frac{1}{2}$. For comparison the IUR sampling with $\alpha \in M^3$ being the normal orientation of test plane $T_\alpha, D_\alpha = D \cap T_\alpha$, $\Psi^\alpha = \Psi \cap T_\alpha$, yields the variance of intensity estimator $var(\frac{4}{\pi}\Psi^\alpha(D_\alpha)) = \lambda d^2 J + [\lambda d^2 \nu^2(D_\alpha)]^2(\frac{2}{3} - \frac{\pi}{16})$. That means VUR sampling yields smaller variance than IUR sampling here.

DISCUSSION

The given Ψ in Example 2 is invariant with respect to rotations which preserve z coordinate, therefore the evaluation of variance in the VUR case is straightforward.

For Ψ without this property the formula $var\Psi_Q^\xi(D) = varE(\Psi_Q^\xi(D) \mid \xi) + Evar(\Psi_Q^\xi(D) \mid \xi)$ has to be used, where the outer variance in the first term is with respect to uniform distribution of ξ. In fact, the two-stage sampling (first step uniform ξ, second step according to Q) is naturally described by conditional probabilities and expectations.

Using systematic sampling of several ξ on M^2 and estimating the integral in (19), (20) as $\int_0^\pi F(\xi)d\xi = b\sum_{k=0}^{m-1} F(a+kb), b = \frac{\pi}{m}, m \geq 1, a \in < 0, b)$ one can use recent methods by **Cruz-Orive(1993)** to get estimation variance. In the model-based approach the covariance of $\Psi_Q^\xi(D_\xi)$ in pairs of sampling points is desired to apply it.

For practical estimation the measure Φ_Q is further discretized. There are two possibilities. The first natural method makes use of the representation of Φ_Q by means of intersection counting, see (6) and (7). Special form of Q_i in (17) leads to cycloidal test lines **(Baddeley, 1985)** and cycloidal test surfaces **(Gokhale, 1990)**.

The second method is based on formula (9) and estimation of $\mathcal{F}_{Q\mathcal{R}}$ on vertical planes measuring fibre tangent angles. The functions $\mathcal{F}_Q(m)$ in (18) are symmetric around $\frac{\pi}{2}$ and for projection purposes it suffices to consider them on the interval $< 0, \frac{\pi}{2})$ only. Estimation procedure like this for vertical projections is described in **Cruz-Orive and Howard (1991)**, p.108.

Discretization of Φ_Q contributes to the estimation variance expressed above. General evaluation of this contribution is still an open problem, special results for different sampling designs than those discussed here are e.g. in **Kiêu, Vedel-Jensen (1993)**.

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