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ORIGINAL PAPER

A RELATIONSHIP BETWEEN THE PARAMETER κ OF A VON MISES DISTRIBUTION AND INTERSECTION DENSITIES MEASURED ON TWO PERPENDICULAR TEST LINES

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ABSTRACT

The fraction of the two intersection densities of test lines along and perpendicular to a known prevalent direction is considered as a function of the concentration parameter κ of the bimodal von Mises distribution. This is done similar to Weibel(1980), who derived the corresponding function for the Mariott distribution.
A table and the asymptotic behaviour are given. Small deviations of the test line system to the prevalent direction are considered. Finally it is shown, that a three dimensional initial problem, raised by biopedologists can be treated with the appropriate formula and also with the results given in the paper of Mathieu et al.(1983), depending on the assumed model.

INTRODUCTION

This paper is concerned with the problem of estimating the concentration parameter κ of a bimodal von Mises distribution, assuming that the angle of the prevalent direction is known. Because of technical constraints which are given for example by a microscope, the parameter κ should be estimated by measuring the intersection densities of two perpendicular test lines.
Weibel has solved this problem for the (bimodal) Mariott distribution following an idea of Cruz-Orive. The concentration parameter K of the Mariott distribution however, allows only to examine cases of mild degree of anisotropy because it is bounded ($K \leq 1$). That's why biologists (Mathieu et al.,1983) and biopedologists raised again this problem. The biopedological background shall be outlined here.
Leaves falling on the ground are situated in a high degree of anisotropy. In deeper horizons of the humus profile (for example in a depth of 5 - 30 mm) there are still leaf residues, but situated in a milder degree of anisotropy (lamellar structure).
In plane sections of the ground (polished blocks), cutted perpendicularly to the surface in different directions, traces of the leaves are obtained. The directions of these traces are assumed to be von Mises distributed (bimodal) with the preferred orientation parallel to the surface of the ground and a fix parameter κ for each horizon.

First the method of Weibel and Cruz-Orive will be extended to the case of the bimodal von Mises distribution as Mathieu et al. (1983) have done for the Fisher axial distribution. Then the results of small deviations from the prevalent direction are regarded. Finally the spatial case is considered corresponding to the above application. Two different models are compared.

THE FUNCTION $X = f(\kappa)$

Following the notation of Weibel (1980), who always considers the orientation on half the orientation circle, the density function of the bimodal von Mises distribution is given by

$$M(\psi) = \frac{1}{\pi I_0(\kappa)} e^{\kappa\cos 2\psi} \quad , \quad \psi \in (-\frac{\pi}{2}, \frac{\pi}{2}] \ , \tag{1}$$

where $I_0(\kappa)$ is a modified Bessel function (see for example Abramowitz and Stegun 1970). The results of this paper are given for $\kappa \geqq 0$ but can easily be extended to negative κ because

$$f(-\kappa) = \frac{1}{f(\kappa)} \qquad \text{(see eq.4)}.$$

Weibel shows that

$$B_A = \gamma_{01}^{-1} I_{L1} \tag{2}$$

and

$$B_A = \gamma_{02}^{-1} I_{L2} \tag{2'}$$

hold, where B_A is the length density of some anisotropic curves in the plane and I_{L1} (I_{L2}) is the intersection density of these curves hitted with a test line perpendicular (parallel) to the preferred orientation of the curves. If the directions of the line elements are von Mises distributed then

$$\gamma_{01}(\kappa) = \frac{1}{\pi I_0(\kappa)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\kappa\cos 2\theta} \cos\theta d\theta$$

and

$$\gamma_{02}(\kappa) = \frac{1}{\pi I_0(\kappa)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\kappa\cos 2\theta} |\sin\theta| d\theta = \frac{1}{\pi I_0(\kappa)} \int_0^{\pi} e^{\kappa\cos 2\theta} \sin\theta d\theta .$$

Defining the value X as

$$X := \frac{I_{L1}}{I_{L2}}$$

it is easy to derive that

$$X = \frac{\gamma_{01}(\kappa)}{\gamma_{02}(\kappa)} =: f(\kappa) \ .$$

For the graph of $X = f(\kappa)$ see Figure 1. It is compared with the corresponding function of the bimodal Mariott distribution, which can easily be identified because the graph is only defined for values $\kappa \in [0,1]$ (only positive values of κ are considered).

The following representations of X are useful and were received by elementary transformations. Essentially, representation (3) is the fraction of the error function divided by Dawson's integral and in this way X may be calculated by tables given in Abramowitz and Stegun (1970).

$$X = \frac{\int_0^{\frac{\pi}{2}} e^{\kappa \cos 2\theta} \cos\theta d\theta}{\int_0^{\frac{\pi}{2}} e^{\kappa \cos 2\theta} \sin\theta d\theta} = \frac{\int_0^{\sqrt{2\kappa}} e^{\kappa - u^2} du}{\int_0^{\sqrt{2\kappa}} e^{u^2 - \kappa} du}$$

$$= \frac{\int_0^{\sqrt{2\kappa}} e^{-u^2} du}{e^{-2\kappa} \int_0^{\sqrt{2\kappa}} e^{u^2} du} \quad (3)$$

$$= e^{2\kappa} \frac{\sum_{n=0}^{\infty} \frac{(-2\kappa)^n}{n!(2n+1)}}{\sum_{n=0}^{\infty} \frac{(2\kappa)^n}{n!(2n+1)}} \quad (4)$$

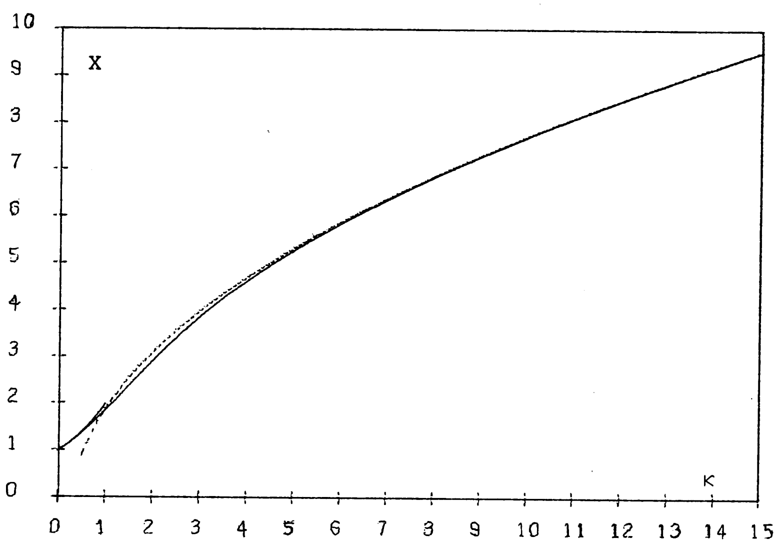

Fig. 1. a) The graph of $f(\kappa)$, $\kappa \in [0,15]$; b) the approximation by formula (6) (broken line); c) see the text for the graph on [0,1].

Using (3) it can be shown that $\frac{X}{\sqrt{\kappa}}$ tends to $\sqrt{2\pi}$ for $\kappa \to \infty$ and hence the asymptotic behaviour of X is

$$X \simeq \sqrt{2\pi\kappa} \; . \quad (5)$$

This formula has been improved by Bach (personal communication) deriving the asymptotic series

$$X \simeq \frac{\sqrt{2\pi\kappa}}{2}\left(a_0 + \frac{a_1}{\kappa} + \frac{a_2}{\kappa^2} + \ldots\right) ,$$

where $a_0 = 2,\ a_1 = -\frac{1}{2},\quad a_k = \frac{2k-3}{4}\, a_{k-1} - \frac{1}{2}\sum_{j=1}^{k-1} a_j a_{k-j}\ ,\ k \geqq 2$.

On using the first two terms, it is

$$X \simeq \sqrt{2\pi\kappa}\,\left(1 - \frac{1}{4\kappa}\right) \tag{6}$$

(see Figure 1). This leads to acceptable results for $\kappa > 6$ or $X > 6$ respectively. Eq. (6) allows not only to calculate X approximately for given κ but also to calculate κ for given X which might be the more interesting case. Table 1 shows some values of κ for given $X < 10$.

Table 1. The function $X = f(\kappa)$, inverse tabulated.

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	0.0000	0.1432	0.2749	0.3977	0.5137	0.6241	0.7301	0.8326	0.9323	1.0298
2	1.1257	1.2204	1.3142	1.4075	1.5007	1.5940	1.6877	1.7821	1.8773	1.9736
3	2.0712	2.1703	2.2712	2.3739	2.4788	2.5858	2.6953	2.8074	2.9221	3.0397
4	3.1603	3.2839	3.4106	3.5406	3.6739	3.8105	3.9505	4.0939	4.2407	4.3910
5	4.5447	4.7019	4.8625	5.0266	5.1940	5.3649	5.5391	5.7167	5.8976	6.0819
6	6.2695	6.4604	6.6547	6.8522	7.0529	7.2570	7.4643	7.6748	7.8886	8.1057
7	8.3260	8.5495	8.7762	9.0062	9.2394	9.4758	9.7155	9.9583	10.2044	10.4537
8	10.7061	10.9618	11.2207	11.4828	11.7481	12.0166	12.2883	12.5633	12.8414	13.1227
9	13.4072	13.6949	13.9858	14.2798	14.5772	14.8776	15.1812	15.4882	15.7982	16.1099

SMALL DEVIATIONS OF THE PREVALENT DIRECTION

In practice it is obviously difficult to realize the coincidence of the test lines with the preferred direction. So it is of some interest to know something about the error.
Let the test lines be orientated as before but the preferred direction has now a deviation $\alpha \neq 0$. Then the value X depends on α:

$$X(\alpha) = \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\kappa\cos 2(\theta-\alpha)}\cos\theta d\theta}{\int_0^{\pi} e^{\kappa\cos 2(\theta-\alpha)}\sin\theta d\theta} \qquad \left(= \frac{Z(\alpha)}{N(\alpha)} \right)$$

and $X(0) = X$.
The function $X(\alpha)$ obviously has a maximum for $\alpha = 0$.
Hence $\left.\frac{dX(\alpha)}{d\alpha}\right|_{\alpha=0} = 0$, what is easy to verify because

$$Z'(\alpha) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\kappa\cos 2(\theta-\alpha)} 2\kappa\sin 2(\theta-\alpha)\cos\theta d\theta$$

and

$$N'(\alpha) = \int_0^{\pi} e^{\kappa\cos 2(\theta-\alpha)} 2\kappa\sin 2(\theta-\alpha)\cos\theta d\theta$$

have integrands which are symmetrical to the point $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively if $\alpha = 0$. Hence the integrals are vanishing.
Using this, the second derivative of $X(\alpha)$ can be written as $(\alpha = 0)$

$$X''(0) = X(0)\left(\frac{Z''(0)}{Z(0)} - \frac{N''(0)}{N(0)} \right) .$$

The second derivatives of Z and N are

$$Z''(\alpha) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4(\kappa^2\sin^2 2(\theta-\alpha) - \kappa\cos 2(\theta-\alpha))e^{\kappa\cos 2(\theta-\alpha)}\sin\theta d\theta$$

$$N''(\alpha) = \int_0^{\pi} 4(\kappa^2\sin^2 2(\theta-\alpha) - \kappa\cos 2(\theta-\alpha))e^{\kappa\cos 2(\theta-\alpha)}\sin\theta d\theta .$$

Using $\sin^2 2\theta \leqq 1$ and $|\cos 2\theta| \leqq 1$ it follows that

$$\frac{Z''(0)}{Z(0)} = 4\kappa^2 \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 2\theta \cdot e^{\kappa\cos 2\theta}\cos\theta d\theta}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\kappa\cos 2\theta}\cos\theta d\theta}$$

$$- 4\kappa \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2\theta \cdot e^{\kappa\cos 2\theta}\cos\theta d\theta}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\kappa\cos 2\theta}\cos\theta d\theta} \leqq 4\kappa^2 + 4\kappa \quad (\kappa > 0)$$

and similarly

$$\frac{N''(0)}{N(0)} \leqq 4\kappa^2 + 4\kappa \quad (\kappa > 0) .$$

The Taylor expansion of second order

$$X(\alpha) = X(0) + X'(0)\alpha + X''(0)\alpha^2$$

leads to an expression of the relative error

$$\left|\frac{X(\alpha) - X(0)}{X(0)}\right| \leqq 8(\kappa^2 + \kappa)\alpha^2 \quad (\kappa > 0)$$

if the terms of higher order are neglected.
This shows in which way a small deviation of the prevalent direction can influence the measured X.

SOME CONCLUSIONS

a) If the parameter κ is known then the length density B_A can be determined by eq. (2). This equation is preferred because the coefficient γ_{01} can be transformed in the following way

$$\gamma_{01} = \frac{1}{\pi I_0(\kappa)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\kappa\cos 2\theta}\cos\theta d\theta = \frac{2e^{\kappa}}{I_0(\kappa)\sqrt{2\kappa}} \int_0^{\sqrt{2\kappa}} e^{-u^2} du$$

$$= \frac{e^{\kappa}}{I_0(\kappa)\sqrt{2\pi\kappa}} \operatorname{erf}(\sqrt{2\kappa}) \; .$$

All the functions appearing in the last expression are well known (erf(.) = error function) and tabulated; see for example Abramowitz and Stegun (1970).

b) The problem of the biopedologists suggests to regard the line elements as surface elements hitted by a section. All above was done under the assumption that the traces of the leaves are bimodal von Mises distributed. But, considering the spatial structure of the problem, the Fisher axial distribution seems also to be an acceptable assumption for the normals of the surface elements.
The probability element of this distribution is

$$dF(\theta,\phi|\chi) = c(\chi)\cdot e^{\chi\cos 2\theta}\sin\theta d\theta d\phi \;, \; \theta \in [0,\tfrac{\pi}{2}] \;, \; \phi \in [0,2\pi) \;,$$

where the pole is at $\phi = \theta = 0$.
Following Mardia (1972,p.233) or Downs (1966) respectively the transformation

$$\cos\theta = r\cdot\cos\theta' \quad , \quad \sin\theta\cos\phi = r\cdot\sin\theta' \qquad (7)$$

leads to

$$dF(r,\theta'|\chi) = c(\chi)\cdot e^{\chi r^2\cos 2\theta'}\cdot e^{-\chi(1-r^2)}\frac{2r}{\sqrt{1-r^2}}\, d\theta' dr \;,$$

$\theta' \in [-\frac{\pi}{2},\frac{\pi}{2}]$, $r \in [0,1)$.

This shows that the corresponding conditional distribution of θ' given $r = r_0$ is a bimodal von Mises distribution and particularly for given $r = 1$ with the same concentration parameter.
The density of the angle distribution for the traces of the leaves in sections parallel to the pole direction is

$$dF(\theta'|\chi) = \int_{r=0}^{1} r\cdot dF(r,\theta'|\chi) \;,$$

which does not represent a von Mises distribution.
The probability of hitting the trace of a leaf with the test line parallel to the pole direction in the section (I_{L1}) is given by

$$a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta'\cdot dF(\theta'|\chi)$$

$$= a\cdot c(\chi) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r\cdot\cos\theta'\cdot e^{\chi r^2\cos 2\theta' - \chi(1-r^2)}\frac{2r}{\sqrt{1-r^2}}\, dr d\theta'$$

$$= a\cdot c(\chi) \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\theta\cdot e^{\chi\cos 2\theta}\sin\theta d\theta d\phi \;,$$

where a is a norming factor.
The last term is received using the transformation (7) backwards.

On the one hand it does not surprise this result being equal to that of Weibel (1980, eq.10.63) which gives the probability of hitting surface elements by the test line. The corresponding result holds for a test line perpendicular to the pole direction in the section (I_{L2}).

On the other hand Mathieu et al.(1983) considered line elements hitted by test planes assuming that the line elements are Fisher axial distributed.

Both kinds of consideration are exchangeable because lines and planes are interacting. Therefore it is possible to estimate χ by I_{L1}/I_{L2} using table 1 or figure 1 given in the paper of Mathieu et al.(1983).

Comparing that figure with the figure 1 of the present paper, the similarity of the results under the two different assumptions is remarkable.

Finally, assuming the Fisher axial distribution, the surface density S_V can be calculated by

$$S_V = \gamma_1^{-1}(\chi,0) \cdot I_{L1}$$

where $\gamma_1(\chi,0)$ is tabulated in Weibel (1980,p.298).

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