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GENERAL UNIVARIATE SHAPE DESCRIPTORS

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ABSTRACT

In this paper a general univariate morphological image representation scheme is proposed as a theoretical basis for analyzing images. Here emphasis is given on the generation of a set of non-overlapping segments of the image via repeated erosions and set transformations.

Notations: $\ominus$: Erosion; $\oplus$: Dilation; $\bigcirc$: Opening ; © : closing ;

Key words: Mathematical morphology, Pecstrum, Gecstrum.

INTRODUCTION

In many image analysis applications there is a need to develop an image representation scheme which contains serious important aspects of the image in a compressed form, In particular, by developing the necessary mathematical tools, we are usually able to transform an image into a set of simpler images which contain sufficient information about the shape, size, orientation and geometry of the image under consideration. The representation scheme can then be effectively used for the design of automated image analysis and computer vision schemes.

A good number of sources (Serra, 1982; Bronskill and Venetsanopoulas, 1988; Ghosh and Chanda, 1993; Ghosh and Chanda, 1995; Ghosh, 1996; Kresh and Malah, 1994; Pitas and Venetsanopoulas, 1992) are available for various univariate morphological representation schemes. A good shape representation scheme should have the following properties :

1. It should conform with our intuitive notions of 'simpler' components of a 'complex' picture.
2. It should have a well defined mathematical characterization.
3. It must be **mathematically practicable** i.e. it should be efficient and easy to use for various image analysis and computer vision applications. This usually requires the representation to be invariant under translation and scaling.
4. It should be **information preserving** i.e. it must be a unique mapping. For this, exact reconstruction of the original object is possible.
5. It must be **compact** i.e. it should be non-redundant and provide high data compression. Non redundant representation is defined as allowing a reconstruction of the original object, however, removal of any one of its elements would violate this reconstruction.

In this paper we present a general class of morphological univariate image representations and we study its properties. A theory is developed for the morphological univariate representation of images in a general form and to derive useful results and properties for its behaviour. First we are dealing with the univariate notion of general information preserving image description. Then the discussion is on general morphological non-information preserving univariate image analysis descriptor, called the *Gecstrum*, and its relation with the *Pecstrum*. Several properties of good image representation techniques are studied and conditions are derived for the invertibility, translation invariance, and efficiency of the representation. The detailed proof is beyond the scope of this paper and can be seen in Ghosh (1996).

REPRESENTATION SCHEME

This section deals with the theory which has developed for the univariate case in Shih and Pu (1992). The only difference here is that we take a single shape of structuring element with different sizes. The basis of the univariate image representation theory relies upon the generation of a set of nonoverlapping segments of the image via repeated erosions and set transformations, which in turn produces a decomposition that guarantees the exact reconstruction of the original image. Let A be a set representing the original discrete and binary image, and B be the structuring element.

A fundamental image decomposition procedure is to define the sets $S_n(A)$ by

$$S_n(A) = (A \ominus nB) - (A \ominus (n+1)B) \text{ for } n = 0, 1, \ldots, N \tag{1}$$

where '$-$' stands for the set difference and

$$nB = \underbrace{B \oplus B \oplus \ldots\ldots \oplus B}_{(n+1)times} \text{ for } n = 0, 1, \ldots, N \tag{2}$$

$N = max\{n; A \ominus nB \neq \phi\}$. So equation 2 means $0B = B$, $1B = B \oplus B$, $2B = B \oplus B \oplus B$ and so on.

From (2) it is easily seen that,

$$A \ominus (n+1)B = [A \ominus nB] \ominus B \subseteq A \ominus nB \text{ for } n = 0, 1, \ldots, N \tag{3}$$

By using (1) and (3) we get,

$$S_{n1}(A) \cap S_{n2}(A) = \phi \text{ for } n1 \neq n2 \tag{4}$$

Observe that,

$$S_n(A) \subseteq A \text{ for } n = 0, 1, \ldots, N \tag{5}$$

$$\cup_{n=0}^{N} S_n(A) = A \tag{6}$$

therefore the image A is decomposed into a sequence $\{S_n(A),\ n = 0, 1, \ldots, N\}$ of $(N+1)$ non-verlapping segments which guarantee exact reconstruction. Although the previous decomposition satisfies our requirements, it does not result in a useful image representation, since no redundant information is removed from the image A under consideration.

Let us define a sequence of general set transformations $\{\Psi_n[\bullet], n = 0, 1, \ldots N\}$ such that,

$$A \ominus (n+1)B \subseteq \Psi_n[A \ominus (n+1)B] \subseteq A \ominus nB \tag{7}$$

for $n = 0, 1, \ldots, N$

for every image $A \subseteq Z^2$, and if $R_n(A)$ are the subsets such that

$$R_n(A) = A \ominus nB - \Psi_n[A \ominus (n+1)B] \text{ for } n = 0, 1, \ldots, N \tag{8}$$

Now clearly from (1), (7) and (8)

$$R_n(A) \subseteq S_n(A) \text{ for } n = 0, 1, \ldots, N \tag{9}$$

The motivation for restriction (7) is to obtain a collection of disjoint subsets $\{R_n(A), \text{ for } n = 0, 1, \ldots, N\}$ that satisfy the inequality (9). The sequence $\{R_n(A), \text{ for } n = 0, 1, \ldots, N\}$ is also invertible, i.e., there exists a sequence of set transformations$\{F_n[\bullet], \text{ for } n = 0, 1, \ldots, N\}$ such that

$$A = \cup_{n=0}^{N} F_n[R_n]$$

Note the image decomposition in terms of $\{R_n(A), \text{ for } n = 0, 1, \ldots, N\}$ may provide an efficient image representation scheme. Now from this a image representation $R(A)$ of A is defined by

$$R(A) = \{R_0(A), R_1(A), \ldots, R_N(A)\} \tag{10}$$

We shall see in the following, $R(A)$ is an important representation which decomposes image A into $(N+1)$ disjoint subsets (i.e. $\{R_n(A), \text{ for } n = 0, 1, \ldots, N\}$). This set of subsets is proved to contain sufficient information to uniquely represent the original image A. We present a theorem which defines a restriction on the choices of the sequence $\{\Psi_n(\bullet), \text{ for } n = 0, 1, \ldots, N\}$ as a direct consequence of constraint (7) and we establish the invertibility of $R(A)$ under this restriction.

Theorem 1

If the sequence of set transformations $\{\Psi_n[\bullet], n = 0, 1, \ldots,, N\}$ satisfies (1), for every image $A \subseteq Z^2$, then

$$A \ominus (n+1)B \subseteq \Psi_n[A \ominus (n+1)B] \subseteq [[A \ominus (n+1)B] \bigcirc B] \copyright nB \tag{11}$$

for $n = 0, 1, \ldots, N$. Moreover,
$R(A)$ is invertible and

$$A = R^{-1}[R(A)] = \cup_{n=0}^{N} (R_n(A) \oplus nB) \tag{12}$$

It is interesting to note that $[[A \ominus nB] \bigcirc B] \copyright nB \subseteq A \ominus nB$, hence it is a tighter upper bound than $A \ominus nB$.

In this section we shall restrict $\{\Psi_n[\bullet], n = 0, \ldots,, N\}$ to satisfy (11), thereby allowing for a representation $R(A)$ which permits the exact reconstruction of A.

Some examples are as follows.

These are the importatnt special cases of the general morphological image representation.

Example 1 *Generalised Morphological Skeleton (Maragos and Schaffer,1986)* :
If

$$\Psi_n(X) = X \oplus B \text{ for } n = 0, 1, \ldots, N.$$

then

$$R_n(A) = A \ominus nB - [A \ominus nB] \bigcirc B$$

Example -2 *Reduced Morphological Skeleton (Maragos and Schaffer,1989)* :
If

$$\Psi_n(X) = [X \oplus B] \copyright nB \text{ for } n = 0, 1, \ldots, N.$$

then

$$R_n(A) = A \ominus nB - [[A \ominus nB] \bigcirc B] \copyright nB$$

Example -3
If

$$\Psi_n(X) = X \oplus [B©nB] \text{ for } n = 0, 1, \ldots, N.$$

then

$$R_n(A) = A \ominus nB - [[A \ominus (n+1)B] \oplus [B©nB]$$

We have already stated some of the choices for $\Psi_n[\bullet]$ and now we give some of the properties of $R(A)$.

First, we state that $R(A)$ is a translation invariant transformation, when $\{\Psi[\bullet], n = 0, 1, \ldots, N\}$ is a sequence of translation invariant mappings.

Proposition 1 :

$$\text{If } \Psi_n[A \oplus \{z\}] = \Psi_n[A] \oplus \{z\}, \text{ for } n = 0, 1, \ldots, N,$$

then,

$$R(A \oplus \{z\}) = R(A) \oplus \{z\}$$

for every $z \in Z^2$ where Z is set of integers.

Proposition 2 :

$$a) R_{n_1}(A) \cap R_{n_2}(A) = \phi, \text{ for } n_1 \neq n_2$$

and

$$b) R_n(A) \subseteq A \text{ for } n = 0, 1, \ldots, N.$$

Proposition 2 is the resulting morphological image representation subsets $R_n(A)$, for $n = 0, 1, \ldots, N$ are disjoint and anti-extensive.

GECSTRUM

The introduction of G-spectrum by Shih and Pu (1992) as a useful shape description tool, based on the theory developed by Gautias and Schonfeld (1991), is primarily for its less redundancy property compared to other existing shape-size descriptors. From this concept we develop the *Gecstrum*, based on the morphological erosion and other set transformations, as a measurement for quantifying the geometric shape of discrete multidimensional images with the help of a single set of structuring elements of the same shape but different sizes.

The formal definition of the *Gecstrum* is given by

Definition 1 :

$$Gecstrum = \{G_0(x), G_1(x), \ldots, ., G_N(x)\} \tag{13}$$

where

$$G_n(A) = \frac{\text{Card}(A \ominus nB) - \text{Card}(\Psi_n[A \ominus (n+1)B])}{\text{Card } (A)} \text{ for } n = 0, 1, \ldots, N \tag{14}$$

By the suitable choice of the sequence of transformations $\{\Psi_n[\bullet], n = 0, 1, \ldots, N\}$ that satisfy (7)

$$A \ominus nB - \Psi_n[A \ominus (n+1)B] \subseteq (A \ominus nB) - (A \ominus (n+1)B) \quad \text{for } n = 0, 1, 2, \ldots, N. \tag{15}$$

Hence

$$\begin{aligned}\text{card}[A \ominus nB &- \Psi_n[A \ominus (n+1)B]]\\ &\leq \text{card}[(A \ominus nB) - (A \ominus (n+1)B)]\end{aligned}$$

for $n = 0, 1, 2, \ldots, N.$ (16)

According to (14) and(16), *Gestrum* is less redundant than $\frac{R_n(A)}{\text{card}(A)}$; i.e.

$$G_n(A) \leq \frac{\text{card}[A \ominus nB] - \text{card}[A \ominus (n+1)B]}{\text{card}(A)} = \frac{R_n(A)}{\text{card}(A)} \tag{17}$$

It has been shown that the upper bound of the set transformations $\{\Psi_n(A)\}$ which satisfy equation (7) is $A \ominus nB \bigcirc B \copyright nB$. Also equation (18) is satisfied by

$$A \bigcirc kB = \cup_{n=k}^{N}[(A \ominus nB - \Psi[A \ominus (n+1)B]) \oplus nB \tag{18}$$

The difference between two successive openings is

$$(A \bigcirc nB - A \bigcirc (n+1)B \supseteq (A \ominus nB - \Psi_n[A \ominus (n+1)B]) \tag{19}$$

$$\Leftrightarrow P_n(A) \geq G_n(A). \tag{20}$$

where P_n is the nth element of the *Pecstrum*. Hence the *Gecstrum* has less redundancy than the *Pecstrum*.

Properties

The properties of the *Gecstrum* are now presented and discussed in this subsection.

Proposition 3

For a given image A, each element of the *Gecstrum* is a positive valued function. That is

$$G_n(A) \geq 0 \text{ for } n = 0, 1, \ldots, N \tag{21}$$

Proof: From equation (7) we know that

$$A \ominus nB \supseteq \Psi[A \ominus (n+1)B]$$

By applying the cardinality to both sides yields

$$\text{card}[A \ominus nB] \geq \text{card}[\Psi[A \ominus (n+1)B]]$$

Because card(A) $\geq$, we have

$$\frac{\text{card}[A \ominus nB] - \text{card}[\Psi[A \ominus (n+1)B]]}{\text{card}(A)} \geq 0$$

According to equation (14), the result obtained. ∎

As stated in proposition 3, the *Gecstrum* is a set of positive values which gives the quantative feature of an image based upon its geometry. The redundant rate function (RRT) (discussed in the next proposition) is an indicator of how much redundant information can be reduced by using the *Gecstrum*. The RRT can also be used in the matching procedure in object recognition.

Proposition 4

With a compact region of support, the summation of the *Gecstrum* is equal to one minus the redundant reduction rate (RRT). That is

$$\sum_{n=0}^{N} G_n(A) = 1 - \text{ RRT}(A). \tag{22}$$

$$\text{where RRT}(A) = \frac{1}{\text{Card }(A)} \sum_{n=0}^{N} \text{ Card }(\Psi[A \ominus (n+1)B]) - \text{Card }(A \ominus (n+1)B). \tag{23}$$

The summation of the *Gecstrum* is used to determine the degree of redundancy for an image representation. The smaller the value of $\sum_{n=0}^{N} G_n(A)$ the more redundant information is removed from the image. For an image A, the RRT(A) will be varied with respect to the different set transformations. By employing the above concept, we are able to select a suitable set transformation which leads to the best performance on image coding.

Proposition 5

If $\Psi_n[A \oplus \{z)] = \Psi_n[A] \oplus \{z\}$ for $n = 0, 1, \ldots, N$ then the *Gecstrum* is translation invariant i.e.

$$G_n(A \oplus \{z\}) = G_n(A) \text{ for } n = 0, 1, \ldots, N \tag{24}$$

where z is any integer.
Proof:

$$\begin{aligned} G_n(A \oplus \{z\}) &= \frac{\text{Card}((A \oplus \{z\}) \ominus nB) - \text{Card}(\Psi[(A \oplus \{z\}) \ominus (n+1)B])}{\text{Card }(A(\oplus\{z\}))} \\ &= \frac{\text{Card}((A \ominus nB) \oplus \{z\}) - \text{Card}(\Psi[(A \ominus (n+1)B]) \oplus \{z\})}{\text{Card }(A(\oplus\{z\}))} \\ &= \frac{\text{Card}(A \ominus nB) - \text{Card}(\Psi[A \ominus (n+1)B])}{\text{Card }(A)} \\ &= G_n(A) \end{aligned}$$

■

The translation invariance property is an essential criterion for a good shape description method. The next proposition is that normalised *Gecstrum* is a scale invariant shape descriptor.

Proposition 6

The *Gecstrum* is scaling invariant if the set A is normalised. That is

$$G_n(M(\xi A) = G_n(M(A)) \quad \text{for} \quad n = 0, 1, 2, \ldots, N \tag{25}$$

where ξ is an unknown scaling factor and $M(A)$ a normalization function which is defined as

$$M(A) = \frac{\tau}{\text{Card }(A)} \tag{26}$$

where τ is a pre-defined value.
From Proposition 4, if we perform the normalization (note that Card $(M(A)) = \tau$) on the images with various scaling factors ξ, the *Gecstrums* of ξA and A are the same. This implies that the normalisation according to a pre-defined value τ can produce the scaling invariant version of the *Gecstrum*.

Proposition 7

The first k elements of the *Gecstrum* are zeros,

$$G_n(A) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots, k-1, \tag{27}$$

iff the following equations are satisfied :

$$A \bigcirc kB \;=\; A \tag{28}$$

$$\Psi_n[A \ominus (n+1)B] \;=\; (A \ominus nB) \bigcirc B \copyright nB$$

$$\text{for} \quad n = 0, 1, \ldots, k-1. \tag{29}$$

If we can find a sequence of set nB, for $n = 0, 1, \ldots, N$ which satisfy equations (28) and (29), the recognition problem can be simplified by matching only $N - k + 1$ elements of the *Gecstrum*.

Proposition 8
If the set of structuring elements is chosen to be isotropic, the *Gecstrum* can be regarded as rotation invariant.

There is a relation between *Pecstrum* and *Gecstrum* which we try to reveal in the next proposition.

Proposition 9
There exists some n such that

$$G_n(A) = 0 \tag{30}$$

iff the following are satisfied

$$\Psi_n[A \ominus (n+1)B] = (A \ominus nB) \bigcirc B \copyright nB \tag{31}$$

and

$$P_n(A) = 0. \tag{32}$$

Proposition 9 tells us that if the transformation $\Psi_n[\bullet]$ is constrained by equation (31) and the nth element of the pattern spectrum is equal to zero, then the nth element of the *Gecstrum* will be equal to zero or vice versa.

CONCLUSION

This paper investigate a general univariate shape descriptor with the help of mathematical morphology (Serra, 1982). The term 'univariate' emphasises the generation of different non-overlapping image shapes from the original image at different scales with the help of different sized structuring elements generated from a single shape image kernel. The bivariate case is under study.

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