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STEREOLOGICAL MODEL TESTS FOR THE SPATIAL POISSON-VORONOI TESSELLATION. I

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ABSTRACT

For the spatial POISSON-VORONOI tessellation, five stereological model tests based on the distribution of the number of cell vertices in random plane sections are proposed. For some chosen sample sizes the quantiles of the distribution of the test variables are estimated by simulation. The power of the model tests is investigated under some special parametric alternative hypotheses. Finally, an application to single-phase alumina ceramics is given.

Keywords: model test, random plane section, spatial POISSON-VORONOI tessellation, stereology, stochastic geometry.

INTRODUCTION

The random *VORONOI tessellation* is an important model of stochastic geometry which seems to be suitable for describing space-filling mosaic-like structures resulting from growth processes. VORONOI tessellations have successfully been used as models in many fields of science, e.g. in materials science, geography, astrophysics, cell biology and geology (Stoyan et al., 1987).

A random spatial VORONOI tessellation is a random division of space into convex polyhedra (cells) defined with respect to a point process of germs. Each cell consists of those points of the space which are nearer to a given germ than to all other germs. If the germs constitute a homogeneous POISSON point process the tessellation is called ***POISSON-VORONOI tessellation***. The only parameter of this model is the intensity $\lambda > 0$, the mean number of points of the underlying POISSON point process per unit volume.

Until now there are no statistical procedures allowing to check the applicability of this model. In the development of model tests it should be taken into account that in practice a spatial mosaic-like structure can typically not be directly observed. Only information from linear or plane sections is available. Therefore, the aim of this and forthcoming papers is to develop stereological model tests for the spatial POISSON-VORONOI tessellation.

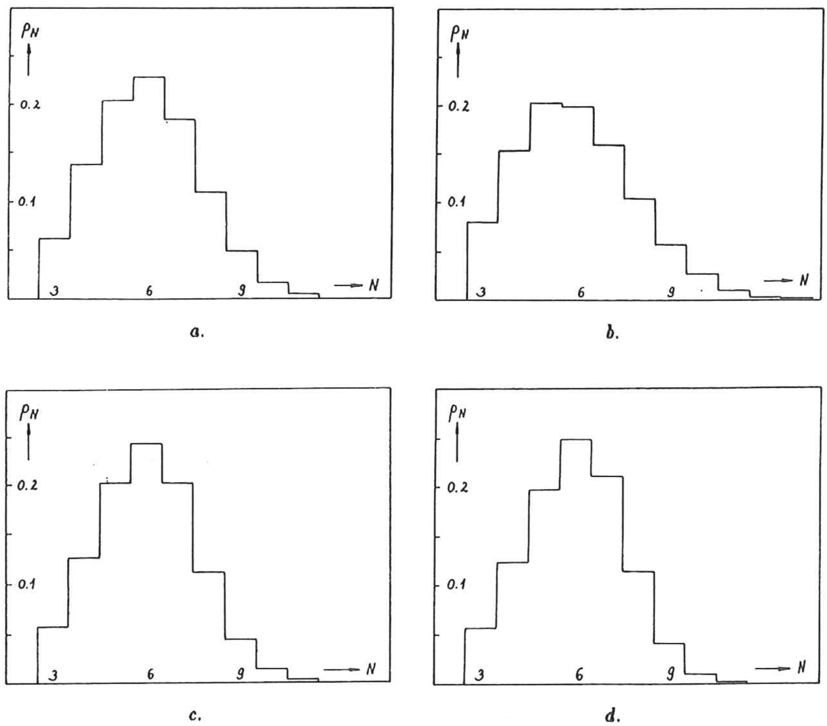

Figure 1. Histograms of relative frequency p_N of the number of cell vertices N in plane sections of several VORONOI tessellations:
PVT (*a*), CVT (*b*), HVT (*c*), and SVT (*d*).

STEREOLOGICAL MODEL TESTS

All model tests proposed here are based on the distribution of the number of cell vertices N in random plane sections of a spatial tessellation. This distribution is independent of the intensity of the germs (hence scale invariant) and discrete (therefore a division into classes is straightforward). Furthermore, in practice the number of cell vertices can easily be determined.

In Figure 1 histograms of relative frequency of N for the POISSON-VORONOI tessellation (PVT) and for typical examples of VORONOI tessellations are shown with respect to a *MATERN cluster point process* (CVT), a *MATERN hard-core point process* (HVT), and a *simple sequential inhibition point process* (SVT), respectively. For a mathematical definition of these point processes see (Diggle, 1983) and (Stoyan et al., 1987). Obviously, the HVT and especially the SVT is in some sense more regular than the PVT whereas the CVT is more irregular. The knowledge about the differences in the shape of these histograms can be used in order to find suitable test variables.

As a first test the well-known χ^2*-test of goodness of fit* is proposed. The probabilities p_N, $N = 3, 4, \dots$, that a plane section cell of a spatial PVT has N vertices were estimated by simulation (Lorz, 1991). In order to approximately satisfy the usual condition $np_i^! \geq 4$ for all class probabilities $p_i^!$ and all considered sample sizes n a division into seven classes with $p_i^! = p_i$, $i = 3, \dots, 8$, and $p_9^! = \sum_{N=9}^{\infty} p_N$ is used (see Table 1). Consequently, the test variable is

$$T_n = \sum_{i=3}^{9} \frac{(H_i^! - np_i^!)^2}{np_i^!}, \qquad (1)$$

where $H_i^!$, $i = 3, \dots, 8$, is the absolute frequency of i-sided section cells and $H_9^!$ is the absolute frequency of the more than eight-sided section cells, respectively.

Table 1. Estimated class probabilities $p_i^!$ for the PVT.

i	$p_i^!$	i	$p_i^!$
3	0.06305593	7	0.18371370
4	0.13579820	8	0.11038764
5	0.20474441		
6	0.22733316	9	0.07496600

For all considered VORONOI tessellations the theoretical mean value of N is equal to six (Stoyan et al., 1987). But they differ considerably with respect to the variance of N. Thus the usual sample dispersion of N

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (N_i - \overline{N})^2, \tag{2}$$

seems to be a suitable test variable for a second model test. Obviously, tessellations with a smaller variance of N are more regular and tessellations with a greater variance of N are more irregular than the PVT.

By analogy, the sample skewness (CHARLIER) of N

$$C_n = \frac{M_n^{(3)}}{[M_n^{(2)}]^{3/2}}, \tag{3}$$

where

$$M_n^{(k)} = \frac{1}{n} \sum_{i=1}^{n} (N_i - \overline{N})^k, \; k = 2, 3, \ldots, \tag{4}$$

denotes the sample moment of order k, can be used as test variable. Here the PVT also takes an intermediate position. The histograms of more regular tessellations (HVT, SVT) are nearly symmetric whereas for more irregular tessellations the histograms of N are considerably right-skewed.

Finally, the sample kurtosis of N

$$K_n = \frac{M_n^{(4)}}{[M_n^{(2)}]^2} - 3 \tag{5}$$

and the test variable

$$G_n = \frac{\sqrt{n \sum_{i=1}^{n} (N_i - \overline{N})^2}}{\sum_{i=1}^{n} |N_i - \overline{N}|} \tag{6}$$

(in analogy with GEARY's test for the normal distribution, see e.g. Rasch et al. (1973)), which are sensitive with respect to the kurtosis of N, are taken in consideration.

In contrast to the χ^2-test of goodness of fit all other tests allow to test PVT versus more regular or more irregular tessellations, respectively. Hence, they can be performed as one-sided tests.

QUANTILES OF THE TEST VARIABLES

Since the number of cell vertices of neighbouring section cells is not independent the quantiles of the χ^2-distribution cannot be used for the χ^2-test of goodness of fit. Furthermore, in this situation the distribution of the test variables S_n^2, C_n, K_n, and G_n is completely unknown.

In order to determine the quantiles for the proposed model tests these quantities were estimated by a computer simulation. The α-quantiles for the levels of significance $\alpha = 0.01$, 0.025, 0.05, and 0.1 were determined for the practical important sample sizes $n = 50$, 100, 150, and 200. About 7000 samples were generated for each sample size n . For details concerning this simulation procedure see (Lorz, 1991) and (Møller et al., 1989).

Table 2. Estimated and χ^2-quantiles for the χ^2-test of goodness of fit.

	$t_{n,1-\alpha}$				$\chi^2_{6,1-\alpha}$
α	$n = 50$	$n = 100$	$n = 150$	$n = 200$	$n \to \infty$
0.01	16.579	16.152	15.895	16.708	16.81
0.025	14.090	13.849	13.808	14.127	14.45
0.05	12.348	11.904	12.005	12.064	12.59
0.10	10.005	10.100	10.059	10.116	10.64

Table 3. Estimated quantiles $s^2_{n,\alpha}$.

	$s^2_{n,\alpha}$			
α	$n = 50$	$n = 100$	$n = 150$	$n = 200$
0.005	1.5465	1.8476	2.0448	2.1190
0.01	1.6494	1.9390	2.1009	2.1800
0.0125	1.6902	1.9696	2.1271	2.2010
0.025	1.8127	2.0908	2.1951	2.2778
0.05	1.9592	2.1918	2.2966	2.3748
0.1	2.1224	2.3200	2.4154	2.4768
0.9	3.7143	3.4645	3.3528	3.2964
0.95	3.9984	3.6752	3.5025	3.4201
0.975	4.2466	3.8460	3.6508	3.5476
0.9875	4.4898	4.0181	3.7731	3.6619
0.99	4.5567	4.0590	3.8177	3.6828
0.995	4.7351	4.1870	3.9396	3.8050

Table 4. Estimated quantiles $c_{n,\alpha}$.

α	$c_{n,\alpha}$ $n = 50$	$n = 100$	$n = 150$	$n = 200$
0.005	-0.3707	-0.1662	-0.0855	-0.0345
0.01	-0.3091	-0.1195	-0.0450	-0.0001
0.0125	-0.2904	-0.1093	-0.0330	0.0085
0.025	-0.2149	-0.0596	0.0039	0.0430
0.05	-0.1420	-0.0111	0.0530	0.0827
0.1	-0.0599	0.0565	0.1056	0.1360
0.9	0.6497	0.5758	0.5426	0.5152
0.95	0.7696	0.6645	0.6137	0.5860
0.975	0.8800	0.7472	0.6766	0.6462
0.9875	0.9955	0.8277	0.7378	0.7001
0.99	1.0260	0.8639	0.7593	0.7187
0.995	1.1312	0.9404	0.8296	0.7833

In the Tables 2, 3, and 4 the empirical quantiles $t_{n,\alpha}$, $s^2_{n,\alpha}$, and $c_{n,\alpha}$ are given of the test variables T_n, S^2_n, and C_n, respectively. The estimated quantiles for the χ^2-test of goodness of fit differ only slightly from the corresponding χ^2-quantiles.

POWER OF THE MODEL TESTS

In order to assess the power of the model tests, values of their power functions were estimated. As special parametric alternative hypotheses the more regular HVT and SVT and the more irregular CVT were chosen, respectively.

The MVT as well as the SVT model can be characterized by the scale parameter $\lambda_{hc} > 0$, the mean number of points of the underlying point process per unit volume, and the shape parameter $p_{hc} = \lambda_{hc} \frac{4}{3}\pi R^3_{hc}$, the mean volume fraction of the hard cores with radius $R_{hc} > 0$ (Lorz, 1990). For the HVT the parameter p_{hc} has to be taken from the interval $[0, \frac{1}{8})$ whereas for the SVT p_{hc} can be chosen between 0 and approximately 0.4. Consequently, with the SVT model a higher degree of regularity in the tessellation can be reached. Since in both models the limiting case $p_{hc} = 0$ corresponds to the PVT, the parametric test problem can be formulated with the simple null hypothesis $p_{hc} = 0$ versus the compound alternative hypothesis $p_{hc} > 0$.

The model parameters of the CVT are the scale parameter $\lambda_{cl} > 0$, the mean number of points of the underlying cluster point process per unit volume, and the shape parameters $N_{cl} > 0$, the mean number of points per cluster, and $R_{cl} > 0$, the cluster radius (Stoyan et al.,

$N_{cl} > 0$, the mean number of points per cluster, and $R_{cl} > 0$, the cluster radius (STOYAN et al., 1987). The scale dependent cluster radius R_{cl} can be replaced by

$$p_{cl} = 1 - exp\left\{\frac{\lambda_{cl}}{N_{cl}} \frac{4}{3}\pi\ R_{cl}^3\right\}, \tag{7}$$

the probability that neighbouring clusters overlap. For $N_{cl} \longrightarrow 1$ and $p_{cl} \longrightarrow 0$ the CVT tends to the PVT. Hence the parametric test problem consists in the simple null hypothesis $N_{cl} = 1$ and $p_{cl} = 0$ versus the compound alternative hypothesis $N_{cl} \neq 1$ and $p_{cl} > 0$.

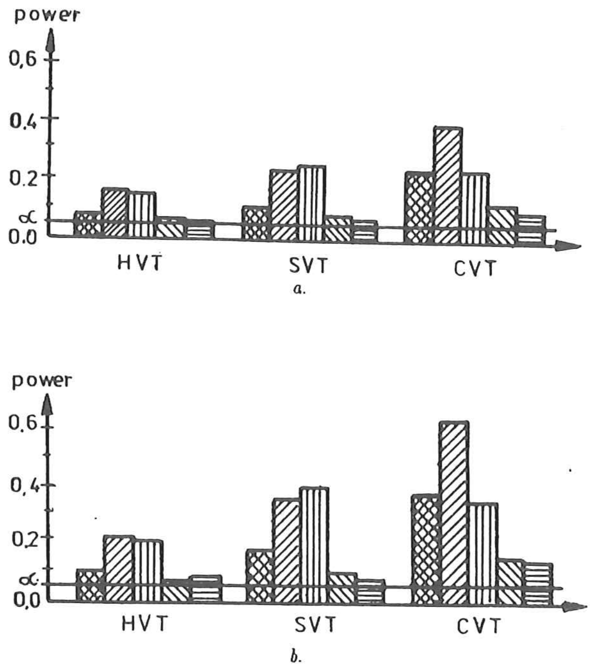

Figure 2. Estimated values of the power functions for the sample sizes $n = 100$ (a) and $n = 200$ (b).

χ^2-test of goodness of fit (1)
one-sided variance test (2)
one-sided skewness test (3)
one-sided kurtosis test (5)
one-sided GEARY-test (6)

$p_{hc} = 0.2$ (SVT), and $N_{cl} = 10$ and $p_{cl} = 0.7$ (CVT) were chosen. The intensities λ_{hc} and λ_{cl} were set to unity. About 7000 samples were generated for each model and sample size. The results for two sample sizes are presented in Figure 2.

The most powerful of the considered tests is the variance test (2), followed by the skewness test (3) and the χ^2-test of goodness of fit (1), where the skewness test is more sensitive with respect to more regular than to more irregular tessellations. The tests (4) and (5) can not be recommended.

As expected the power of the model tests is rather small for small sample sizes. But for sample size $n = 200$ a probability of the error of second kind of less than 40% is reached (CVT). In general, it is easier to distinguish between PVT and more irregular tessellations than between PVT and more regular tessellations.

APPLICATION TO SINGLE-PHASE ALUMINA CERAMICS

The analysis of the microstructure of metals or ceramics is an important problem in materials science. In order to solve this problem it is necessary to find suitable models for the investigated materials.

The spatial POISSON-VORONOI tessellation seems to be an appropriate model to describe certain single-phase microstructures, if the pores can be neglected and the grains are approximately polyhedra. With this model the analysis of the geometrical properties of the spatial structure is very simple. It suffices to determine the mean number P_A of vertices per unit area or the mean boundary length L_A per unit area in a plane section. The model parameter λ can then be estimated by the help of the formulae (Stoyan et al., 1987)

$$\lambda = 0.2008\ P_A^{3/2} \qquad \text{or} \qquad \lambda = 0.0837\ L_A^3. \tag{8}$$

Finally, mean values and variances of geometric characteristics of the grains, e.g. the surface area S and the volume V, can be obtained using (Stoyan et al., 1987; Brakke, 1985)

$$\mathsf{E}(S) = 5,821\ \lambda^{-2/3}, \qquad \mathrm{Var}(S) = 2,191\ \lambda^{-4/3}, \tag{9}$$

$$\mathsf{E}(V) = \lambda^{-1}, \qquad \text{and} \qquad \mathrm{Var}(V) = 0,179\ \lambda^{-2}. \tag{10}$$

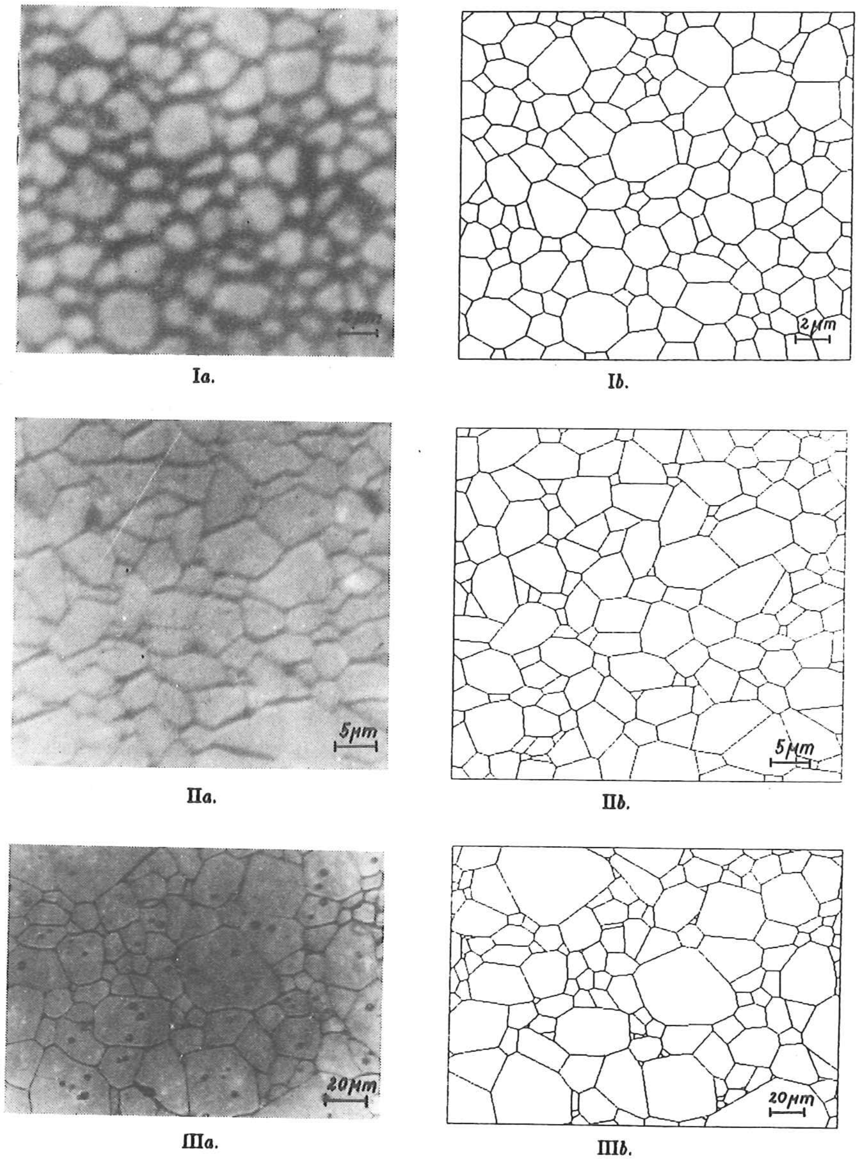

Figure 3. Plane sections of alumina ceramics (*a*) and their schemes as planar tessellations (preprocessings) (*b*).

In Figure 3 parts of plane sections of three specimen of single-phase alumina ceramics are shown together with the corresponding schemes as planar tessellations consisting of polygons. These preprocessings are obtainded by neglecting pores and inclusions, and connecting the cell vertices by straight line segments. Specimen I is rather regular whereas specimen III is rather irregular. Specimen II takes an intermediate position.

In order to check the applicability of the PVT model for each specimen, the five model tests were carried out with a level of significance $\alpha = 0.05$. Both in the simulation procedures and in the application of the model tests, only cells completely inside the observation window were included. Hence, edge effects were treated in the same manner.

Whereas for specimen II the null hypothesis is accepted by all model tests, for specimen I it is only the tests (3), (4), and (5), and for specimen III only test (5) that accept the null hypothesis. Thus for specimen II the PVT model can be accepted and formulae (8), (9), and (10) can be applied for the analysis.

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REFERENCES

Brakke KA. Statistics of three dimensional random VORONOI tessellations. *unpublished manuscript*, Susquehanna Univ., Dept. of Math. Sciences, Selinsgrove, 1985.

Diggle PJ. *Statistical Analysis of Spatial Point Processes*. London: Academic Press, 1983.

Lorz U. Cell-area distributions of planar sections of spatial VORONOI mosaics. *Materials Characterization* 1990; 25: 297–311.

Lorz U. Distributions of cell characteristics of the spatial POISSON-VORONOI tessellation and plane sections. In: Eckhardt U, Hübler A, Nagel W, Werner G, eds. *Geometrical Problems of Image Processing*. Ser. Research in Informatics, Vol. 4, Berlin: Akademie Verlag, 1991: 171–178.

MØller J, Jensen EB, Petersen JS, Gundersen HJG. Modelling an aggregate of space filling cells from sectional data. *Res. Rep. 182*, Univ. Aarhus, Inst. Theor. Statist. (1989).

Rasch D, Enderlein G, Herrendörfer G. *Biometrie – Verfahren, Tabellen, Angewandte Statistik*. Berlin: VEB Deutscher Landwirtschaftsverlag, 1973.

Stoyan D, Kendall WS, Mecke J. *Stochastic Geometry and Its Applications*. Berlin: Akademie-Verlag, 1987.

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