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THE SECOND-ORDER ANALYSIS OF PARTICLE SIZES BY A TEST QUADRAT METHOD

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ABSTRACT

For a set of convex particles on the plane the particle widths (sizes) h_1,h_2 measured in two perpendicular directions form a two-dimensional random variable (h_1,h_2). The coresponding second order moment $E(h_1h_2)$ of (h_1,h_2) characterizes the particle size and shape. The moment $E(h_1h_2)$ is related to the expected value $EN_K(A)$ of the number of counted particles which are totally within a test quadrat of area A. An experimental example of application for a Fe_3C-dispersion in steel is given.

Keywords: convex particle, shape, testing by quadrat, size correlation.

INTRODUCTION

Test quadrats are often used in the image analysis practice (Wiencek and Hougardy, 1987; Rys, 1989). There are also some theoretical investigations in this field (Saltykow, 1974; Miles, 1978).

One of the methods of analysing a two-dimensional set of randomly distributed convex particles by a test quadrat **K** consists on counting those particles which are totally contained within the quadrat **K** (and excluding the particles which intersect the border of **K**), Fig.1.

For a randomly positioned test quadrat **K** (of area A) the number of particles within **K**, $N_K(A)$, is a discrete random variable and its expected value, $EN_K(A)$, is a function of some distributional characteristics of the set of particles. This paper aims to discuss theoretical properties of this function and possible applications.

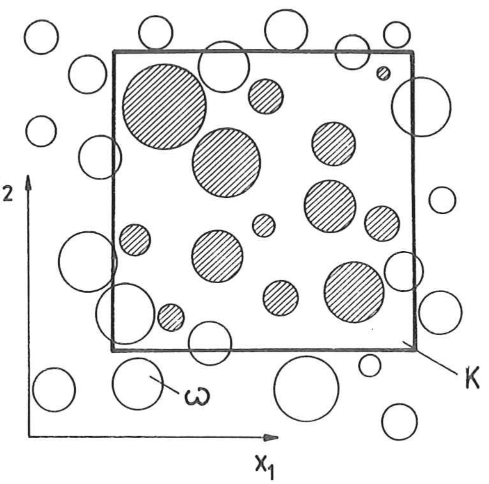

Fig. 1. Test quadrat **K** and the counted particles.

THEORETICAL CONSIDERATIONS

A collection Ω of two-dimensional convex, non overlapping particles is randomly distributed on the plane $\mathbf{R}^2$ forming there an homogeneous and isotropic random set. In the plane a Cartesian (x_1,x_2) coordinate system is given. The particles can be characterized relatively to the (x_1,x_2) coordinate system as follows. With each particle $\omega(k) \in \Omega$ a rectangle of side lengths $h_1(k)$ and $h_2(k)$ (the so-called particle widths) is conjugated, where the sides are tangents to the particle and are parallel to the x_1, x_2 axes, Fig.2 ($k = 1, 2, \ldots$).

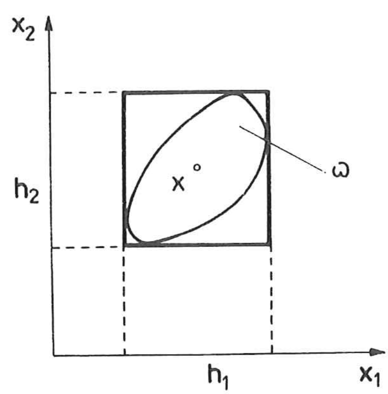

Fig. 2. A convex particle and the conjugated rectangle.

The center $x(k)$ of the rectangle is taken as that of the particle. The set of particles Ω may be represented by the set of the conjugated rectangles.

The set of particles Ω can be described by an homogeneous marked point process (Stoyan et al., 1987) $\Psi_2 = \{[x(k), h_1(k), h_2(k)]\}$. Here $x(k)$ is the center of the k-th particle, and the marks $h_1(k)$, $h_2(k)$ denote its widths. The $x(k)$ form a simple point process in the plane $\mathbf{R}^2$ with the point density N_A. The particle widths form a two-dimensional random variable (h_1,h_2) with fundamental moments $E(h_1)$, $E(h_2)$ and $E(h_1h_2)$. For the process Ψ_2 the expected number $EN(A)$ of points $x(k)$ in a test field $\mathbf{K}$ of area A is equal $A \cdot N_A$. Because of the homogeneity and isotropy assumption the variable (h_1,h_2) is independent of the way in which the coordinate system (x_1,x_2) on the plane has been chosen. This implies the equality $E(h_1) = E(h_2) \equiv E(h)$ where $E(h)$ is the so-called mean caliper diameter. These moments determine the covariance $C_{12} = E(h_1h_2) - E^2(h)$.

In the following analysis it is assumed that the variables h_1 and h_2 are discrete with values: h_{1i}, h_{2j} for $i=1,\ldots,k$ and $j=1,\ldots,l$. The test quadrat $\mathbf{K}$ of sides of length L which are parallel to the x_1, x_2 axes ($L > (h_{1k}$ and $h_{2l}))$ is positioned on the plane $\mathbf{R}^2$ containing the set of particles Ω. Let $\Omega(i,j) \in \Omega$ denotes the subset of Ω consisting of the particles $\omega(i,j)$ of widths h_{1i} and h_{2j} which appear with the density $N_A(i,j)$. A particle $\omega(i,j)$ belong to the interior of $\mathbf{K}$ if its center is in a rectangle $\mathbf{K}(i,j) \subset \mathbf{K}$ with side lengths $L-h_{1i}$ and $L-h_{2j}$, Fig.3.

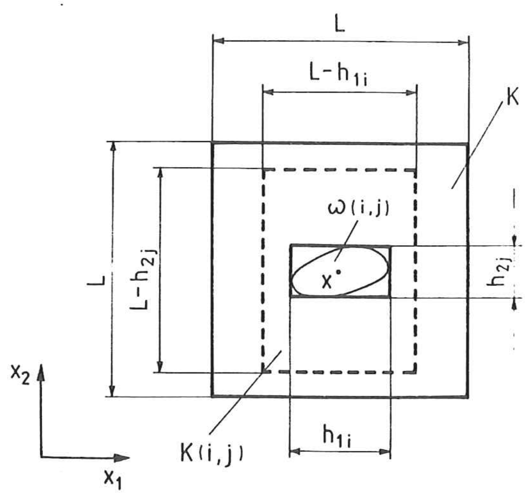

Fig. 3. Test quadrat $\mathbf{K}$ and the rectangle $\mathbf{K}(i,j)$ associated with the particle of $\Omega(i,j)$ set.

For a random position of $\mathbf{K}$ (realized by a random translation) on the plane $\mathbf{R}^2$ the number $N_K(i,j|A)$ of particles of $\Omega(i,j)$ being totally within $\mathbf{K}$ is a discrete random variable with the expected value

$$EN_K(i,j|A) = N_A(i,j)(L-h_{1i})(L-h_{2j}). \tag{1}$$

Then the expected value $EN_K(A)$ for the Ω particles set is

$$EN_K(A) = \sum_{i,j} N_A(i,j)(L-h_{1i})(L-h_{2j}) \tag{2}$$

and expansion of the right side of (2) gives

$$EN_K(A) = L^2\sum_{i,j} N_A(i,j) - L\sum_{i,j} h_{1i}N_A(i,j) - L\sum_{i,j} h_{2j}N_A(i,j) + \sum_{i,j} h_{1i}h_{2j}N_A(i,j).$$

Taking into account the stereological equation

$$N_L = N_A E(h) = \sum_{i,j} h_{1i}N_A(i,j) = \sum_{i,j} h_{2j}N_A(i,j)$$

and the relationships

$$N_A = \sum_{i,j} N_A(i,j)$$

and

$$N_A E(h_1h_2) = \sum_{i,j} h_{1i}h_{2j}N_A(i,j)$$

yields

$$EN_K(A) = L^2N_A\,[1 + E(h_1h_2)/L^2] - 2LN_L. \tag{3}$$

Here N_L is the density of the intersections of particles with a test line.

Equation (3) indicates that $EN_K(A)$ is a function of the two stereological parameters N_A and N_L as well as of the moment $E(h_1h_2)$, it is true also for a general distribution of (h_1,h_2) (not only for the discrete case).
Of course, the expected number $EN(A)$ of particle centers and the expected number $EN(L)$ of intersections (with a test line of length L) are equal to L^2N_A and LN_L respectively. Thus equation (3) can be expressed as

$$EN_K(A) = EN(A)[1 + E(h_1h_2)/L^2] - 2EN(L) \tag{4}$$

Finally the moment $E(h_1h_2)$ can be expressed by the covariance C_{12}:

$$C_{12} = EN_K(A)/EN(A) + 2\lambda + \lambda^2 - 1 \tag{5}$$

with $\lambda = E(h)/L$.

Equation (5) shows that the covariance of the variable (h_1,h_2) is determined by the ratio of the expected number of particles belonging totally to

the interior of **K** and of particle centers in **K**.
For circular particles with diameters d, the covariance C_{12} is equal to the diameter variance $\sigma^2(d)$, which thus can be expressed by equation (5).
With increasing L the value $E(h_1h_2)/L^2$ in (3) decreases to zero rapidly and equation (3) could be used for estimating N_A by two countings: that of particles being totally within the quadrat **K** and that of particles intersecting two parallel borderlines of **K**. The two-countings method is easier than direct estimation based on counting of particle centers due to the difficulties of identification of centers.

EXPERIMENTAL EXAMPLE

A coagulated Fe_3C-dispersion in a steel sample was investigated in order to get some information on particle size and shape. For this purpose the moment $E(h_1h_2)$ was estimated using the test quadrat method. The estimation was performed by using of the following estimator $\langle h_1h_2\rangle$ based on equation (4)

$$\langle h_1h_2\rangle/L^2 = [\bar{N}_K(A) + 2\bar{N}(L)]/\bar{N}(A) - 1 \qquad (6)$$

$\bar{N}(A)$, $\bar{N}_K(A)$ and $\bar{N}(L)$ are estimators of $EN(A)$, $EN_K(A)$ and $EN(L)$ obtained from various test quadrats in different positions by averaging numbers.
For comparison the value $\langle h_1h_2\rangle$ was also established by direct measurement of h_1 and h_2 for the particles.
The micrographs (magnification 2000 times) were made of a polished and etched by picral specimen of carbon steel (Fe–0.6%C) (hardened and annealed in vacuum at 700°C for 600 hours). Fig. 4 presents the microstructure of the Fe_3C-dispersion.

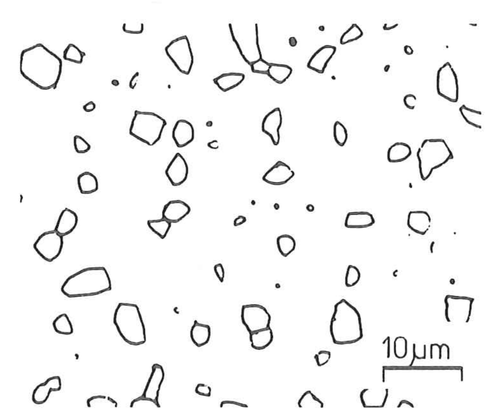

Fig. 4. Microstructure of steel with the Fe_3C-particles.

The metallographic analysis indicates that the Fe_3C-particle sections are approximately convex and randomly distributed in the ferrite matrix. Thus the marked point process assumed above is an acceptable model for the Fe_3C-dispersion. The means $\bar{N}(A)$, $\bar{N}_K(A)$ and $\bar{N}(L)$ for the test quadrat **K** with side length L=20 mm (scale of micrograph at 2000 times of magnification) were determined. The measurements on micrographs were performed by a measurement quadrat $\mathbf{K}_0$ with L_0=80 mm divided into 16 test quadrats $\mathbf{K}$ with L=20 mm. The quadrat $\mathbf{K}_0$ determines a square net containing 16 test-quadrats $\mathbf{K}$ and 10 segments with length L_0. For a randomly positioned quadrat $\mathbf{K}_0$ the particle centers in $\mathbf{K}_0$ as well as the particles being totally within the quadrats $\mathbf{K}$ and the intersections of the particles with the segments of length L_0 were counted. (One has to note that for a particular measurement quadrat $\mathbf{K}_0$ the test quadrats $\mathbf{K}$ and the segment with length L are arranged systematically. Nevertheless for homogeneous and isotropic dispersions the estimation of the means $\bar{N}_K(A)$ and $\bar{N}(L)$ by systematic testing is equivalent to the estimation by a random testing procedure.)

The microstructure was analysed by using of 43 randomly positioned disjoint quadrats $\mathbf{K}_0$. The whole number of particles was 1529. 925 particles were totally in the interior of the 688 test quadrats $\mathbf{K}$ while 871 particles were intersected by the. 430 segments of length L_0. The means together with the corresponding standard deviations s($\cdot$) are equal to:

$\bar{N}(A) = 2.222$, $s(\bar{N}(A)) = 0.049$

$\bar{N}_K(A) = 1.344$, $s(\bar{N}_K(A)) = 0.041$

$\bar{N}(L) = 0.506$, $s(\bar{N}(L)) = 0.015$

The obtained values are presented at the scale of the micrographs (magnification 2000 times). Substitution of these values into equation (6) gives $\langle h_1h_2 \rangle = 24.120\ \mathrm{mm}^2$. The accuracy of the $E(h_1h_2)$ estimation is determined by the precision of $\bar{N}(A)$, $\bar{N}_K(A)$ and $\bar{N}(L)$. The total differential method applied to eq. (6) yields a value of 13.10 mm^2 for the standard deviation of $\langle h_1h_2 \rangle$.

Next, h_1 and h_2 values for two perpendicular directions (parallel to x_1 and x_2 axes) were measured for all 1212 particles in 22 randomly chosen sampling areas of the microstructure. For the obtained pairs (h_1, h_2) the two-dimensional empirical distribution was determined. In Fig.5 shows the plot of the conditional mean $\bar{h}_2|h_1$ as a function of h_1 is given together with the upper and lower standard deviation regions.

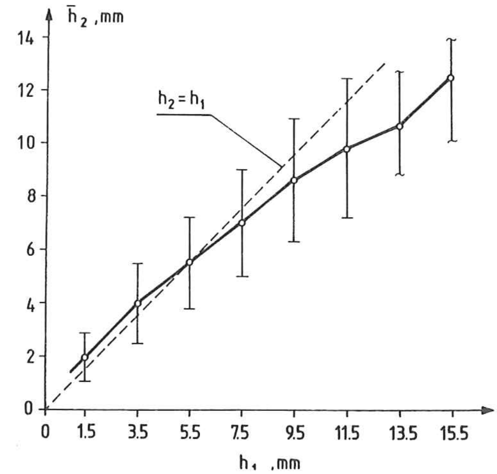

Fig. 5. The conditional mean $\bar{h}_2|h_1$ as a function of h_1 for the Fe_3C-particles.

For randomly distributed particles the moment $E(h_1h_2)$ and the form of $E(h_2)|h_1$ depend on the shape of particles. In the circular case it is $h_1=h_2$ and $E(h_2)|h_1$ is a straight line. The empirical relation in Fig. 5. shows deviations from the linear form, which indicate that some of the particles are slightly elongated as it can be seen in Fig.4. This particle shape characteristic is probably connected with interaction between the particles and matrix grain boundaries during annealing.

The mean $\overline{h_1h_2}$ calculated by a standard statistical procedure for two-dimensional random variables (Sachs, 1978) is equal to 27.90 mm^2 being close to the value above which was obtained by the test quadrat **K** method. The corresponding covariance and the correlation coefficient are C_{12}= 5.52 mm^2 and r_{12}=0.80.

CONCLUSIONS

A counting method for estimation of the moment $E(h_1h_2)$ by equation (6) for two dimensional convex particles is proposed. The estimation procedure requieres counting the particles occupying the interior of the test quadrat **K** (the side length L of which should not be too large compared to the particle sizes) and estimation of the mean $EN_K(A)$ as well as the estimation of the densities N_A and N_L. However, a sufficient number of measurements (different positions of the test quadrat) is necessary to get a reasonable accuracy.

The moment $E(h_1h_2)$ can be taken as a measure of mean shape and size of two-dimensional convex particles.

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