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GEODESIC CONVEXITY

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ABSTRACT

This paper is a resumption of the paper entitled : "On the use of the geodesic metric in image analysis" which was presented by Ch. Lantuéjoul and S. Beucher at the 5th I.S.S. Congress. We recall several features that can be intrinsically defined on 2-D Euclidean subsets (length, geodesic radius). These features satisfy isoperimetric inequalities in the case where the subsets under study are simply connected. As an application, a metallographic case study is presented.

1. GEODESIC DISTANCE FUNCTION

1.1 Existence and Unicity

Let X be a subset of the 2-D Euclidean space in $\mathbb{R}^2$, and let x and y be two points of X. Let $d_X(x,y)$ denote the infimum of the lengths of the arcs in X between x and y if such arcs do exist, and $+\infty$ if not. The function d_X satisfies the 3 axioms of a distance function

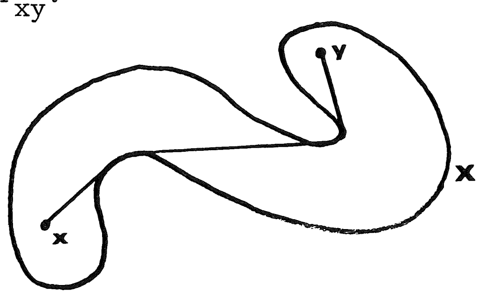
$$d_X(x,y) \geqq 0 \quad \text{and} \quad = 0 \quad \text{iff } x = y$$

$$d_X(x,y) = d_X(y,x)$$

$$d_X(x,z) \leqq d_X(x,y) + d_X(y,z)$$

but may take infinite values.

Suppose now that $d_X(x,y) < +\infty$. If X is *closed*, there is a (simple) arc in X linking x and y, with its length equal to $d_X(x,y)$. Moreover, such an arc is unique if X is *simply connected* (1). We term it geodesic arc between x and y, and denote it Γ_{xy}.



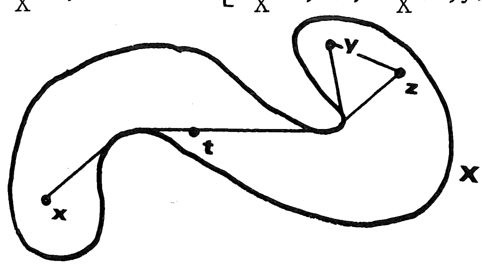
1.2 Geodesic Distance Function

From now on, we assume X to be a non-empty, closed subset of $\mathbb{R}^2$, which is connected, simply connected and compact for d_X. Then d_X is a distance function called *"geodesic distance function"*.

Let $z \in X$. It can be shown that the function $t \to d_X(z,t)$ is *geodesically convex* (2). More precisely, if $t \in \Gamma_{xy}$ such that $d_X(x,t) = \alpha \cdot d_X(x,y)$ $(0 < \alpha < 1)$, then :

$$d_X(z,t) \leqq (1-\alpha)\, d_X(z,x) + \alpha\, d_X(z,y) \qquad (1)$$

$$d_X(z,t) < \max\left[d_X(z,x),\, d_X(z,y)\right] \qquad (2)$$



2. MORPHOLOGICAL APPLICATIONS

2.1 Propagation Function

Imagine that X is a soundproof room. At time 0, a beep is emitted from a point x in X. Sound propagates in all directions within X at a constant speed (taken to be equal to 1). We measure the time T(x) at which all the points in X have received the beep :

$$T(x) = \max_{y \in X} d_X(x,y)$$

The function T is termed *"propagation function"*. Clearly, T is continuous :

$$|T(x) - T(y)| \leqq d_X(x,y)$$

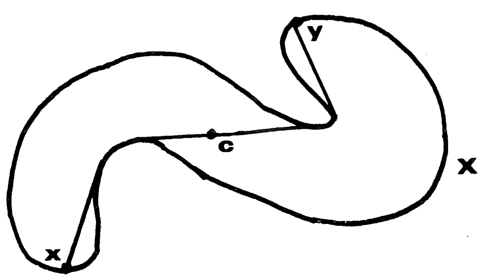
T is also geodesically convex (2): if $t \in \Gamma_{xy}$ such that $d_X(x,t) = \alpha\, d_X(x,y)$ $(0 < \alpha < 1)$, then

$$T(t) \leqq (1-\alpha)\, T(x) + \alpha T(y) \qquad (1')$$

$$T(t) < \max\left[T(x),\, T(y)\right] \qquad (2')$$

2.2 Geodesic Features

Since X is compact, T reaches its extremal values. We write $D(X) = \max_{x \in X} T(x) = \max_{x,y \in X} d_X(x,y)$. D(X) is called *geodesic diameter* or *length* of X. Using the convexity inequality (2'), it can be shown that T reaches its minimal value at a unique point c. This point is called *geodesic center*, and the corresponding value T(c) *geodesic radius*. This value is denoted R(X).



2.3 An Isoperimetric Inequality

The geodesic radius and the geodesic diameter satisfy the following inequalities (3)

$$R(X)\sqrt{3} \leqq D(X) \leqq 2\, R(X)$$

and these inequalities are the best (e.g. an equilateral triangle and a segment). In the specific case where X is convex, we obtain Jung's inequality (4).

Remark : If X is not simply connected, the geodesic diameter and the geodesic radius can be defined exactly as above, but the inequality $R(X)\sqrt{3} \leqq D(X)$ does not necessarily hold. Furthermore, a center is not necessarily uniquely defined (e.g. if X is a circumference, all the points in X are a center and $R(X) = D(X)$).

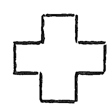
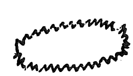
3. A CASE STUDY

The following study has been supported by the "Centre de Recherches de Pont-à-Mousson". It is well known that the lack of malleability of grey cast-iron is related to the presence of long, narrow graphite particles which tend to favour crack propagation. The question is: how can the tendency of a particle to be elongated be quantified?

3.1 The Shape Factor

Let X be a graphite particle. A first method consists in computing the so-called "shape factor" :

$$\phi(X) = \frac{P^2(X)}{4\pi\, A(X)}$$

where A(X) and P(X) respectively stand for the surface area and the perimeter of X. It turns out that this method provides rather poor results. Indeed, arbitrarily shaped particles can take the same shape factor (e.g.  and ), and conversely two particles with the same shape can take entirely different shape factors (e.g.  and ).

3.2 A Length Index

We propose to define a length index as follows :

$$\alpha(X) = \frac{\pi\, D^2(X)}{4\, A(X)}$$

where the length D(X) is used instead of the perimeter P(X). The more elongated the particle, the greater its length index. $\alpha(X)$ is minimal and equal to 1 if and only if X is a disk.

More generally, the length index satisfies the two following inequations :

$$\sqrt{\alpha(X \cup Y)} \leqq \sqrt{\alpha(X)} + \sqrt{\alpha(Y)} \quad \text{when } X \cap Y \neq \emptyset$$

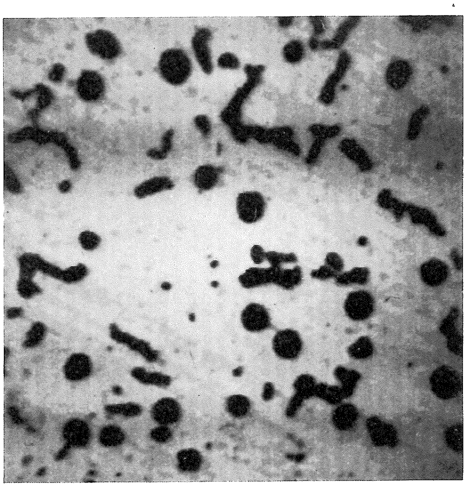
$$\alpha(X \oplus Y) \leqq \max[\alpha(X), \alpha(Y)]$$

where $X \oplus Y$ stands for the Minkowski's sum of X and Y, i.e. $X \oplus Y = \{x+y | x \in X, y \in Y\}$.

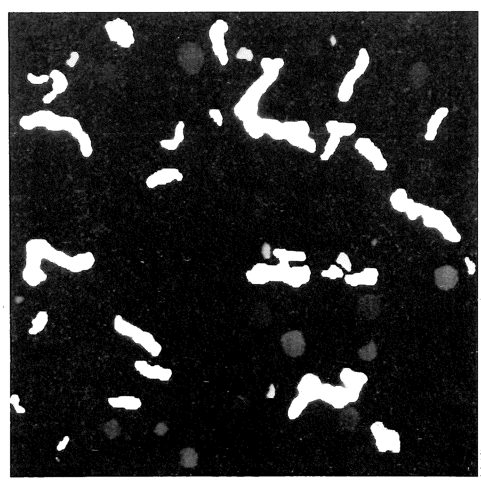
The first experiments have shown that the length index seems to be more reliable than the shape factor. This probably stems from the fact that $\alpha(X)$ is hardly sensitive to the fluctuations of the contour of X. Two photographs have been taken from the study :

(1) original image with the black graphite particles

(2) length index of the particles: the more elongated they are, the brighter they are when displayed.



(1)



(2)

An algorithm to get the propagation function and its subsequent features has been presented in (5).

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