ACTA STEREOL 1987; 6/SUPPL II: 185–192
ISS COMMEMORATIVE-MEMORIAL VOLUME
ORIGINAL SCIENTIFIC PAPER

STEREOLOGICAL ESTIMATION OF MINERAL LIBERATION
I. CONCENTRIC SPHERE MODELS

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ABSTRACT

Problems of mineral liberation are illustrated through models for two-phase spherical particles, sampled by random line and plane sections.

Keywords: Intercept grades, Mineral processing, Random sections, Two-phase particles, Uniformly random sections.

INTRODUCTION

The reason for mineral technology is the processing of ores to separate mixtures of intergrown minerals. Rocks are ground down to give up their mineral grains. This grinding is expensive, so it is essential to have rules which show the earliest instant at which the grinding can cease. As well as the waste of energy and the cost of sharpening the cutting edges, very small particles can be difficult to process. The particles produced would ideally each contain just a single mineral phase. These are the <u>liberated</u> particles. Those that still contain more than one mineral phase are called <u>composite</u>.

The external appearance of particles can seem to show them to be liberated when in fact they contain internal phase boundaries. Nevertheless, as composite particles are broken, liberated pieces are created, and those still composite tend to have less intergrowth of phases and show a greater concentration on one of them. Eventually most particles can be expected to be liberated. However, as already remarked upon, we often cannot afford to wait so long, and a "dust" of liberated particles can be unsuitable for further processing.

At each stage of the grinding we would like an estimate of the proportions of liberated particles of each phase for each size of particle, and to identify the distribution of the composition and the degree of intermingling of phases for the composite particles of each size.

In a mixture of particles of different sizes we are not comparing like with like if we say that half the particles are liberated, when the half that are liberated are all very small and the half that are still composite are all large. Davy(1984) constructs indices of liberation which take into account the differing sizes of the particles. From a mathematical point of view we can take the particles all to be of the same size, since in principle we could use a sequence of sieves to sort the particles. The particles of a particular size are those which pass through one sieve but not the next in the sequence, if the holes get successively smaller.

Similarly, in principle, we need consider particles having only two mineral phases, α and β say, where α can represent the phase of interest and β is a conglomerate of all the other phases. We can carry out separate analyses taking each phase in turn to be the α phase.

Let us consider particles some of which may have a core of one material set in a shell of a second material. These appear to be liberated. This can lead to overestimation of the proportion of liberated particles. Examination of the internal structure by line probe sections or plane sections can reduce this systematic error but will not remove it entirely, since if the section misses the core it will fail to expose the internal phase boundary.

In this paper our concern is with the use of stereological information given by random sections through a random sample of particles. For only a few geometrically structured particles has the distribution of the intercepts been derived theoretically, and then only for sections taken in certain uniformly random ways (Coleman,1978,1981; Jones et al,1978; Jones & Horton,1979). That there is more than one way of taking a line or plane section uniformly at random, and that these give rise to different intercept distributions, was noted in the case of a random chord of a circle as long ago as 1888 by Bertrand(1888). In Coleman(1987) more than 100 different cases for line and plane sections of a sphere are listed. The various randomness mechanisms extend to the taking of line and plane sections through irregularly shaped particles (see for example Coleman,1984). The regular geometric shapes for which the intercept distribution can be found theoretically are not generally those of particles obtained from crushing, and a comparison with the intercepts for synthetic populations of irregularly shaped particles is not close (Moore & Jones,1980,1981).

As a general observation there can be no wholly adequate solution to the problem of mineral liberation without modelling assumptions for the structure of the particles. Attempts have been made to use general theorems of integral geometry such as formulae for the mean intercept length of a random line section. But even this requires us to assume that the particles are convex in shape, and have at most one interface between the two phases, and that this is a planar interface so that the two phases are themselves convex grains. See for example Bodziony & Kraj(1985) for a review of this.

A PROBABILITY MODEL FOR MINERAL LIBERATION

At its simplest the problem can be presented as follows. Let

$$\gamma = V_V(\alpha)$$

be the volumetric grade of phase α in a particle, that is to say, the fraction of the volume that is phase α. We suppose that the population of particles has a volumetric grade distribution whereby, for a particle taken at random, it is liberated and phase α (so $\gamma = 1$) with probability p_A, it is liberated and phase β (so $\gamma = 0$) with probability p_B, or it is composite with probability $p_C = 1-p_A-p_B$. Those particles in the population that are composite will have a distribution of volumetric grades:

$$p_C(c)dc = P(\ \gamma \text{ is in the narrow range } (c,c+dc)) \quad (0<c<1),$$

where $P(A)$ denotes the probability of the event A. The density $p_C(c)$ can be treated as a generalized probability density which allows a discrete probability distribution

$$P(\gamma = \gamma_i) \quad (i = 1,2,\ldots,k),$$

$$P(\gamma = 0) = p_B, \quad P(\gamma = 1) = p_A$$

to be represented as a convex combination of delta functions. In the continuous case we have a mixed distribution with a density over the range 0 to 1 and atoms of probability at 0 and at 1.

The problem is to estimate p_A, p_B and the density $p_C(c)$ from the intercept data. For line sections through each of a random sample of particles this intercept data is generally the linear grade G of the intercept, the fraction of its length that is in phase α,

$$G = L_L(\alpha).$$

For plane sections it is the area grade, the fraction of its area that is in phase α,

$$G = A_A(\alpha).$$

The statistics at our disposal for the estimation are the proportions of liberated intercepts having $G = 1$ and $G = 0$ respectively, and the values of the intercept grades G of the composite intercepts.

We illustrate the above with a model in which all the particles are spheres, and the composite particles are concentric spheres. In a second paper the technically more difficult case in which the phase interface in the spherical particles is a plane is treated.

CONCENTRIC SPHERE MODELS

Example 1

The particles are all spheres of radius one. Some are entirely of phase β and some have a central core of phase α surrounded by a shell of phase β.

For a line section that penetrates the core of a composite particle we can measure the total intercept length in the particle, u_0, and the intercept length in the core, u_1. Simple triangle geometry gives the core radius r

$$r = \sqrt{(1 - (u_0^2 - u_1^2)/4)}$$

Similarly for a plane section that passes through the core we measure the diameter of the disc section of the particle, u_0, and of the core section, u_1. Then the core radius r takes the same form as for the line section. In each case we know the core radius exactly. All that remains is to identify the core radii of those particles which are composite but for which the section misses the core.

Let us suppose that we have a sample of n particles, and that of the n random intercepts, n_0 pass through a core, and for these n_0 we obtain values for the core radii. These core radii are sorted into k size classes $D_1,\ldots,D_k$ centred on the values $r_1,\ldots,r_k$ respectively. If there are m_j in the jth size class then there will be another unknown number $n_j - m_j$ of particles with core radii in this size class but for which the intercepts missed the core. Let

"C" be the event "the core is hit", and let

"R_j" be the event "the core radius is in the size class centred on r_j". Then, with some approximation because of the grouping into classes and the assumption that each core radius in the classhas the precise value r_j, the probability that the core is hit given that the core radius is in the size class D_j

$$p_j = P(C|R_j) = m_j/n_j .$$

The value of p_j is known theoretically for a uniformly random section of a sphere, m_j is observed, and so n_j is determined approximately. That is to say,

$$\hat{n}_j = m_j/p_j$$

is our estimate of the total number out of the sample of n particles that have their core radii in the size class centred on the value r_j. The " ^ " denotes that the value $\hat{n}_j$ is an estimate of n_j. We can then estimate the total number of particles in the sample that have a core by

$$\hat{n}_c = \Sigma \hat{n}_j .$$

We can introduce a safeguard to prevent its value exceeding that of n, i.e.

$$\hat{n}_c = \Sigma \hat{n}_j \quad (\Sigma \hat{n}_j \leqslant n); \qquad \hat{n}_c = n \quad (\Sigma \hat{n}_j \geqslant n).$$

The number which are liberated, $n_L = n - n_c$ is estimated by $n - \hat{n}_c$. The ratios $(\hat{n}_j / \hat{n}_c)$ estimate the distribution of the core radii for the composite particles. If there are many repeated values of the core radii we can take this to be evidence that the core radii take their values from a discrete distribution. We can then avoid grouping into size classes and use the exact values r_j. If the data appear to come from a continuous distribution we may be able to apply some smoothing to the estimates. In both cases it may be possible to fit a parametric distribution. Corrections for grouping, appropriate choice of size classes and the statistical estimation of standard errors can all be made before presenting the results.

The above is based on the use of the full information contained in the intercept data.

Table 1. Distribution of the distance X to random chords and random plane sections of spheres of unit radius. In this table $v = \sqrt{(1 - x^2)}$. For each uniform randomness, we give alternative notations in brackets, and write $F(x) = P(X \leqslant x)$.

Line sections		Plane sections	
Randomness	F(x)	Randomness	F(x)
S (γ)	$1 - v$	I (IUR, μ)	x
I (IUR, μ), β	x^2	W (ν), β	$\frac{3}{2}(x - \frac{1}{3}x^3)$
W (ν)	$1 - v^3$	β_α	$\frac{15}{8}(x - \frac{2}{3}x^3 + \frac{1}{5}x^5)$
α	$1 - v^4$	α_β	$\frac{35}{16}(x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7)$
T (λ)	$1 - v^6$	T (λ)	$\frac{315}{128}(x - \frac{4}{3}x^3 + \frac{6}{5}x^5 - \frac{4}{7}x^7 + \frac{1}{9}x^9)$
⊥	x^3	⊥	x^3

Example 2

When we have information on just the intercept grades we need further probability arguments. This situation can arise when the particles which are concentric spheres are of various sizes. The core radius is no longer the relevant parameter, since it is not in correspondence with the volumetric grade γ. The intercept grades take values u_1/u_0 (= L_L for a line section, = A_A for a plane section).

We sort the n_0 composite intercept grades that are our observations into k size classes $D_1, D_2, \ldots, D_k$ centred on the values $r_1, r_2, \ldots, r_k$. This is for line sections. For plane sections the intercept grades are sorted into the size classes $D'_1, \ldots, D'_k$ centred on $r_1^2, \ldots, r_k^2$. If we take the square root of the intercept grades, these can be sorted into the same size classes as for the line sections.

Let "G_i" be the event "the intercept grade is in the class D_i or D'_i". Then we can obtain theoretical expressions for the probability of G_i when the core radius is in the size class D_j, i.e. for

$$q_{ij} = P(G_i | R_j).$$

Without any loss of generality let us scale each particle's size so that its outer radius is 1 and its inner radius is $r = \gamma^{1/3}$. If the size class D_i runs from $r = a$ to $r = b$, the event G_i occurs if the distance of the intercept from the particle centre lies in the range

$$\left(\frac{r^2 - b^2}{1 - b^2}\right)^{\frac{1}{2}} = x_b \text{, say, to } \left(\frac{r^2 - a^2}{1 - a^2}\right)^{\frac{1}{2}} = x_a \text{, say,}$$

This follows from triangle geometry. Let this distance from the centre be the random variable X. Examples of the distribution of X are given in Table 1 adapted from Coleman(1987). We note that the probabilities needed for the full information case of Example 1 are also given by this table:

$$p_j = P(C | R_j) = P(X \leqslant r_j) = F(r_j).$$

If the boundaries of the size class D_i are a and b, then

$$q_{ij} = P(G_i | R_j) = P(x_b \leqslant X \leqslant x_a) = F(x_a) - F(x_b)$$

where $r = r_j$ in the formulae for x_a and x_b.

We again approximate by assuming that if a core radius has its value in D_j then its value is exactly r_j, the centre of the class. With this approximation we have the probability relationship (known as the law of total probability)

$$P(G_i) = \sum_j P(G_i | R_j) P(R_j).$$

Suppose that we observe w_i of the n intercepts to have grades in the class D_i or D_i', then we can estimate $P(G_i)$ by w_i/n. The probabilities $P(G_i | R_j) = q_{ij}$ are known theoretically from the above, so we have a set of simultaneous equations for $P(R_j)$, which we can write as n_j/n. Our equations for estimating $\{n_j\}$ are therefore

$$w_i = \sum_j q_{ij} \hat{n}_j.$$

This can be written in matrix form

$$\underset{\sim}{w} = \underset{\approx}{Q} \, \hat{\underset{\sim}{n}},$$

which has the solution

$$\hat{\underset{\sim}{n}} = \underset{\approx}{Q}^{-1} \underset{\sim}{w} .$$

Example 3

The methods of Example 2 can be applied to a population of spherical particles of four types in a mixture of two groups: the first is liberated particles of phase β and those with a core of α surrounded by β as in Example 2; the second group is the reverse case, liberated particles of phase α and those with a core of β surrounded by α . The intercepts will clearly show to which group a particle belongs, so the results for each group can be analysed separately.

Example 4

The results can be extended to the case of particles of the following sorts: liberated spheres of phase α , cores of phase β_1 surrounded by α , cores of β_2 surrounded by α , of β_3 surrounded by α , and so on. From the composite intercepts of each type we estimate the distributions of core sizes, and the numbers for which the intercept misses the core. By subtraction from the observed total of intercepts that miss the cores, we estimate the number of particles that are liberated of phase α. It is easy to see how, by Examples 3 and 4, even more elaborate models involving concentric sphere particles can be analysed.

ACKNOWLEDGEMENTS

An early version of this paper was presented at the Colloquium on Mathematical Morphology, Stereology and Image Analysis in Prague in September 1982, when I was a guest of the Czechoslovak Academy of Sciences on a Royal Society - Czechoslovak Academy of Sciences Study Visit. A brief report was given as Coleman(1983) in Saxl(1983).

I was introduced to these problems by Dr R. Horton, formerly of the Department of Mineral Resources Engineering at Imperial College, and this work is based on a close collaboration with Dr S. W. Moore, currently of that Department.

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