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SIGNED LINE INTERCEPT COUNT

Robert T. DeHoff

Department of Materials Science and Engineering
University of Florida
Gainesville, FL 32611

ABSTRACT

Elements of surfaces that bound a phase (β) in a two phase mixture (α+β) may be classified as:

a. convex (++) if both principle curvatures are positive;
b. concave (--) if both are negative; and
c. saddle (+-) if one is positive and the other negative.

The traces of these surfaces that form the boundaries of the areas on a representative two dimensional section may also be:

a. convex (+) if the local curvature is positive; or
b. concave (-) if it is negative.

Line intercept counts may be tabulated separately for sections with convex (P_{L+}) and concave (P_{L-}) segments of boundary. This paper presents a derivation of fundamental stereological formulae that relate these counting measurements to three dimensional geometric properties of the structure they sample.

THEORETICAL DEVELOPMENT

All sections through convex (++) surface elements are convex (+); all sections through concave (--) surface elements are concave (-). However, sections through saddle surface (+-) elements may be either convex (+) or concave (-). Thus, P_{L+} arises from a mixture of intersections with convex and saddle surface, while P_{L-} arises from intersections with a mixture of concave and saddle surface. The following development derives the separate contributions of convex, concave and saddle surface to each of these counts.

In general, the line intercept count is a ratio of measures:

$$P_L(\text{event}) = \frac{\text{Measure of the set of lines that satisfy some event}}{\text{Measure of the length of lines in the sample}}$$

$$= \iiiint_{Event} dG / \iiiint_{Sample} L_o dG \qquad (1)$$

where dG is the density of lines in three dimensional space (Santalo)

$$dG = dxdy \sin\phi \, d\theta d\phi \qquad (2)$$

where x and y are positional coordinates, and θ and ϕ are orientation coordinates on the unit sphere. Insert equation (2) in equation (1):

$$P_L \text{ (event)} = \iiiint_{Event} dxdy \sin\phi \, d\theta d\phi \; / \; \iiiint_{Sample} L_o dxdy \sin\phi \, d\theta d\phi \qquad (3)$$

The domain of integration of the denominator is the set of lines that intersect the sample; its evaluation is straightforward:

$$\iiiint_{Sample} L_o \, dxdy \sin\phi \, d\phi d\theta = 2\pi V_o \qquad (4)$$

where V_o is the volume of the sample.

The events of interest, and their domains of integration are different for the six combinations of contributions of the three classes of surfaces to the convex and concave counts, see Columns 1 and 2 of Table 1. In all cases, the measure of the set of lines of a fixed orientation that intersect an element of area is given by

$$dxdy = dS\cos\phi \qquad (5)$$

where dS is the area of the element, and ϕ is the angle between the test line direction and the vector normal to the tangent plane at dS. Equation (3) may be written explicitly in generic form:

$$P_{Li}(jk) = \frac{1}{2\pi V_o} \iint_{S_{jk}} \int_o^{\pi/2} 4\int_{\theta_{min}(jk)}^{\theta_{max}(jk)} \cos\phi \, d\theta \, d\phi \, dS$$

$$P_{Li}(jk) = \frac{1}{\pi V_o} \iint_{S_{jk}} [\theta_{max}(jk) - \theta_{min}(jk)] dS$$

where the index i may be (+) or (-), and (jk) may be (++), (--) or (+-). Integration over ϕ introduces a factor of 1/2. Domains of integration of θ corresponding to each event are different for each case, as explicitly shown in Column 3 of Table 1. For saddle surface, θ_o gives the local direction in the tangent plane where the curvature on a sectioning plane changes from positive to negative; this direction is called the local "asymptotic direction", and is related to the local principle curvatures by, (Struik)

$$\theta_o = \tan^{-1} \sqrt{-(K_1/K_2)} \quad (7)$$

It is useful to define a value for θ_m for all three classes of surface. If θ_m is the upper limit of the variable θ that yields positive intersections on a sectioning plane, then $\theta_m = \pi/2$ for convex surface (++), $\theta_m = 0$ for concave surface (--), and the variable asymptotic direction angle, $\theta_m = \theta_o$ for saddle surface (+-). An average value of θ_m for all surface in the system may be reasonable defined to be:

$$\overline{\theta}_m \equiv [\iint_{S++} \frac{\pi}{2}\, dS + \iint_{S--}(0)dS + \iint_{S+-}\theta_o dS]/(S_{++} + S_{--} + S_{+-}) \quad (8)$$

With this definition, the total line intercept count in each category may be written

$$P_{L+} = \frac{1}{\pi}\overline{\theta}_m S_V \quad ; \quad P_{L-} = \frac{1}{\pi}(\frac{\pi}{2} - \overline{\theta}_m)S_V \quad (9)$$

Note that

$$P_L = P_{L+} + P_{L-} = \frac{1}{2}S_V \quad (10)$$

which recovers the familiar fundamental equation for the total line intercept count.

The new information supplied by the signed line intercept count is the average value of the angle θ_m defined in equation (8). Combination of equations (9) and (10) gives:

Table 1. Contribution to signed line intercept counts from each class of surface.

Section (i)	Surface (jk)	θ_{min} to θ_{max}	$P_{Li}(jk)$
Convex (+)	Convex (++)	0 to $\frac{\pi}{2}$	$\frac{1}{\pi V_o}\iint_{S++}\frac{\pi}{2}\,dS$
	Concave (--)	0 to 0	0
	Saddle (+-)	0 to θ_o	$\frac{1}{\pi V_o}\iint_{S+-}\theta_o dS$
Concave (-)	Convex (++)	0 to 0	0
	Concave (--)	0 to $\frac{\pi}{2}$	$\frac{1}{\pi V_o}\iint_{S--}\frac{\pi}{2}dS$
	Saddle (+-)	θ to $\frac{\pi}{2}$	$\frac{1}{\pi V_o}\iint_{S+-}(\frac{\pi}{2} - \theta_o)dS$

$$P_{L+}/P_L = \bar{\theta}_m/(\frac{\pi}{2}) \tag{11}$$

SUMMARY

The separate tabulation of the line intercept count for convex and concave segments of boundaries on representative sections through three dimensional structures yields information about the average value of θ_m, the angle that gives the asymptotic direction for saddle elements, and is defined to be $\pi/2$ for convex elements and 0 for concave elements. The relationships that establish this connection are stereologically fundamental, in the usual sense that their validity does not depend upon geometric assumptions.

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