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THE STEREOLOGY OF SPHERICAL SHELLS

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ABSTRACT

A specimen containing spherical particles each of which has a concentric spherical core is sectioned by an IUR plane. The profiles in the plane are a mixture of discs and annular rings. Formulae are given which relate the size distributions of the profiles to that of the particles.

Keywords: Wicksell's corpuscle problem, stereological unfolding, spherical shell particles, shell thickness distribution, embryology.

INTRODUCTION

The problem comes from embryology. We are presented with a flask containing a colourless liquid, which we are told contains a suspension of invisible colourless cells. We seek their size distribution. First an opaque material that binds to the surface of the cells is introduced. The cells are rendered visible but their sizes cannot be directly observed because they are hidden within an opaque coating. A transparent resin is added which solidifies the contents of the flask. We now have a solid block containing coated cells in a transparent matrix. The stereological procedure is to take a plane section at random through the block. The result is a probability sample of sections of cells. The randomness is to ensure that there is no systematic bias. The randomness needs qualification since, even if it is expressed as "uniformly at random", this does not make for a unique specification. See for example Bertrand (1888) or Coleman (1989b). Here we require the randomness designated IUR (*isotropic uniform random)* by stereologists.

The profiles are either annular rings or discs, depending on whether or not the sectioning plane cuts the core of the shell. We observe the sizes of the rings and discs, and their intensities (the numbers of each per unit area of the section). The problem is to obtain estimates of the size distribution of the shells, that is to say, the joint distribution of their inner and outer radii, and their intensity (the expected number of shells per unit volume of the specimen). This is thus a generalization of the Wicksell corpuscle problem (Wicksell, 1925). Indeed, if the cores were not there, or the shells had zero thickness, we would have Wicksell's formulation.

The problem of this paper was first discussed in Bogataj (1980). Coleman (1986) gave a distribution-free estimation procedure for shells sectioned by a thin slice and was based on methods developed by Saltykov (see Weibel, 1980, Chapter 6). Simple adaptive methods of selecting the size classes have been demonstrated in a simulation exercise (Yan, 1989).

Recently, new stereological tools (Gundersen et al., 1988) and the need to study non-spherical particles have diminished the importance of Wicksell sampling. Despite this, the new formulae given below solve a problem of geometrical probability that has been awaiting solution for some time.

THE MODEL AND RESULTS

The shells

A typical shell will have an outer radius a and an inner radius b taken from a distribution which has the joint distribution function F_S, with the marginal expectations μ_A = Ea and μ_B = Eb . The shell centres will have intensity λ_S . We let $f_S(a,b)$ denote the probability density function (pdf) for (a,b) (expressed as a convex combination of delta functions if the distribution is discrete).

The ring profiles

A typical ring profile will have an outer radius a_0 and an inner radius b_0 . We let F_R denote the joint distribution function for these radii, and let $f_R(a_0,b_0)$ denote the corresponding pdf. The intensity (the expected number of ring centres per unit area of the section) will be denoted by λ_R . If a shell having radii pair (a,b) has its centre a distance z from the plane section, then, for a ring profile, the section must cut its core so we must have b > z. From simple geometry

$$(a_0,b_0) = (\surd(a^2-z^2),\surd(b^2-z^2)), \tag{1}$$

$$z = \surd(a^2-a_0^2) = \surd(b^2-b_0^2), \tag{2}$$

$$a = \surd\{b^2+(a_0^2-b_0^2)\}, \quad b = \surd\{a^2-(a_0^2-b_0^2)\}. \tag{3}$$

THEOREM 1.

$$f_R(a_0,b_0) = \frac{1}{\mu_B}\int_{a=a_0}^{\infty} \frac{a_0b_0}{\surd(a^2-a_0^2)\ \surd\{a^2-(a_0^2-b_0^2)\}} f_S(a,\surd\{a^2-(a_0^2-b_0^2)\})\ da$$

(4)

$$= \frac{1}{\mu_B}\int_{b=b_0}^{\infty} \frac{a_0b_0}{\surd\{b^2+(a_0^2-b_0^2)\}\ \surd(b^2-b_0^2)} f_S(\surd\{b^2+(a_0^2-b_0^2)\},b)\ db$$

for $0 \le b_0 \le a_0 < \infty$.

$$\lambda_R = 2\lambda_S\mu_B . \tag{5}$$

The disc profiles

A typical disc profile will have a radius r. We let F_D denote the distribution function for the radii, and let $f_D(r)$ denote the corresponding pdf. The intensity (the expected number of disc centres per unit area of the section) will be denoted by λ_D . If a shell having radii pair (a,b) has its centre at a distance z from the plane section, then, for a disc profile, the section must miss its core so we must have $b < z < a$. From simple geometry

$$r = \sqrt{(a^2-z^2)}. \tag{6}$$

THEOREM 2.

$$f_D(r) = \frac{1}{\mu_A-\mu_B}\int_{a=r}^{\infty}\int_{b=0}^{\sqrt{(a^2-r^2)}} \frac{r}{\sqrt{(a^2-r^2)}}\, f_S(a,b)\, db\, da \tag{7}$$

for $0 \le r < \infty$.

$$\lambda_D = 2\lambda_S(\mu_A-\mu_B) . \tag{8}$$

We note that the relationship between the distributions of the (outer) radii of the profiles and of the shells is given by the Wicksell formula. This is verified by noting that

$$\frac{\mu_B}{\mu_A} f_R(a_0)\Big]_{a_0=r} + \frac{\mu_A-\mu_B}{\mu_A} f_D(r)$$

$$= \frac{1}{\mu_A}\int_{a=r}^{\infty} \frac{r}{\sqrt{(a^2-r^2)}}\left\{\left[\int_{b=\sqrt{(a^2-r^2)}}^{a} + \int_{b=0}^{\sqrt{(a^2-r^2)}}\right] f_S(b|a)\, db\right\} f_S(a)\, da$$

$$= \frac{1}{\mu_A}\int_{a=r}^{\infty} \frac{r}{\sqrt{(a^2-r^2)}}\, f_S(a)\, da . \tag{9}$$

where $f_R(a_0)$ is the marginal pdf for a_0 when we integrate $f_R(a_0,b_0)$ with respect to b_0. The right hand side is the form of $f_P(r)$, the profile radius density, given by Wicksell's integral, while the left hand side is a mixture of the ring profile outer radius density and the disc profile radius density, the mixture being taken in the ratio λ_R to λ_D .

THE INVERSE PROBLEM

We have expressions for the size distributions of the profiles in terms of that for the shells. We seek to invert the relationships by deriving an expression for $f_S(a,b)$ in terms of $f_R(a_0,b_0)$ and $f_D(r)$. We can go some way towards a formal inversion by obtaining $f_S(a,b)$ in terms of $f_R(a_0,b_0)$. In practice however the numerical inversion of integral relationships like these is notoriously unstable, and direct substitution of an estimate of f_R cannot be relied upon to give a satisfactory estimate of f_S .

THEOREM 3.

$$f_S(a,b)/(ab) = -\frac{2\mu_B}{\pi}\int_{b_0=b}^{\infty} \frac{1}{\sqrt{(b_0^2-b^2)}} \frac{\partial}{\partial b_0}\left\{f_R(a_0,b_0)/(a_0b_0)\right\} db_0 , \qquad (10)$$

on $0 \le b \le a < \infty$, where, in the integrand, we take $a_0 = \sqrt{\{b_0^2+(a^2-b^2)\}}$.

We cannot complete the inversion by obtaining f_S in terms of f_D , since the disc diameters do not contain sufficient information about the shells.

Methods of unfolding (as this inversion is referred to by stereologists) are briefly reviewed in Coleman (1989a). In particular, if f_S is from a parametric family of distributions, f_R and f_D will be from parametric families also, and they will be parameterized by the same parameter as f_S. The data from f_R and f_D will allow identification of this parameter.

REMARKS ON THE PROOFS OF THE THEOREMS

Theorems 1 and 2

These were proved using the stochastic geometry method, as set out in Stoyan, Kendall & Mecke (1987). The features are modelled as a stationary marked point process (SMPP) of geometric objects. The points give the locations of the features, the marks describe their sizes. The coordinates of a location and a mark when taken together specify a point on a manifold. The process of features is thus modelled as a process of points, the associated point process (APP). The probabilistic description of the features can then be made in terms of the counting measure of the point process. An operation on the SMPP will lead to another process, the derived SMPP. In the case of stereology, this is the process of profiles seen in the section. By comparing the counting measures of the corresponding APPs and using factorizations that arise from invariance properties, relationships are obtained between the distributions of the sizes of the geometric objects and of the derived profiles. The proofs are not appropriate for presentation in this paper, but may be obtained as a technical report from the author.

The first order characteristics of SMPPs are the intensity (the expected number per unit volume) and the mark distribution, and in respect of these it does not matter whether geometrical probability or the stochastic geometry method is used. Geometrical probability in principle requires only elementary probability theory and very simple geometry, but the mixture of profiles is awkward. The discs have a one-dimensional size distribution, while that of the annular rings is two-dimensional. No such difficulty arises in the stochastic geometry method. It keeps the discs and the rings separate.

Theorem 3

From the geometry of the shells

$$z^2 = a^2 - b^2 = a_0^2 - b_0^2 . \tag{11}$$

We transform the pdf for (a, b) to that for (b, z), and the pdf for (a_0, b_0) to that for (b_0, z). Then the mapping can be taken from b to b_0 keeping z fixed. This mapping can be written in the form of the integral equation for the Wicksell's corpuscle problem, which has a well known formal solution (Wicksell, 1925). This gives the pdf for (b, z) in terms of that for (b_0, z). We then transform back for the pdf for (a, b) in terms of that for (a_0, b_0). The key relationships are:

$$f_R(a_0, b_0) = \frac{1}{\mu_B} \int_{b=b_0}^{\infty} \frac{a_0 b_0}{a \sqrt{(b^2-b_0^2)}} f_S(a, b) \, db , \tag{12}$$

where, in the integrand, $a = \sqrt{\{b^2+(a_0^2-b_0^2)\}}$. This gives

$$g_R(b_0) = \frac{b_0}{\mu_B} \int_{b=b_0}^{\infty} \frac{1}{\sqrt{(b^2-b_0^2)}} g_S(b) \, db , \tag{13}$$

where

$$g_R(b_0) = f_R(b_0, z)/(b_0^2+z^2) , \quad g_S(b) = f_S(b, z)/(b^2+z^2) . \tag{14}$$

This is just the Wicksell integral equation, which has formal solution

$$g_S(b) = - \frac{2\mu_B}{\pi} b \int_{b_0=b}^{\infty} \frac{1}{\sqrt{(b_0^2-b^2)}} \frac{d}{db_0} \left\{ \frac{g_R(b_0)}{b_0} \right\} db_0 , \tag{15}$$

where

$$\left\{ \frac{2\mu_B}{\pi} \right\}^{-1} = \int_{b_0=0}^{\infty} \frac{g_R(b_0)}{b_0} \, db_0 . \tag{16}$$

This restores to

$$f_S(a, b)/(ab) = - \frac{2\mu_B}{\pi} \int_{b_0=b}^{\infty} \frac{1}{\sqrt{(b_0^2-b^2)}} \frac{\partial}{\partial b_0} \left\{ f_R(a_0, b_0)/(a_0 b_0) \right\} db_0 , \tag{17}$$

on $0 \le b \le a < \infty$, where, in the integrand, $a_0 = \sqrt{\{b_0^2+(a^2-b^2)\}}$.

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