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N_V - STEREOLOGY FOR PARALLEL SECTIONS

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ABSTRACT

A system of convex perticles distributed in space is characterised by the particle density N_V. The intersection of the particle system with a pair of objects (points, parallel lines or parallel planes) separated by a distance λ, is characterised by a pair density function, i.e., the density of pairs of particle intersections as a function of λ. Stereological formulae for N_V, related to the pair density functions are given. They form a basis for development of stereological methods for N_V - estimation by counting measurements made on parallel sections.

Keywords: convex particles, erosion, parallel sections, particle size, random closed set, random function.

INTRODUCTION

For about 25 years, the problem of the particle density N_V estimation by counting measurements made on the intersection of a particle system with the pair of: points (Wiencek, 1989), parallel lines (Duvalian, 1972; Ryś and Wiencek, 1983) and parallel planes (Bodziony et al., 1972; Wiencek and Hougardy, 1983 and 1988; Sterio, 1984) have been analysed often and some interesting results are in this field of stereology.

The aim of this work is to present a more general approach to this problem which is based on a stochastic model in which particle structure is considered as a random closed set. In this context, the stereology for simple sections (for point, line or plane sections) can be developed by the use of the ergodic random functions. The stereology for the parallel sections includes three parts, as follows: (i) the stereology for simple sections, (ii) the transformation of the structure by erosion with a pair of parallel objects into an eroded set, and (iii) applying the stereology in (i) to the eroded set, which leads to the solution in the stereology for parallel sections.

Space and geometrical objects

In Euclidean space $\mathbf{R}^3$, a Cartesian coordinate system is given, Fig. 1. Let $x = (x_1, x_2, x_3)$ be a point in $\mathbf{R}^3$, the point $o = (0,0,0)$ is the origin. The space $\mathbf{R}^3 = \{x = (x_1, x_2, x_3)\}$. $\mathbf{R}^r$ $(r = 0,\ldots, 3)$ is a subspace of $\mathbf{R}^3$. $\mathbf{R}^o = o$, $\mathbf{R}^1 = \{x = (x_1, 0, 0)\}$ and $\mathbf{R}^2 = \{x = (x_1, x_2, 0)\}$. Another subspace $\mathbf{L}^{3-r}$ is the orthogonal complement of $\mathbf{R}^r$ in $\mathbf{R}^3$, the special cases are, $\mathbf{L}^o = o$, $\mathbf{L}^1 = \{x = (0,0,x_3)\}$, $\mathbf{L}^2 = \{x = (0, x_2, x_3)\}$ and $\mathbf{L}^3 = \mathbf{R}^3$.

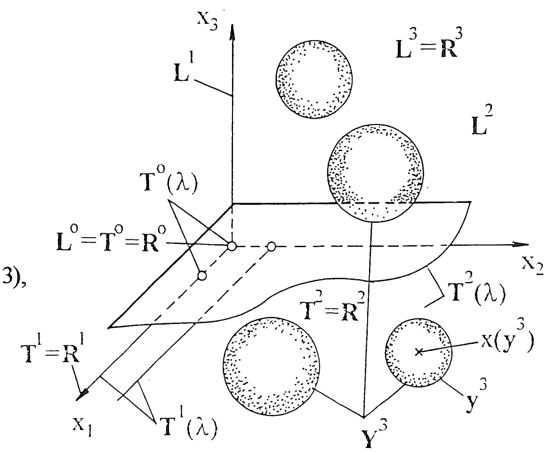

Fig. 1. The spaces $\mathbf{R}^r$ and $\mathbf{L}^{3-r}$ $(r = 0,\ldots,3)$, particle structure $\mathbf{Y}^3$ and geometrical objects $\mathbf{T}^r$ and $\mathbf{T}^r(\lambda)$ $(r = 0, 1, 2)$.

M^r $(r = 0,\ldots, 3)$ will be a measure of an r - dimensional domain in $\mathbf{R}^r$ or in $\mathbf{L}^{3-r}$ (here, the superscript r in M^r is an index and not a power, this convention will be used below). In the particular cases: $M^o = 1$, $M^1 = L$ is the length of a segment, $M^2 = A$ is the plane area, and $M^3 = V$ is the volume.

The following r - dimensional geometrical objects $\mathbf{T}^r$ $(r = 0, 1, 2)$ will be defined, the point $\mathbf{T}^o = \mathbf{R}^o$, the line $\mathbf{T}^1 = \mathbf{R}^1$ and the plane $\mathbf{T}^2 = \mathbf{R}^2$. Next, for a given λ $(\lambda > 0)$, the object $\mathbf{T}^r_\lambda$ is a parallel one to $\mathbf{T}^r$ at a distance $x_{r+1} = \lambda$. $\mathbf{T}^r(\lambda) = \mathbf{T}^r \cup \mathbf{T}^r_\lambda$ is a pair of parallel objects, i.e., a pair of points $\mathbf{T}^o(\lambda)$, a pair of lines $\mathbf{T}^1(\lambda)$ and a pair of planes $\mathbf{T}^2(\lambda)$, Fig. 1. The objects $\mathbf{T}^r$ and $\mathbf{T}^r(\lambda)$ will be used for generation of intersections with a structure in $\mathbf{R}^3$, while the space $\mathbf{L}^{3-r}$ will be used, for generation of particle projections or for defining any functions.

The particle structure model

Geometrically, a particle y^3 will be regarded as a convex body to which is assigned a so-called reference point $x(y^3)$ (e.g. the center of mass). The particle is characterised quantitatively by a set of numbers $Z(y^3) = Z$, for instance, $Z = V$ is the volume.

Let x_i $(i = 1,2,...)$ be a point in the space $\mathbf{R}^3$ and $\mathbf{y}_i^3$ a particle with its reference point x_i. The particle structure $\mathbf{Y}^3$ in $\mathbf{R}^3$ can be written as follows

$$\mathbf{Y}^3 = \bigcup_{i=1}^{\infty} \mathbf{y}_i^3 . \tag{1}$$

In the structure $\mathbf{Y}^3$, the particles do not overlap. The particle structure $\mathbf{Y}^3$ will be considered as a realization of an isotropic and ergodic random set (Lantuejoul, 1990; Stoyan et al., 1995) and will be called the random particle structure $\mathbf{Y}^3$ (denoted also by $\mathbf{Y}^3$), i.e., the structure $\mathbf{Y}^3$ is a realization of the random structure $\mathbf{Y}^3$. By definition, a random structure $\mathbf{Y}^3$ is homogeneous (stationary). Its particle reference points form a homogeneous point field in $\mathbf{R}^3$ with the point density N_V (i.e. the average number of points per unit volume, as a result of averaging in the set of possible realizations of the random $\mathbf{Y}^3$). It will be assumed that N_V is the particle density of $\mathbf{Y}^3$. The other parameter of the random structure $\mathbf{Y}^3$ will be a Z-density, Z_V (i.e., the average sum of the particles parameter Z per unit volume, as the result of averaging in the set of possible realizations of $\mathbf{Y}^3$).

Because of ergodicity, the parameters N_V and Z_V are also characteristics of every structure $\mathbf{Y}^3$. In this case, the parameters are considered as the results of averaging in the whole space $\mathbf{R}^3$. Therefore, from the structural parameters point of view it is not necessary to distinguish between the random structure and its realization. However, in situations where the differentiation is of significance it will be obvious from the context of the problem being analysed.

In the following sections, two stereologies of the $\mathbf{Y}^3$ structure will be considered. The first, for simple sections generated by $\mathbf{T}^r$ and the second, for parallel sections generated by $\mathbf{T}^r(\lambda)$. Because the random structure $\mathbf{Y}^3$ is homogeneous and isotropic, the fixed objects $\mathbf{T}^r$ or $\mathbf{T}^r(\lambda)$ can be used for generation of the sections.

SIMPLE SECTIONS

The set $\mathbf{y}^r = \mathbf{y}^3 \cap \mathbf{T}^r$ $(r = 0, 1, 2)$, a non - empty intersection of the particle $\mathbf{y}^3$ with the object $\mathbf{T}^r$, is an r - dimensional particle. The structure $\mathbf{Y}^r = \mathbf{Y}^3 \cap \mathbf{T}^r$, an intersection of $\mathbf{Y}^3$ with $\mathbf{T}^r$ is a set of non - overlapping particles $\mathbf{y}_j^r$ $(j = 1,2, ...)$ which are distributed in space $\mathbf{R}^r = \mathbf{T}^r$. The special cases are as follows, $\mathbf{Y}^o$ is a point $\mathbf{y}^o$ in $\mathbf{R}^o$, $\mathbf{Y}^1$ is a system of chords $\mathbf{y}^1$ in $\mathbf{R}^1$ and $\mathbf{Y}^2$ is a system of convex profiles $\mathbf{y}^2$ in $\mathbf{R}^2$. Formally, the structure $\mathbf{Y}^r$ can by written analogously to $\mathbf{Y}^3$ by a formula of the type (1).

For a given structure $\mathbf{Y}^r$, the particle reference points form a point field in the space $\mathbf{R}^r$, with a point density N_{M^r}, which is assumed to be the particle density of $\mathbf{Y}^r$ in $\mathbf{R}^r$. The special cases are as follows, for r=0, N_N is the point density of $\mathbf{Y}^o$ (here, the subscript N in N_N denotes M^o); then, for r=1, N_L is the chord density of $\mathbf{Y}^1$ and for r=2, N_A is the profile density of $\mathbf{Y}^2$. N_{M^r}, will be the only parameter of the structure $\mathbf{Y}^r$.

For a particle $\mathbf{y}^3$ of the structure $\mathbf{Y}^3$, the parameter Z will be chosen with regard to the intersection with the $\mathbf{T}^r$ object. Let the set $y^3 |\mathbf{L}^{3-r}$ (r = 0, 1, 2) be the projection of $\mathbf{y}^3$ onto $\mathbf{L}^{3-r}$ and M^{3-r} be the measure of this set. Because M^{3-r} is the measure of all non-empty intersections of the set $\mathbf{y}^3$ with the object $\mathbf{T}^r$, it will be assumed as a parameter of the particle $\mathbf{y}^3$ itself and be denoted by X^{3-r}, i.e., $Z = X^{3-r}$. Consequently, the X^{3-r} - density, X_V^{3-r} is a parameter of the structure $\mathbf{Y}^3$.

The special cases for the parameters of the structure $\mathbf{Y}^3$ are as follows, for r=0, V_V is the volume density; for r=1, A_V is the projected area density and for r=2, L_V is the projected length density. Table 1, presents the parameters X_V^{3-r} and N_{M^r} for the respective r's, r = 0, 1, 2.

Table 1. The structural parameters of $\mathbf{Y}^3$ and $\mathbf{Y}^r$ (r = 0, 1, 2)

r	X_V^{3-r}	N_{M^r}
0	V_V	N_N
1	A_V	N_L
2	L_V	N_A

The fundamental equation for simple sections stereology

For a given structure $\mathbf{Y}^3$ and object $\mathbf{T}^r$ (r = 0,1,2) in $\mathbf{R}^3$, the structures $\mathbf{Y}^3$ and $\mathbf{Y}^r = \mathbf{Y}^3 \cap \mathbf{T}^r$ are not independent and their parameters X_V^{3-r} and N_{M^r}, respectively, satisfy the following equation (Wiencek, 1996):

$$X_V^{3-r} = N_{M^r} \qquad \text{for } r = 0, 1, 2. \qquad (2)$$

Equation (2) is of fundamental meaning in simple section stereology. The special forms of equation (2) indicate that the parameters in a given row of Table 1, are equal to each other. A possible proof of equation (2) will be given below.

Let T_o^r $(r = 0, 1, 2)$ be a closed set in $\mathbf{R}^r$ which is connected with the origin o. It is the point $T_o^o = o$ or an r-dimensional unit segment:

$$T_o^r = \{x \in \mathbf{R}^r : 0 \leq x_i \leq 1,\ i = 1, ..., r\} \qquad \text{for } r = 1, 2\,.$$

In particular, $T_o^1 \subset \mathbf{R}^1$ is a segment and $T_o^2 \subset \mathbf{R}^2$ is a quadrat. In contrast to the geometrical objects $\mathbf{T}^r$ (e.g., the plane $\mathbf{T}^2$) the set T_o^r is denoted by a non-bold letter T. A number $M(T_o^r)$, which characterises the size of T_o^r, is equal to one, i.e., $M(T_o^r) = 1$.

The set $Q_r = T_o^r \times \mathbf{L}^{3-r}$ is a subspace of $\mathbf{R}^3$, which is determined as the Cartesian product of T_o^r and $\mathbf{L}^{3-r}$. $Q_o = \mathbf{R}^3$ while the other Q_r's can be written as follows:

$$Q_r = \{x \in \mathbf{R}^3 : 0 \leq x_i \leq 1,\ i = 1, ..., r\} \qquad \text{for } r = 1, 2.$$

Fig. 2. shows the Q_r $(r = 0, 1, 2)$ and the respective T_o^r and $\mathbf{L}^{3-r}$, as well.

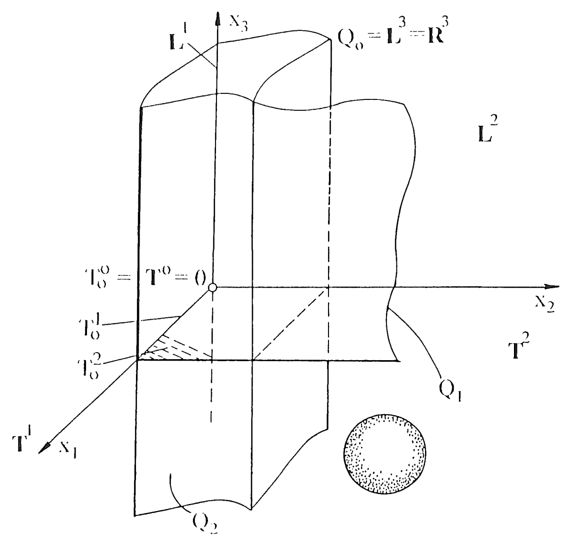

Fig. 2. The sets T_o^r and subspaces Q_r $(r = 0, 1, 2)$.

For a given structure $\mathbf{Y}^3$ and a subspace Q_r, the set

$$\mathbf{Y}^3(Q_r) = \{\mathbf{y}_i^3 \in \mathbf{Y}^3 : x_i \in Q_r\}$$

where x_i is the reference point of the particles $\mathbf{y}_i^3$. $\mathbf{Y}^3(Q_r)$ is a particle structure which belongs to Q_r. Because of the $\mathbf{Y}^3$ - homogeneity, the stereological parameters of the structures $\mathbf{Y}^3$ and $\mathbf{Y}^3(Q_r)$, i.e., N_V and X_V^{3-r} are the same.

Let $\mathbf{y}_j^r$ (j = 1,2,...) be a particle of the structure $\mathbf{Y}^r \subset \mathbf{R}^r$ and x_j be the reference point of such a particle $\mathbf{y}_j^3 \in \mathbf{Y}^3$ for which $\mathbf{y}_j^r = \mathbf{y}_j^3 \cap \mathbf{T}^r$. For a given $\mathbf{Y}^r$ and T_o^r, the set

$$\mathbf{Y}^r(T_o^r) = \{\mathbf{y}_j^r \in \mathbf{Y}^r : x_j \in Q_r\} \qquad \text{for } r = 0, 1, 2$$

is a particle structure which belongs to T_o^r. The structures $\mathbf{Y}^r$ and $\mathbf{Y}^r(T_o^r)$ are characterised by the parameter N_{M^r}. It is important to note that for a homogeneous random structure $\mathbf{Y}^r$ the parameter N_{M^r} is independent of the convention used for defining the particle reference point. However, by all the unbiased countings for an averaging procedure used to determine N_{M^r}, the reference points are to be chosen in an unique manner.

The random function

Let $\mathbf{T}_x^r$ and T_x^r $(r = 0, 1, 2; x \in \mathbf{L}^{3-r})$ be the object $\mathbf{T}^r$ and the set T_o^r, after a translation by x in both cases. $\mathbf{Y}^3 \cap \mathbf{T}_x^r$ is an r-dimensional structure in the space $\mathbf{R}_x^r = T_x^r$. For a given structure $\mathbf{Y}^3 \cap \mathbf{T}_x^r$ and the set T_x^r, the set $\mathbf{Y}^r(T_x^r)$ is a particle structure which belongs to the T_x^r.

Let k (k = 0, 1, ...) denote the number of particles which belongs to $\mathbf{Y}^r(T_x^r)$. For x as a variable k(x) is a function, which takes integer values and is continuous almost everywhere in $\mathbf{L}^{3-r}$. Let Ω be a domain in $\mathbf{L}^{3-r}$ and $M(\Omega) = M$ be its measure. The mean $\langle k \rangle$ of the k(x) function, which is given by

$$\langle k \rangle = \lim_{M \to \infty} M^{-1} \int_{\Omega} k(x)\, dx$$

is the result of averaging in the space $\mathbf{L}^{3-r}$.

It seems obvious that the mean $\langle k \rangle$ is equal to the parameter X_V^{3-r} (here, considered as the result of averaging in the space Q_r),

$$\langle k \rangle = X_V^{3-r}. \tag{3}$$

For a random structure $\mathbf{Y}^3$, the function k(x) can be considered as a realization of an ergodic random function (stochastic process) $\{k_x, x \in \mathbf{L}^{3-r}\}$. For x = 0 (at the origin *o*), k_o is a discrete random variable with its mean (the result of averaging in the set of possible realizations), being aqual to $\langle k \rangle$, because of ergodicity. By definition, k is equal to the number of particles which belongs to $\mathbf{Y}^r(T_o^r)$. Consequently, the mean of k_o is equal to the particle density N_{M^r},

$$\langle k \rangle = N_{M^r}. \tag{4}$$

The substitution of (3) into (4) results in (2).
It is important to notice that in the simple section stereology presented which is characterised by equation (2), the particle density N_V does not occur explicitely.

PARALLEL SECTIONS

An intersection of the particle structure $\mathbf{Y}^3$ with a pair of parallel objects $\mathbf{T}^r(\lambda)$ $(r = 0, 1, 2)$ separated by a distance λ leads to a stereology in which it is possible to give formulae for the particle density N_V. The transformation of the structure $\mathbf{Y}^3$ by erosion with the parallel objects $\mathbf{T}^r(\lambda)$ generates an eroded set. It will be shown that the $\mathbf{Y}^3$ structure stereology for parallel sections is equivalent to the eroded set stereology for simple sections. When applying the stereology for simple sections given above to the eroded sets, it is possible to derive equations of the stereology for parallel sections.

The particle size

For a given particle $\mathbf{y}^3$ and object $\mathbf{T}^r(\lambda)$, the event $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$ ocuurs only when the events $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$ and $\mathbf{y}^3 \cap \mathrm{T}^r_\lambda \neq \emptyset$ also occur. The event $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$ can be chosen as the basis for a particle size definition with regard to the object $\mathbf{T}^r(\lambda)$. It seems obvious that the event $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$ can occur only when λ is less than a given value D_r. This D_r value will be used as the size of particle $\mathbf{y}^3$ with regard to the object $\mathbf{T}^r(\lambda)$. A possible analytical expression for the particle size is given by

$$D_r \overset{\mathrm{def}}{=} \sup[\lambda : \mathbf{T}^r(\lambda) \cap \mathbf{y}^3_x \neq \emptyset,\ x \in \mathbf{L}^{3-r}],$$

where: $\mathbf{y}^3_x = \mathbf{y}^3 + x = \{z + x : z \in \mathbf{y}^3;\ x \in \mathbf{L}^{3-r}\}$ is the $\mathbf{y}^3$ translated by x, when $x(\mathbf{y}^3) = o$. The particle size D_r is the largest distance between two parallel objects $\mathbf{T}^r$, $\mathrm{T}^r_{\lambda=D} \in \mathbf{T}^r(\lambda = D)$ which are tangential to the particle. It is easy to show that the particle size defined above is equal to the length of the largest chord of projection $\mathbf{y}^3 \,|\mathbf{L}^{3-r}$ in the x_{r+1} co-ordinate direction. The special cases are as follows, D_o is the largest chord of $\mathbf{y}^3$ in the x_1 direction; D_1 is the largest chord of projection $\mathbf{y}^3 \,|\mathbf{L}^{3-r}$ in the x_2 direction and D_2 is the projected length of $\mathbf{y}^3$ onto $\mathbf{L}^1$. (In the following, the index r in D_r will be neglected.)

It is important to notice that for a single particle the above size definition is rather ambiguous. However, when a particle is a member of the collection $\mathbf{Y}^3$ such a size has a well-defined value.

It will be assumed that a given particle y_i^3 of the structure $\mathbf{Y}^3$ is characterised by the size D_i. Consequently, the structure $\mathbf{Y}^3$ is characterised by the particle size distribution $N_V(D)$ which gives the density of particles with a size less than D. The main properties of the $N_V(D)$ are as follows: $N_V(0) = 0$ and $N_V(D) = N_V$ for $D>D_m$ (D_m is the largest size). An integral form of the parameter X_V^{3-r} can be expressed by $N_V(D)$, as follows:

$$X_V^{3-r} = \int_0^{D_m} \langle X^{3-r}(D)\rangle \, dN_V(D), \tag{5}$$

whenever this integral exists. $\langle X_V^{3-r}(D)\rangle$ denotes the mean of X^{3-r} for the particles of size D.

The pair density function

For a given structure $\mathbf{Y}^3$ and object $\mathbf{T}^r(\lambda)$, the intersection $\mathbf{Y}^3 \cap \mathbf{T}^r(\lambda)$ is composed of two r-dimensional structures $\mathbf{Y}^r = \mathbf{Y}^3 \cap \mathbf{T}^r$ and $\mathbf{Y}_\lambda^r = \mathbf{Y}^3 \cap \mathbf{T}_\lambda^r$,

$$\mathbf{Y}^3 \cap \mathbf{T}^r(\lambda) = \mathbf{Y}^r \cup \mathbf{Y}_\lambda^r .$$

For $\lambda < D_m$ the structures $\mathbf{Y}^r$ and $\mathbf{Y}_\lambda^r$ are conjugated by the events $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$.

In the structure $\mathbf{Y}^r$, the particles $\mathbf{y}^r = \mathbf{y}^3 \cap \mathbf{T}^r$ for which $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$ will be distinguished and denoted by $\mathbf{y}^r(\lambda)$. There is $\mathbf{y}^r(\lambda) = \mathbf{y}^3 \cap \mathbf{T}^r$ when $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$ occurs.

The set $\mathbf{Y}^r(\lambda) \subset \mathbf{Y}^r$ is a set of the $\mathbf{y}^r(\lambda)$ particles and $N_{M^r}(\lambda)$ is its particle density. The special cases are as follows, $N_N(\lambda)$ is the $\mathbf{y}^o(\lambda)$ point density of $\mathbf{Y}^o(\lambda)$; $N_L(\lambda)$ is the $\mathbf{y}^1(\lambda)$ chord density of $\mathbf{Y}^1(\lambda)$ and $N_A(\lambda)$ is the $\mathbf{y}^2(\lambda)$ profile density of $\mathbf{Y}^2(\lambda)$. It will be assumed that $N_{M^r}(\lambda)$ is also the density of the pair of particle intersections of $\mathbf{Y}^3 \cap \mathbf{T}^r(\lambda)$ in relation to the space $\mathbf{R}^r \supset \mathbf{T}^r$. The special cases are as follows, $N_N(\lambda)$ is the density of pair of points of $\mathbf{Y}^3 \cap \mathbf{T}^o(\lambda)$; $N_L(\lambda)$ is the density of pair of chords of $\mathbf{Y}^3 \cap \mathbf{T}^1(\lambda)$ and $N_A(\lambda)$ is the density of pairs of profiles of $\mathbf{Y}^3 \cap \mathbf{T}^2(\lambda)$. The set $\mathbf{Y}^r(\lambda)$ represents the set $\mathbf{Y}^3 \cap \mathbf{T}^r(\lambda)$ with respect to the parameter $N_{M^r}(\lambda)$. For λ as a variable, $\mathbf{Y}^3 \cap \mathbf{T}^r(\lambda)$ and $\mathbf{Y}^r(\lambda)$ are the sets which depend on λ, their characteristic is the so-called pair density function $N_{M^r}(\lambda)$. It is, for $\lambda \to 0$: $\mathbf{Y}^3 \cap \mathbf{T}^r(\lambda) \to \mathbf{Y}^r$ and $N_{M^r}(\lambda) \to N_{M^r}$.

Erosion and the eroded set

The set $\mathbf{Y}^r(\lambda)$ mentioned above is a transformed one of $\mathbf{Y}^r = \mathbf{Y}^3 \cap \mathbf{T}^r$ by $\mathbf{T}^r(\lambda)$. Another set transformation will be related directly to the particle set $\mathbf{Y}^3$.

Given a particle $\mathbf{y}^3$ with reference point $x(\mathbf{y}^3) = o$ and an object $\mathbf{T}^r(\lambda)$ for $\lambda < D$. A transformation $\mathbf{y}^3_r(\lambda)$ of the particle $\mathbf{y}^3$ by $\mathbf{T}^r(\lambda)$ is given by

$$\mathbf{y}^3_r(\lambda) = \bigcup_{x \in \mathbf{L}^{3-r}} \{ \mathbf{y}^3 \cap \mathbf{T}^r_x : \mathbf{y}^3 \cap \mathbf{T}^r_x(\lambda) \neq \emptyset \}, \qquad (6)$$

where $\mathbf{T}^r_x(\lambda)$ is the object $\mathbf{T}^r(\lambda)$ translated by x and $x \in \mathbf{L}^{3-r}$. It is obvious that $\mathbf{y}^3_r(\lambda)$ is a convex subset of $\mathbf{y}^3$, i.e., $\mathbf{y}^3_r(\lambda) \subset \mathbf{y}^3$.
It should by noted that the transformation defined by (6) is eqivalent to the morphological erosion of $\mathbf{y}^3$ by an $r+1$ - dimensional convex object $\mathbf{t}^{r+1}(\lambda)$, which is bounded by $\mathbf{T}^r(\lambda)$. It can be described as follows

$$\mathbf{y}^3_r(\lambda) = \mathbf{y}^3 \ominus \mathbf{t}^{r+1}(\lambda),$$

where $\ominus$ is the usual symbol for the erosion (Serra, 1982). Because of equivalence of the transformations above, the one given by (6) will be considered as an erosion of $\mathbf{y}^3$ by $\mathbf{T}^r(\lambda)$ and described by using $\ominus$, as follows

$$\mathbf{y}^3_r(\lambda) = \mathbf{y}^3 \ominus \mathbf{T}^r(\lambda),$$

where $\mathbf{y}^3_r(\lambda)$ is the eroded particle. As an example, Fig. 3 shows a circle eroded by a pair of points $\mathbf{T}^o(\lambda)$ and a pair of lines $\mathbf{T}^1(\lambda)$.

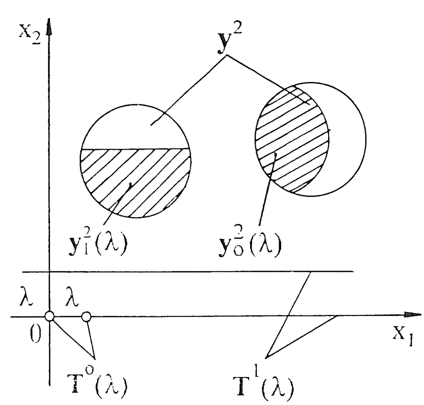

Fig. 3. The eroded particles

As a convex set, the eroded particle $\mathbf{y}_r^3(\lambda)$ can be treated as a particle with the following property: the occurrence of the event $\mathbf{y}_r^3(\lambda) \cap \mathbf{T}^r \neq \emptyset$ is equivalent to the occurrence of the event $\mathbf{y}^3 \cap \mathbf{T}^r(\lambda) \neq \emptyset$. It follows

$$\mathbf{y}^r(\lambda) = \mathbf{y}_r^3(\lambda) \cap \mathbf{T}^r. \tag{7}$$

For a given structure $\mathbf{Y}^3$ and $\mathbf{T}^r(\lambda)$, the eroded set

$$\mathbf{Y}_r^3(\lambda) = \mathbf{Y}^3 \ominus \mathbf{T}^r(\lambda)$$

is the set $\mathbf{Y}^3$ eroded by $\mathbf{T}^r(\lambda)$, and is a set of the eroded particles $\mathbf{y}_{r,i}^3(\lambda)$ $(i = 1, 2, \ldots)$. It can be described by a formula of type (1) in which the particle reference points are the same as for $\mathbf{Y}^3$. The eroded set $\mathbf{Y}_r^3(\lambda)$ is homogeneous but not isotropic. From (7) it follows that for a given set $\mathbf{Y}_r^3(\lambda)$ and the object $\mathbf{T}^r$, the following fundamental relation is satisfied

$$\mathbf{Y}^r(\lambda) = \mathbf{Y}_r^3(\lambda) \cap \mathbf{T}^r. \tag{8}$$

This means that with respect to the set $\mathbf{Y}^r(\lambda)$ the intersections of $\mathbf{Y}^3$ with $\mathbf{T}^r(\lambda)$ are equivalent to the intersection of $\mathbf{Y}_r^3(\lambda)$ with $\mathbf{T}^r$.
For a given λ, the eroded set $\mathbf{Y}_r^3(\lambda)$ is a set of convex particles $\mathbf{y}_r^3(\lambda)$ which are arranged in the space $\mathbf{R}^3$. Therefore, their quantitative description can be made analogously to that of the particle structure $\mathbf{Y}^3$, i.e., with regard to the intersections with the object $\mathbf{T}^r$.

A particle $\mathbf{y}_r^3(\lambda)$ will be characterised by the parameter $X^{3-r}(\lambda)$ which is equal to the M^{3-r}-measure of its orthogonal projections onto $\mathbf{L}^{3-r}$. Consequently, a stereological parameter of the eroded set $\mathbf{Y}_r^3(\lambda)$ will be the $X^{3-r}(\lambda)$ - density, $X_V^{3-r}(\lambda)$. The special cases are as follows: for $r=0$, $V_V(\lambda)$ is the volume density of $\mathbf{Y}_o^3(\lambda)$; for $r=1$, $A_V(\lambda)$ is the projected area density of $\mathbf{Y}_1^3(\lambda)$ and for $r=2$, $L_V(\lambda)$ is the projected length density of $\mathbf{Y}_2^3(\lambda)$. Then, for the variable λ, the eroded set $\mathbf{Y}_r^3(\lambda)$ which depends on λ is characterised by the so-called erosion function $X_V^{3-r}(\lambda)$.

It follows from (8), that the intersection $\mathbf{Y}^r(\lambda)$ of the eroded set $\mathbf{Y}_r^3(\lambda)$ with the object $\mathbf{T}^r$ which depends on λ is characterised by the pair density function $N_{M^r}(\lambda)$ only. Table 2. presents the erosion functions $X_V^{3-r}(\lambda)$ and the pair density functions $N_{M^r}(\lambda)$ for the respective r's, $r = 0, 1, 2$.

Table 2. The erosion functions $X_V^{3-r}(\lambda)$ and the pair density functions $N_{M^r}(\lambda)$ (r = 0, 1, 2)

r	$X_V^{3-r}(\lambda)$	$N_{M^r}(\lambda)$
0	$V_V(\lambda)$	$N_N(\lambda)$
1	$A_V(\lambda)$	$N_L(\lambda)$
2	$L_V(\lambda)$	$N_A(\lambda)$

A fundamental equation for parallel sections stereology

For a given λ, the eroded set $\mathbf{Y}_r^3(\lambda)$ and its intersection $\mathbf{Y}^r(\lambda)$ with the object $\mathbf{T}^r$ are characterised by the parameters $X_V^{3-r}(\lambda)$ and $N_{M^r}(\lambda)$, respectively. Because the particles of the eroded set $\mathbf{Y}_r^3(\lambda)$ are convex, the parameters $X_V^{3-r}(\lambda)$ and $N_{M^r}(\lambda)$ satisfy formula (2) in the following form (Wiencek, 1996):

$$X_V^{3-r}(\lambda) = N_{M^r}(\lambda) \qquad \text{for } r=0,1,2. \qquad (9)$$

For λ as a variable, equation (9) is an equation with the erosion function $X_V^{3-r}(\lambda)$ and the pair density function $N_{M^r}(\lambda)$, which is the fundamental stereological relation for systems of convex particles with respect to their intersections with an object $\mathbf{T}^r(\lambda)$ which depends on λ. The special forms of equation (9) indicate that the functions in a given row of Table 2. are equal to each other. Formula (2) is a particular case of (9), namely, for $\lambda = 0$ equation (9) transforms into (2).

The erosion function $X_V^{3-r}(\lambda)$

The erosion function $X_V^{3-r}(\lambda)$, is a characteristic of the eroded set $\mathbf{Y}_r^3(\lambda)$ which depends on λ. The basis properties of the $X_V^{3-r}(\lambda)$ are, as follows: it is a continuous, non-negative, monotonic and non-increasing function of λ in the interval $[0, D_m]$. Further properties of the $X_V^{3-r}(\lambda)$ function are closely connected to the size distribution $N_V(D)$.

For a given λ, D(λ) will denote the size of an eroded particle $\mathbf{y}_r^3(\lambda)$ with respect to the intersections with an object $\mathbf{T}^r(\lambda)$. It will be defined analogously to the size D of a particle $\mathbf{y}^3$ and is given by the largest distance between the components of the object $\mathbf{T}^r(\lambda)$, which are tangential to $\mathbf{y}_r^3(\lambda)$. As a result, the size D(λ) is connected with D as follows

$$D(\lambda) = D - \lambda \qquad \text{for } D>\lambda. \qquad (10)$$

For a given λ, a particle $\mathbf{y}^3_{r,i}(\lambda)$ (i = 1,2, ...) of the eroded set $\mathbf{Y}^3_r(\lambda)$ is characterised by the size $D_i(\lambda)$. The size distribution $N_V[D(\lambda)]$ gives the density of particles with a size less than $D(\lambda)$. The size distribution of $\mathbf{Y}^3_r(\lambda)$ is closely connected to the one of $\mathbf{Y}^3$. Because the erosion by $\mathbf{T}^r(\lambda)$ removes all the particles of $\mathbf{Y}^3$ whose size D is less than λ, the functions $N_V(D)$ and $N_V[D(\lambda)]$ are related:

$$N_V[D(\lambda)] = N_V(D) - N_V(D=\lambda) \qquad \text{for } D>\lambda. \tag{11}$$

Let, for a given λ, $\langle X^{3-r}[D(\lambda)]\rangle$ denote the mean of the $X^{3-r}(\lambda)$ - parameter for the eroded particles $\mathbf{y}^3_{r,i}(\lambda)$ which have the same size $D(\lambda)$. An integral form of the erosion function $X^{3-r}_V(\lambda)$ can be given by formula (5) when applied to the eroded set $\mathbf{Y}^3_r(\lambda)$ and by taking into consideration the size distribution $N_V[D(\lambda)]$, it results in:

$$X^{3-r}_V(\lambda) = \int_0^{D_m(\lambda)} \langle X^{3-r}[D(\lambda)]\rangle \, dN_V[D(\lambda)]. \tag{12}$$

After substitution in (12) for $D(\lambda) = D - \lambda$ and taking into consideration (11), one gets the equation

$$X^{3-r}_V(\lambda) = \int_\lambda^{D_m} \langle X^{3-r}(D,\lambda)\rangle \, dN_V(D), \tag{13}$$

where $\langle X^{3-r}(D,\lambda)\rangle$ denotes here the $\langle X^{3-r}[D(\lambda)]\rangle$ as a function for the particles of the structure $\mathbf{Y}^3$. Equation (13) expresses the erosion function $X^{3-r}_V(\lambda)$ by the characteristics of the structure $\mathbf{Y}^3$, the size distribution $N_V(D)$ and $\langle X^{3-r}(D,\lambda)\rangle$ which is determined by the shape of the particles. Consequently the erosion function given by equation (13) can be interpreted as a characteristic of the particle structure $\mathbf{Y}^3$ with respect to the intersections with the object $\mathbf{T}^r(\lambda)$ which depends on λ.

A stereological equation

The substitution of (13) into (9) results in

$$N_{M^r}(\lambda) = \int_\lambda^{D_m} \langle X^{3-r}(D,\lambda)\rangle \, dN_V(D). \tag{14}$$

Equation (14) connects the size distribution $N_V(D)$ of a structure $\mathbf{Y}^3$ with the pair density function $N_{M^r}(\lambda)$ of the set $\mathbf{Y}^3 \cap \mathbf{T}^r(\lambda)$ which depends on λ. This is a fundamental stereological equation in the stereology of a particle structure $\mathbf{Y}^3$ intersected by parallel sections. The practical use of (14) is possible if the $\langle X_V^{3-r}(D,\lambda)\rangle$ function is known explicitely.

THE PARTICLE DENSITY N_V

In the stereology for parallel sections, the formulae for the particle density N_V result from equation (14). For the case of parallel plane sections, a more general analysis for convex particle systems is possible. In other cases, an approach based on a system of spheres will be given. A more detailed analysis will be given for the particular $\mathbf{T}^r(\lambda)$ objects in an opposite sequence of the r superscript, i.e., for r = 2, 1, 0.

The pair of planes $\mathbf{T}^2(\lambda)$

The size D of a convex particle $\mathbf{y}^3$ with regard to the pair of planes $\mathbf{T}^2(\lambda)$ is the projected length onto $\mathbf{L}^1$. The particle structure $\mathbf{Y}^3$ is a system of convex particles which is characterised by the particle density N_V and the size distribution $N_V(D)$.

For a given D and λ ($\lambda<D$), the $\mathbf{y}^3$ particle parameter $X^{3-r}(\lambda)$ for $r=2$ with respect to the intersections with the $\mathbf{T}^2(\lambda)$ is equal to the projected length $L(\lambda) = D(\lambda)$ of the eroded particle $\mathbf{y}_2^3(\lambda)$.

The set $\mathbf{Y}^3 \cap \mathbf{T}^2(\lambda)$ which depends on λ is characterised by the pair density function $N_A(\lambda)$. The function $\langle X^{3-r}(D,\lambda)\rangle$ for $r=2$ is equal to $D(\lambda)$. The substitution of $D(\lambda)$ given by (10) into (14) for $r=2$ results in

$$N_A(\lambda) = \int_{\lambda}^{D_m} (D-\lambda)\, dN_V(D). \tag{15}$$

Equation (15) is a stereological equation for a convex particle structure $\mathbf{Y}^3$ investigated with parallel plane sections. It is a more general version of equation (16) in (Wiencek and Hougardy, 1988). Differentiating (15) with respect to λ gives

$$N_A^{(1)}(\lambda) = N_V(\lambda) - N_V$$

and for $\lambda=0$

$$N_V = -N_A^{(1)}(\lambda=0). \tag{16}$$

Formula (16) for the particle density N_V is a fundamental relation in the stereology of a convex particle structure $\mathbf{Y}^3$ intersected with the parallel planes.

The pair of lines $\mathbf{T}^1(\lambda)$

A spherical particle $\mathbf{y}^3$ is characterised by a diameter D. The structure $\mathbf{Y}^3$ is a system of spheres with particle density N_V and a diameter distribution $N_V(D)$.

For a given D and λ ($\lambda < D$), the $\mathbf{y}^3$ sphere parameter $X^{3-r}(\lambda)$ for r = 1 with respect to the intersections with the pair of lines $\mathbf{T}^1(\lambda)$ is equal to the projected area $A(\lambda)$ of the eroded sphere $\mathbf{y}_1^3(\lambda)$, which is given by the formula

$$A(\lambda) = 0{,}5\,D^2\left[\arccos\left(\frac{\lambda}{D}\right) - \frac{\lambda}{D}\sqrt{1-\left(\frac{\lambda}{D}\right)^2}\,\right]. \tag{17}$$

The set $\mathbf{Y}^3 \cap \mathbf{T}^1(\lambda)$ which depends on λ is characterised by the pair density function $N_L(\lambda)$. The function $\langle X^{3-r}(D,\lambda)\rangle$ for r = 1 is equal to $A(\lambda)$. Substituting (17) into (14) for r = 1 gives

$$N_L(\lambda) = 0{,}5 \int_{\lambda}^{D_m} D^2\left[\arccos\left(\frac{\lambda}{D}\right) - \frac{\lambda}{D}\sqrt{1-\left(\frac{\lambda}{D}\right)^2}\,\right] dN_V(D). \tag{18}$$

Equation (18) is a stereological equation for a system of spheres $\mathbf{Y}^3$ intersected by parallel lines.

The Mellin transformation of $N_L(\lambda)$

For a given complex number s the Mellin transformation $\mathbf{M}(s)$ of $N_L(\lambda)$ is, as follows:

$$\mathbf{M}(s) = \int_{\lambda}^{\infty} \lambda^{s-1} N_L(\lambda)\, d\lambda, \tag{19}$$

if this integral converges. Substituting for $N_L(\lambda)$ from (18) and performing the integration yields for s = −2, 1, 2, ... :

$$\mathbf{M}(s) = \frac{I_{s+2}}{s(s+1)} N_V\langle D^{s+2}\rangle, \tag{20}$$

where $\langle D^k \rangle$ is the k-th moment of D and

$$I_k = \frac{\sqrt{\pi}}{2} \frac{\Gamma[(k+1)/2]}{\Gamma[(k/2)+1]} \qquad \text{for } k = 0, 1, 2, \ldots, \text{(Bach, 1967; Wiencek, 1996).}$$

For $s = -2$, $\langle D^0 \rangle = 1$, $I_0 = \frac{\pi}{2}$ and equation (20) gives the following formula for N_V:

$$N_V = \frac{4}{\pi} \mathbf{M}(-2). \tag{21}$$

Formula (21) for the particle density N_V is a fundamental relation in the stereology of a system of spheres $\mathbf{Y}^3$ intersected by a pair of parallel lines. It is an exact version of Duvalian's formula (Duvalian, 1972).

The pair of points $\mathbf{T}^o(\lambda)$

The $\mathbf{Y}^3$ structure is again a set of spheres with a particle density N_V and a diameter distribution $N_V(D)$.

For a given D and λ ($\lambda < D$), the $\mathbf{y}^3$ sphere parameter $X^{3-r}(\lambda)$ for $r = 0$ with respect to the intersections with the pair of points $\mathbf{T}^o(\lambda)$ is equal to the volume $V(\lambda)$ of the eroded sphere $\mathbf{y}^3_o(\lambda)$, which is given by the formula

$$V(\lambda) = \frac{\pi}{12}(D-\lambda)^2 (2D+\lambda). \tag{22}$$

The set $\mathbf{Y}^3 \cap \mathbf{T}^o(\lambda)$, which depends on λ, is characterised by the pair density function $N_N(\lambda)$. Here, the function $\langle X^{3-r}(D,\lambda) \rangle$ for $r = 0$ is equal to $V(\lambda)$. Substituting (22) into (14) for $r = 0$ gives

$$N_N(\lambda) = \frac{\pi}{12} \int_{\lambda}^{D_m} (D-\lambda)^2 (2D+\lambda) \, dN_V(D). \tag{23}$$

Equation (23) is a stereological equation for a system of spheres $\mathbf{Y}^3$ intersected by a pair of points. Differentiating (23) with respect to λ gives

$$N_N^{(2)}(\lambda) = \frac{\pi}{2} \lambda \, [N_V - N_V(D=\lambda)]. \tag{24}$$

For $\lambda = 0$, $N_N^{(2)}(\lambda = 0) = 0$, and the following limit exists

$$\lim_{\lambda \to 0} \lambda^{-1} N_N^{(2)}(\lambda) = N_N^{(3)}(\lambda = 0). \tag{25}$$

Taking into consideration both (25) and (24) gives the following formula for N_V (Wiencek, 1989):

$$N_V = \frac{2}{\pi} N_N^{(3)}(\lambda = 0). \tag{26}$$

Formula (26) for the particle density N_V is a fundamental relation in the stereology of a system of spheres $\mathbf{Y}^3$ intersected by a pair of points.

DISCUSSION AND CONCLUSIONS

The stereology of a particle structure for parallel sections depending on the distance λ by which the sections are separated, has been given. The general result has a form of the stereological equation (14) which is satisfied by the functions: particle size distribution $N_V(D)$ and the pair density function $N_{M^r}(\lambda)$. The particular cases of (14) are the equations: (15), (18) and (23). On the basis of these equations it was possible to derive formulae for the particle density N_V which are related to the pair density function $N_{M^r}(\lambda)$. The particular forms are presented by formulae (16), (21) and (26). Formulae (16) in the stereology of a convex particles system intersected with parallel planes is the most general result. Whereas, formulae (21) and (26) only characterise the stereology of sphere systems for pairs of parallel lines and pairs of points, respectively. These formulae could be used as a basis for the development of stereological methods for N_V - estimation by counting measurements made on parallel sections. The empirical pair density functions, if given, are in a discrete form for λ_i $(i = 1, 2, ..., n)$. Therefore, in order to make possible the use of these formulae in practice their appropriate numerical forms are needed. In the case of formulae (16) and (26) such a form can be developed by numerical differentiation. However, in the case of formula (21) a numerical approximation of the $N_L(\lambda)$ -function should be made (Duvalian, 1972). It should be noted that the approximation of the derivative in (16) by a simple difference quotient (e.g., the forward-difference formula) results in a form which is equivalent to the disector method for a convex particle system (Sterio, 1983; Stoyan et al., 1995).

Finally, it should be pointed out that the equations (15), (18) and (23) may be interpreted as integral equations for a known pair density function and an unknown size distribution $N_V(D)$. Some analytical as well as numerical solutions of such equations exist (Wiencek, 1996).

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