THE STEREOLOGICAL ESTIMATION OF WEIGHTED FIRST AND SECOND MOMENT OF PARTICLE VOLUME

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Almost all existing stereological methods developed for particle aggregates are directed towards the estimation of the number distribution of a one-dimensional particle size parameter z. Typically, the size parameter z is a linear size parameter. Alternative interesting choices are particle surface area and volume. The number distribution is determined by the probability function

$$g_N(z) = \frac{\text{number of particles of size } z}{\text{total number of particles}}. \qquad (1)$$

The value $g_N(z)$ is the probability that a uniform random particle chosen among the total of N particles is of size z (each particle has probability $1/N$ of being chosen). A basic problem in the stereological estimation of the number distribution is that it is not possible to sample particles with equal probability with a lower-dimensional probe. This is probably one of the reasons why most existing methods for obtaining information on the number distribution are based on the assumption that the particles are of the same, known and simple shape.

In contrast to this, it is possible, by means of point-sampling, to collect a sample of particles in which the volume-weighted distribution of size is correctly represented. The volume-weighted distribution is determined by the probability function

$$g_V(z) = \frac{\text{total volume of particles of size } z}{\text{total particle volume}}. \qquad (2)$$

The value $g_V(z)$ is the probability that a uniform random point chosen in the total particle aggregate is inside a particle of size z.

In this paper, we present stereological methods of estimating the first and second moment in the volume-weighted distribution, when z = particle volume is chosen as size parameter. These moments are called the weighted first and second moment of particle volume and are denoted by $\bar{v}_w$ and $\overline{v^2_w}$. The power of the stereological methods to be presented is that they can be applied to, in principle, arbitrarily shaped particles, if the different separated parts on a plane section through the particle aggregate, which belong to the same particle, can be identified.

In some fields of science, weighted distributions are, in fact, the type of size distribution being investigated. For instance, the sieving distributions in geology. Estimates of weighted moments can also be used in a quantification of departures from convexity of the particles. Furthermore, estimates of $\bar{v}_w$ and $\overline{v_w^2}$ can be combined with an estimate of mean volume, thereby obtaining estimates of usual unweighted second and third moments of particle volume.

The unweighted and weighted first moment of particle volume are identical, if all particles are of identical volume, and the relation between these moments involve only the coefficient of variation in the number distribution of particle volume. If one has prior knowledge that the variation in particle volume is not too high, the estimate of $\bar{v}_w$ can therefore be used as an estimate of usual (unweighted) mean volume.

The estimators of weighted moments of particle volume rely on a point-sampling procedure, which ensures the correct weighted representation of particles in the sample. This type of sampling has, surprisingly enough, not until recently been described in the stereological literature. The other important tool in the development of the estimators is an integral geometric formula due to Blaschke and Petkantschin. This formula gives a decomposition of the density of $q+1$ independent and uniform random points in a 3-dimensional compact object, $1 \leq q \leq 3$. For $q = 1$, an estimator of $\bar{v}_w$ is obtained which can be given in explicit form for arbitrarily shaped particles and involves measurements of cubed distances along lines. The case $q = 2$ leads to an estimator of $\overline{v_w^2}$, which is more difficult to determine. If the particles are triaxial ellipsoids, however, the estimator of $\overline{v_w^2}$ is a simple function of the axes of the section ellipses.

Replication is needed in order to reduce the variability of the estimators. In particular, it is usually important to use lines and planes with random orientations. If this requirement is impracticable the estimators may still be used, if the particles are randomly orientated. A purely model-based derivation of the estimators can thus be given for particle aggregates which can be described by a stationary and isotropic marked point process.