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ORIENTATION ANALYSIS IN SECOND-ORDER STEREOLOGY

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ABSTRACT

Probability distributions on orientation spaces appear frequently in anisotropic stereology (Philofsky and Hilliard, 1967, Kanatani, 1984). In the present paper special cases of integral transformations where two such distributions appear are studied. One corresponds to the rose of directions of a stationary random process of geometrical objects in $\mathbb{R}^3$; the other corresponds to the orientations of probes. In comparison with earlier results (Beneš, 1995; Beneš et al., 1995) spherical harmonics are tried to be used for the evaluation. As an application a new formula for the variance of intensity estimator of a Boolean surface process is obtained, based on projections.

Keywords: Anisotropy, projection measure, spherical harmonics, surface process.

INTRODUCTION

In the stereology of anisotropic structures we deal with orientation distributions. These are mathematicaly described by probability measures on the space of orientations. Let the rose of direction $\mathcal{R}$ be the orientation distribution of a fibre or surface structure. When getting an information about the structure we use as probes sections or projections in different orientations. Let Q denotes the orientation distribution of probes and $\mathcal{F}_Q(m)$ its cosine transform. The equation

$$I_n(Q, \mathcal{R}) = \int_M \mathcal{F}_Q^n(m)\mathcal{R}(\mathrm{d}m) \tag{1}$$

and its application will be studied in our paper. In fact the case of exponent $n = 1$ was thoroughly investigated (Philofsky and Hillard 1967; Kanatani 1984) to get and invert the first order equations between rose of directions and rose of intersections. We investigate the case $n = 2$ which appears in second-order problems, e.g. when evaluating the variances of intensity estimators. A general theory based on projection measures was developed (Beneš 1995; Beneš et al. 1995) to study anisotropic structures. The new results in this paper are derived in the three-dimensional space using spherical harmonics for evaluation of (1) in the case $n = 2$. Moreover the estimation variance of an intensity estimator based on a projection for a Boolean model of compact surfaces is explicitly derived.

VARIANCE OF THE PROJECTION MEASURE

Let $(\mathbb{R}, \mathcal{B}, \nu)^d$ be a d-dimensional Euclidean space with Borel σ-algebra and Lebesgue measure ν. Let $(M, \mathcal{M})^d$ be a measurable space of axial orientations represented by vectors on a unit hemisphere in $\mathbb{R}^d$ and Q be a given probability measure on $\mathcal{M}^d$. The cosine transformation is defined by $\mathcal{F}_Q(l) = \int_{M^d} \mid \cos \sphericalangle(l, m) \mid Q(\mathrm{d}m)$, $l \in M^d$. Let Φ be a stationary random fibre or surface process in $(\mathbb{R}, \mathcal{B}, \nu)^d$ (Stoyan et al., 1987) with intensity constant λ. We denote φ a realization of Φ. For a point $x \in \varphi$ we denote as the weight $m(x) \in M^d$ the tangent (normal) orientation of a fibre (surface) at x, respectively; $\mathcal{R}$ is the weight distribution called the rose of directions.

Definition. The random measure Φ_Q on $\mathbb{R}^d$ defined by

$$\Phi_Q(B) = \int_B \mathcal{F}_Q(m(x))\Phi(\mathrm{d}x), \qquad B \in \mathcal{B}^d, \tag{2}$$

is called the projection measure.
It is interpreted as the total projection length (surface area) of the process Φ in B, averaged with respect to the distribution Q of projection orientations. Since many estimators of surface and length intensity are derived from Φ_Q with various Q (Beneš, 1995), we study first the properties of Φ_Q. Our main aim is to get formulas for variance of Φ_Q to be able to describe the efficiency of stereological intensity estimators.
The following regularity condition is assumed to be fulfilled whenever dealing with a pair $Q, \mathcal{R}$ of probability measures on $\mathcal{M}$ throughout the paper: For any $m, l \in M$ such that the inner product $\langle l, m\rangle = 0$ it holds either $\mathcal{R}(\{m\}) = 0$ or $Q(\{l\}) = 0$. This ensures that projections are correctly defined.
The basic properties of Φ_Q are well-known (Beneš et al., 1995), it has intensity $\lambda_Q = \lambda\mathcal{F}_{\mathcal{R}Q}$ and the orientation weight $m(x) \in M$ of Φ_Q induced by Φ has distribution $\mathcal{R}_Q(L) = \int_L \frac{\mathcal{F}_Q(l)}{\mathcal{F}_{\mathcal{R}Q}}\mathcal{R}(\mathrm{d}l)$, $l \in \mathcal{M}$. Here $\mathcal{F}_{\mathcal{R}Q} = \int_M \mathcal{F}_Q(l)\mathcal{R}(\mathrm{d}l)$.
The variance $var\Phi_Q(B)$ is equal to

$$var\Phi_Q(B) = \lambda_Q^2\left(\int g_B(x)\mathcal{K}_Q(\mathrm{d}x) - [\nu(B)]^2\right), \tag{3}$$

cf. Stoyan et al. (1987). Here $g_B(x) = \nu(B \cap B_{-x})$ and $\mathcal{K}_Q$ is the reduced second moment measure of Φ_Q,. Generally $var\Phi_Q(B)$ depends on the joint two-point weight distribution (Beneš, 1995), we restrict ourselves to Poisson processes where explicit results can be obtained.

Proposition 1. *Let Φ be a Boolean model of fibres (surfaces) which are subsets of straight lines (hyperplanes). Under the condition that a fibre (surface) with orientation m hits the origin let δ_0^m be its probability distribution. Let Q on $\mathcal{M}$ be arbitrary, $B \in \mathcal{B}$ bounded. Then*

$$var\Phi_Q(B) = \lambda \int\int\int \mathcal{F}_Q^2(m)g_B(x)\varphi(\mathrm{d}x)\delta_0^m(\mathrm{d}\varphi)\mathcal{R}(\mathrm{d}m). \tag{4}$$

Proof. See Beneš et al. (1995).
Specially for the Poisson line (hyperplane) process and a ball B the following formulas were obtained in Beneš (1995), respectively

$$\begin{aligned} var\Phi_Q(B) &= 2\lambda \int \mathcal{F}_Q^2(m)\mathcal{R}(\mathrm{d}m)\int_0^\infty g_B(r)\mathrm{d}r, \\ var\Phi_Q(B) &= \mathcal{O}_{d-1}\lambda \int \mathcal{F}_Q^2(m)\mathcal{R}(\mathrm{d}m)\int_0^\infty r^{d-2}g_B(r)\mathrm{d}r, \end{aligned} \tag{5}$$

since $g_B(x) = g_B(r, l) = g_B(r)$ is independent on l in polar coordinates. Here $\mathcal{O}_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$. Our aim is to evaluate $var\Phi_Q(B)$ for a Boolean model of compact surfaces.
Consider Φ a Boolean model with $k-$dimensional compact sets K in $\mathbb{R}^d$. Let Λ_0 denotes the distribution of the typical set centered in the origin and H^k the k-dimensional Hausdorff measure on $\mathcal{B}^d$. Introduce moreover Λ_0^m the distribution on the set of centered compacts which have the weight m in origin and η_{K_0} the measure on $\mathcal{B}^d$ defined by $\eta_{K_0}(.) = H^k(K_0 \cap .)$. The following measure plays an important role in the theory of Boolean models (Rataj, 1995).

$$\mu_f(B|m) = \frac{\int\int \eta_{K_0}(B+y)\eta_{K_0}(\mathrm{d}y)\Lambda_0^m(\mathrm{d}K_0)}{\int H^k(K_0)\Lambda_0^m(\mathrm{d}K_0)}, \qquad B \in \mathcal{B}^d \tag{6}$$

Comparing formulas (4) and (6) we obtain for a Boolean model of compact fibres (surfaces) the following factorized formula including $I_2(Q,\mathcal{R})$ from (1):

Corollary 1. *Under the conditions of Proposition 1 it holds*

$$\int \varphi(B)\delta_0^m(\mathrm{d}\varphi) = \mu_f(B|m).$$

If moreover Λ_0^m *does not depend on* m *then* μ_f *does not depend on* m *and we have*

$$var\Phi_Q(B) = \lambda \int_{M^d} \mathcal{F}_Q^2(m)\mathcal{R}(\mathrm{d}m) \int_{\mathbb{R}^d} g_B(x)\mu_f(\mathrm{d}x) \tag{7}$$

Proof. Formula (6) integrates $\eta K_0(B+y)$ with respect to H^k-volume weighted distribution which corresponds to the integration with respect to δ_0^m. (7) then follows from (4).

Corollary 2. For a ball B we have

$$var\Phi_Q(B) = \lambda \int_{M^d} \mathcal{F}_Q^2(m)\mathcal{R}(\mathrm{d}m) \int_0^\infty g_B(r)\mathrm{d}f(r), \tag{8}$$

where $f(r) = \mu_f(B_r)$ and B_r is a ball of radius r centered in origin.

Proof. Let B have the radius q and $0 = r_0 \leqq r_1 \leqq \cdots \leqq r_n = 2q$ be the partition of interval $\langle 0, 2q\rangle$. Denote $C_{r_i} = \{x : r_{i-1} \leqq \|x\| < r_i\}$. Then $\int g_B(x)f(\mathrm{d}x) = \int_{B_{2q}} g_B(x)f(\mathrm{d}x) = \sum_{i=1}^n \int_{C_{r_i}} g_B(x)f(\mathrm{d}x) \approx \sum_{i=1}^n g_B(r_i)f(C_{r_i}) = \sum_{i=1}^n g_B(r_i)\{f(B_{r_i}) - f(B_{r_{i-1}})\} \longrightarrow \int_0^\infty g_B(r)\mathrm{d}f(r)$ for $n \to \infty$.

Remark. The assumption that Λ_0^m does not depend on m means specially that the segment length (surface area) is independent on its orientation which is the case frequent in real structures.

ORIENTATION ANALYSIS USING SPHERICAL HARMONICS

As it could be seen in the previous part the integral $I_2(Q,\mathcal{R}) = \int \mathcal{F}_Q^2(m)\mathcal{R}(\mathrm{d}m)$ is the important term for describing variances of projection measures. We tried to use spherical harmonics for its evaluation. We restrict to the case $d = 3$, then it holds (using polar coordinates $(\theta,\phi) \in M^3$, where θ denotes the colatitude and ϕ the longitude)

$$\mathcal{R}(\mathrm{d}m) = \{a_0^{(0)} + \sum_{n=1}^{\infty}\{a_0^{(n)}P_n(\cos\theta) + \sum_{k=1}^{n}(a_k^{(n)}\cos k\phi + b_k^{(n)}\sin k\phi)P_{n,k}(\cos\theta)\}\}\sin\theta\mathrm{d}\theta\mathrm{d}\phi,$$

where P_n denote the Legendre polynomials, $P_{n,k}$ the associate Legendre polynomials and $a_k^{(n)}, b_k^{(n)}$ determined coefficients (Smirnoff, 1951). In stereological practice the estimation of $\mathcal{R}$ is a hard problem, see Kanatani(1984), Cruz-Orive et al.(1985).
In the case where $\mathcal{R}$ is the distribution which is symmetric around z-axis, the terms including k vanish. Suppose that $\mathcal{R}$ has a density ρ hence $\mathcal{R}(\mathrm{d}m) = \rho(\theta,\phi)\sin\theta\mathrm{d}\theta\mathrm{d}\phi = \rho(\theta)\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ due the symmetry of $\mathcal{R}$. We can write now

$$\rho(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \tag{9}$$

where $a_n = \frac{2n+1}{2}\int_0^\pi \rho(\theta)P_n(\cos\theta)\mathrm{d}\theta$. Using the notation $\mathcal{F}_Q(m) = \mathcal{F}_Q(\theta,\phi)$ we continue, cf.(1)

$$\begin{aligned} I_2(Q,\mathcal{R}) &= \int_0^\pi\int_0^\pi \mathcal{F}_Q^2(\theta,\phi)\rho(\theta)\sin\theta\mathrm{d}\theta\mathrm{d}\phi = \\ &= \sum_{n=0}^{\infty} a_n \int_0^\pi\int_0^\pi \mathcal{F}_Q^2(\theta,\phi)P_n(\cos\theta)\sin\theta\mathrm{d}\theta\mathrm{d}\phi = \sum_{n=0}^{\infty} a_n b_n, \end{aligned} \tag{10}$$

where $b_n = \int_0^\pi \int_0^\pi \mathcal{F}_Q^2(\theta,\phi) P_n(\cos\theta)\sin\theta \mathrm{d}\theta\mathrm{d}\phi$.
Among the rotational-symmetric distributions we study in detail the parametric family of Dimroth-Watson distributions with parameter κ, $-\infty < \kappa < +\infty$

$$\mathcal{R}(\mathrm{d}m) = \frac{\exp(2\kappa\cos^2\beta)}{2\pi U_0(\kappa)} \sin\beta\mathrm{d}\beta\mathrm{d}\gamma, \tag{11}$$

where $U_0(\kappa) = \int_0^1 \exp(2\kappa x^2)\mathrm{d}x$. In this case it is

$$\begin{aligned} a_n &= \frac{2n+1}{4\pi U_0(\kappa)} \int_0^\pi \exp(2\kappa\cos^2\theta) P_n(\cos\theta)\sin\theta\mathrm{d}\theta = \\ &= \frac{2n+1}{2\pi U_0(\kappa)} \int_0^1 \exp(2\kappa x^2) P_n(x)\mathrm{d}x \end{aligned}$$

for $n = 0, 2, \ldots$ even and $a_n = 0$ otherwise.

Proposition 2. *Let $(\omega,\psi) \in M^3$, ω the colatitude, ψ the longitude. Let $Q(.) = \delta_{(\omega,\psi)}(.)$ corresponds to a fixed projection direction (ω,ψ). Then for $\mathcal{R}$ in (11) it holds*

$$\int \mathcal{F}_Q^2(m)\mathcal{R}(\mathrm{d}m) = \frac{1}{3} + \frac{\cos^2\omega + \cos 2\omega}{6U_0(\kappa)} \int_0^1 (3x^2-1)\exp(2\kappa x^2)\mathrm{d}x.$$

Proof. For $Q = \delta_{(\omega,\psi)}$ it holds $\mathcal{F}_Q(\theta,\phi) = |\sin\theta\sin\omega\cos(\phi-\psi) + \cos\theta\cos\omega|$. Now multiplying $\mathcal{F}_Q^2(\theta,\phi)$ by $P_{2n}(\cos\theta)\sin\theta$ and integrating over θ and ϕ, we get three terms which are

$$\begin{aligned} J_1 &= \sin^2\omega \int_0^\pi \cos^2(\phi-\psi)\mathrm{d}\phi \int_0^\pi P_{2n}(\cos\theta)\sin^3\theta\mathrm{d}\theta = \frac{\pi}{2}\sin^2\omega \int_{-1}^1 (1-x^2)P_{2n}(x)\mathrm{d}x \\ J_2 &= \sin 2\omega \int_0^\pi \cos(\phi-\psi)\mathrm{d}\phi \int_0^\pi P_{2n}(\cos\theta)\sin^2\theta\cos\theta\mathrm{d}\theta = \\ &= \sin 2\omega \int_0^\pi \cos(\phi-\psi)\mathrm{d}\phi \int_{-1}^1 P_{2n}(x)x\sqrt{1-x^2}\mathrm{d}x \\ J_3 &= \cos^2\omega \int_0^\pi \mathrm{d}\phi \int_0^\pi P_{2n}(\cos\theta)\sin\theta\cos^2\theta\mathrm{d}\theta = \pi\cos^2\omega \int_{-1}^1 x^2 P_{2n}(x)\mathrm{d}x \end{aligned}$$

Hence it follows easily that $b_0 = \frac{2}{3}\pi$, $b_2 = \frac{2}{15}\pi(\cos^2\omega + \cos 2\omega)$ in (10) and all others $b_{2n} = 0$. Using $P_0(x) = 1$ and $P_2(x) = \frac{1}{2}(3x^2-1)$ we get $a_0 = \frac{1}{2\pi}$ and $a_2 = \frac{5}{4\pi U_0(\kappa)} \int_0^1 (3x^2-1)\exp(2\kappa x^2)\mathrm{d}x$.

We succeeded to get a closed formula which in fact can be obtained by a direct integration, too. Other choices of Q important in practice, e.g. based on finitely many probe orientations, do not always lead to explicit formulas.

Example. Let $Q = \frac{1}{3}(\delta_x + \delta_y + \delta_z)$ which is a natural sampling design based on three perpendicular probe orientations. Hence it easily follows $\mathcal{F}_Q(\theta,\phi) = \frac{1}{3}(|\sin\theta\cos\phi| + |\sin\theta\sin\phi| + |\cos\theta|)$ and

$$\int_0^\pi \mathcal{F}_Q^2(\theta,\phi)\mathrm{d}\phi = \frac{1}{9}(\pi + 2\sin^2\theta + 4|\sin 2\theta|). \tag{12}$$

Multiplying (12) by $P_{2n}(\cos\theta)\sin\theta$ and integrating over θ we obtain for the third term in (12) $\int_0^\pi |\sin 2\theta| P_{2n}(\cos\theta)\sin\theta\mathrm{d}\theta = 2\int_{-1}^1 |x|\sqrt{1-x^2}P_{2n}(x)\mathrm{d}x$ which is non zero for all n. To evaluate (10) by means of spherical harmonics we need to use numerical integration for each b_n.
On the other hand direct integration (1) for $\mathcal{R}$ in (11) yields

$$I_2(Q,\mathcal{R}) = \frac{1}{9}\left(1 + \frac{2}{\pi U_0(\kappa)} \int_0^1 (1-x^2)\exp(2\kappa x^2)\mathrm{d}x + \frac{4}{\pi U_0(\kappa)} \int_0^1 x\sqrt{1-x^2}\exp(2\kappa x^2)\mathrm{d}x\right),$$

which can be obtained by a single numerical integration.

BOOLEAN MODEL OF RANDOM SURFACES

In Beneš (1995), Beneš et al. (1995) several estimators of intensity of fibre and surface processes are investigated, based on projection measures. Their variances are derived from the variances of projection measures. We extend the known explicit results to the case of a Boolean surface model in $\mathbb{R}^3$. Besides the orientation factor in (8) we need to evaluate $\int g_B(r)\mathrm{d}f(r)$ assuming that B is ball.

Consider the model of Boolean random surface process in $\mathbb{R}^3$ where compact sets are circular disks with random radius x. Let $G(x)$ denote the distribution of this radius. In the first step we must evaluate the function

$$f(r,x) = \int_{\mathbb{R}^3} \eta_{K_0(x)}(B(r)+y)\eta_{K_0(x)}(\mathrm{d}y), \tag{13}$$

cf.(6), where $K_0(x)$ is a disk with radius x. Denote $S(r,x,y)$ the intersection area of two planar circles with radii r, x, respectively, and distance y between centres. Then $S(r,x,y) = \pi x^2$ for $x \leqq r$, $y \leqq r-x$; $S(r,x,y) = \pi r^2$ for $r \leqq x$, $y \leqq x-r$ and $S(r,x,y) = r^2 \arccos \frac{r^2+y^2-x^2}{2ry} + x^2 \arccos \frac{x^2+y^2-r^2}{2xy} - \frac{1}{2}\sqrt{D}$ else, where $D = (x+r+y)(x+r-y)(x-r-y)(r-x-y)$.

Using polar coordinates we get from (13) $f(r,x) = 2\pi\int_0^x yS(r,x,y)\mathrm{d}y$. After a long calculation we get

$$f(r,x) = \begin{cases} \pi^2 x^4 & 0 \leqq x \leqq \frac{r}{2}, \\ \pi^2 x^4 + 2\pi x^2(r^2-x^2)\arccos\frac{r}{2x} - \frac{\pi r}{4}(r^2+2x^2)\sqrt{4x^2-r^2} & x \geqq \frac{r}{2} \end{cases} \tag{14}$$

and hence

$$\frac{\partial f(r,x)}{\partial r} = \begin{cases} 0 & 0 \leqq x \leqq \frac{r}{2}, \\ 4\pi x^2 r \arccos\frac{r}{2x} - \pi r^2\sqrt{4x^2-r^2} & x \geqq \frac{r}{2} \end{cases}. \tag{15}$$

According to these facts we get

Proposition 3. *Let K_0 be a circular disk with random radius x having a distribution $G(x)$ which is independent on the orientation of K_0. Then for $f(r)$ from Corollary 2*

$$f(r) = \frac{1}{\overline{G}}\int_0^\infty f(r,x)\,\mathrm{d}G(x), \tag{16}$$

where $\overline{G} = \pi\int_0^\infty y^2\,\mathrm{d}G(y)$ and

$$\int_0^\infty g_B(r)\,\mathrm{d}f(r) = \frac{\pi}{\overline{G}}\int_0^\infty\int_{\frac{r}{2}}^\infty g_B(r)\{4x^2 r\arccos\frac{r}{2x} - r^2\sqrt{4x^2-r^2}\}\mathrm{d}G(x)\mathrm{d}r. \tag{17}$$

Example. Consider the Boolean model with random disks. We know that $\hat{\lambda} = \frac{\Phi_Q(B)}{\nu(B)\mathcal{F}_{\mathcal{R}Q}}$ is an unbiased intensity estimator. Using (4) we can estimate the variance of this estimator.

Suppose that the disks have a fixed radius R. Then in (16) $f(r) = \frac{1}{\pi R^2}f(r,R)$, using the derivative (15)

$$f'(r) = \begin{cases} 4r\arccos\frac{r}{2R} - \frac{2r^2}{R}\sqrt{1-(\frac{r}{2R})^2}, & 0 \leqq r \leqq 2R \\ 0, & r > 2R. \end{cases}$$

The integral $\int_0^\infty g_B(r)\mathrm{d}f(r)$ is then equal to

$$\int_0^\infty g_B(r)\mathrm{d}f(r) = \int_0^{2R} g_B(r)\left\{4r\arccos\frac{r}{2R} - \frac{2r^2}{R}\sqrt{1-(\frac{r}{2R})^2}\right\}\mathrm{d}r \tag{18}$$

and tends to $2\pi\int_0^\infty rg_B(r)\mathrm{d}r$ for $R\to\infty$ (disks tend to planes) in accordance to the second formula in (5).

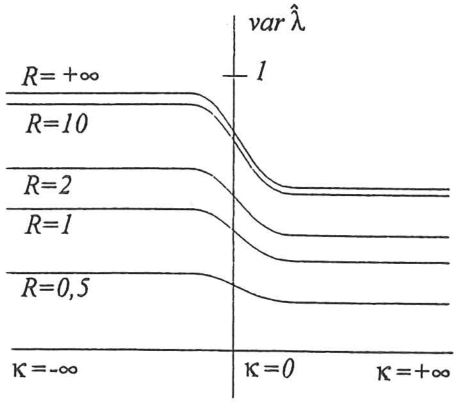

Fig.1: Variance of $\hat{\lambda}$ for different values of κ in (11) and R.

Let B be a ball with radius 1. In this special case we evaluate the variance $var\hat{\lambda}$ of the intensity estimator $\hat{\lambda}$ for $\mathcal{R}$ in (11) and Q in Proposition 2. For $\omega = 0$ and intensity $\lambda = 1$ we get this variance using (8), see Fig.1.

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