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PROPAGATION FUNCTION: TOWARDS CONSTANT TIME ALGORITHMS

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ABSTRACT

Let P_n be the regular n-sided polygon inscribed in a circle of radius 1 in the plane. The distance from x to y induced by P_n is the smallest size of the homothety of P_n centered at x and containing y. On X, simply connected planar set, the propagation function T^n_X is defined by $T^n_X(x) = \sup_{y \in X} d^n_X(x,y)$ where $d^n_X(x,y)$ is the geodesic distance in X, that is the lower bound of the length (induced by P_n) of the paths entirely lying inside X and linking x to y. Efficient algorithms for T^n_X are based on the following remark: the farthest points to any x in X may have only a few possible locations Y. In this paper, it is shown that, as in the convex case, there exists a set Y with at most n elements, such that $T^n_X(x) = \sup_{y \in Y} d^n_X(x,y)$. In the case of the square lattice equipped with the 4-connectivity distance, this theorem leads to an algorithm computing the propagation function by means of at most 7 geodesic balls, whatever the shape of X.

Key words: Propagation function, Geodesic distance, Digital metrics.

INTRODUCTION

The notion of **geodesic distance** has been introduced in mathematical morphology to take into account the fact that the paths linking two points (x, y) in a set X may be very long although the Euclidean distance from x to y is small (fig. 1). The geodesic distance d_X in X is defined as the lower bound of all the paths linking x and y which are totally included in X. This distance highly depends on the shape of X. Some mathematical properties may be found in (Maisonneuve and Lantuéjoul, 1984), (Lantuéjoul and Beucher, 1981) or (Schmitt and Mattioli, 1994) and their use in image analysis in (Lantuéjoul and Maisonneuve, 1984).

In this paper, we study a very useful function based on the geodesic distance, introduced in (Lantuéjoul and Maisonneuve, 1984) and called the **propagation function** T_X (fig. 2). It is defined as the distance to the farthest point in X:

$$\forall\, x \in X,\ T_X(x) \ = \ \sup\ \{d_X(x,y) \mid y \in X\} \qquad [1]$$

Function T_X gives rise to many morphological notions on an object X, such as:

Ends: they may be defined as the regional maxima (Maisonneuve, 1982) of T_X.

Length: the value of the global maximum of T_X is the maximal distance of two points in X according to the geodesic distance d_X.

Stretching factor (Maisonneuve and Lantuéjoul, 1984): it is defined as $\varrho(X) = \pi L^2(X)/4S(X)$ where $S(X)$ stands for the Euclidean surface area of X and $L(X)$ for the previously defined length of X.

All these notions correspond to the intuition we have and are very robust with respect to noise perturbation of the boundaries of the objects.

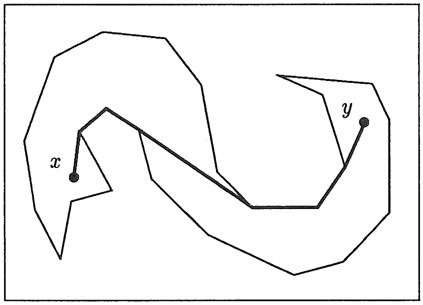


Fig. 1. Notion of geodesic distance.

DIGITAL DISTANCES

In order to compute in practice the propagation function on images, we have to measure the distances between pixels according to digital distances. Let us state in detail the construction of the propagation function in the case of these metrics. The best way to study these distances is to find a distance function in the continuous plane, whose restriction to the lattice precisely is the digital distance.

These continuous (non-euclidean) distances depend on a regular n-sided polygon P_n (n even) inscribed in the circle of radius 1. The derived distance d^n between any two points is then:

$$d^n(x, y) = \inf\{\lambda \mid y - x \in \lambda P_n\} \qquad [2]$$

Examples are given in fig. 2. The restriction of such metrics to the lattice may be interpreted as the distance in the sense of a weighted graph structure.

The length $\mathcal{L}(\gamma)$ of a **path** $\gamma : [a, b] \to \mathbb{R}^2$ is defined as for the Euclidean distance, by substituting to the Euclidean distance the d^n distance :

$$\mathcal{L}(\gamma) = \sup\left\{\sum_{i=1}^{p} d^n(\gamma(s_{i-1}), \gamma(s_i)),\ a = s_0 < s_1 < \cdots < s_{p-1} < s_p = b\right\} \qquad [3]$$

Now let X be a set in the plane. The d^n-geodesic distance d^n_X associated to d^n is as usual the lower bound of the length of paths linking x to y and totally included in X.

We know that in a simply connected closed set, for the Euclidean distance there always exists only one path of minimal length, called **geodesic arc**. Under the same assumptions on X, but for the d^n metrics, there may exist **many** geodesic arcs, one of them being the Euclidean one, which may be used to compute the d^n-geodesic distance d^n_X (Schmitt, 1989).

Then the propagation function T^n_X for d^n is:

$$\forall\, x \in X,\ T^n_X(x) = \sup\ \{d^n_X(x, y) \mid y \in X\} \qquad [4]$$

STRONG AND WEAK CONVEXITY

The existence of many geodesic arcs between two points leads to the definition of two distinct notions of convexity (Mattioli and Schmitt,1992).

The strong convexity: X is strongly convex if all the geodesic arcs between points of X are totally included in X.

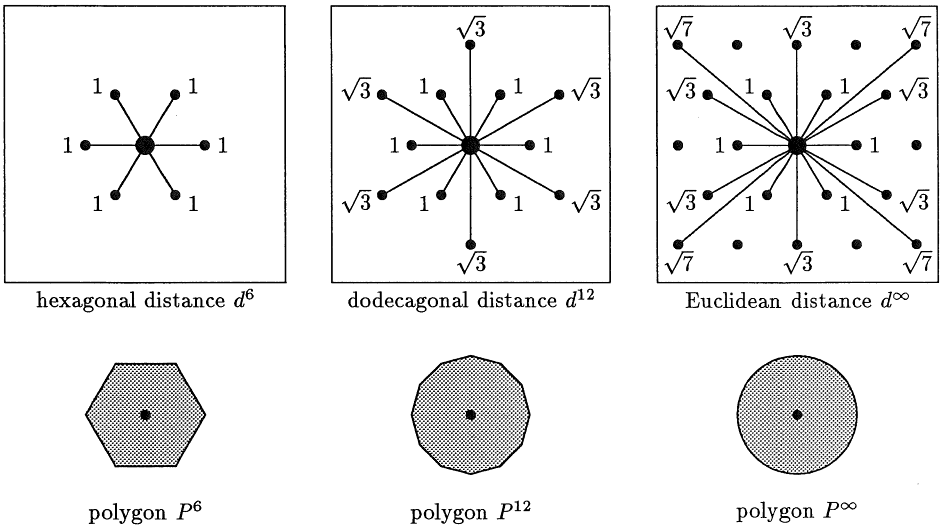


Fig. 2. Different graph structures on the hexagonal lattice and associated polygon.

The weak convexity: X is weakly convex if at least one of the geodesic arcs between points of X is totally included in X.

Remark: the intersection of strong convex sets is a strong convex set, but the intersection of weak convex sets is not always weakly convex (it may even be disconnected).

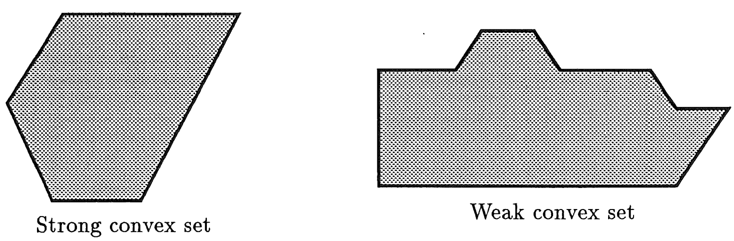


Fig. 3. Two notions of convexity for the hexagon P^6.

THE EXISTING ALGORITHM FOR T_X^n

The algorithm has been designed for X being a simply connected compact polygon the edges of which are parallel to those of P_n. It is based on the following idea: the farthest points cannot be anywhere. At first glance, they are on the boundary of X. For the Euclidean distance, this set may be minimal, as in the case of the unit disk. But for the d^n metrics, the following theorem shows that farthest points are very rare (Schmitt, 1989):

Theorem 1: *Let us denote the angle of two consecutive edges of P_n by $\alpha = \pi(1 - \frac{2}{n})$. At least one point y maximizing $d_X^n(x, y)$ has one of the two configurations (fig. 4):*

1- y is a corner point with angle $< \alpha$

2- y is any point on an edge, its end points being corner points with angle α and the isosceles triangle based on that edge and with opposite angle $\pi - \alpha$ being included in X.

In the convex case, more precisely, if X is weakly convex there exist at most n points verifying the configurations stated in theorem 1.

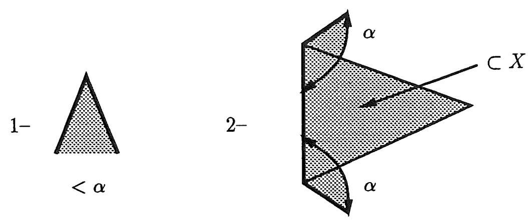


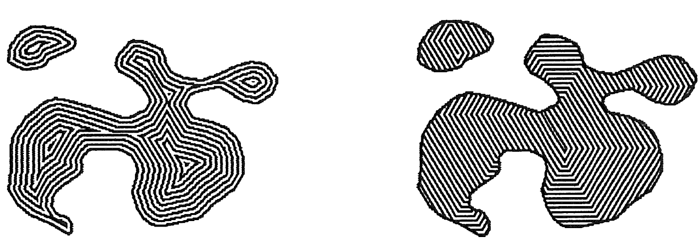
Fig. 4. Farthest point configurations.

THE FUNDAMENTAL THEOREM

A first remark is that, for any simply connected compact X, the lower section $T_\lambda = \{x,\ T^n_X(x) \leq \lambda\}$ of T^n_X are strongly d^n-geodesic convex sets: T_λ is the intersection of all d^n-geodesic balls of radius λ which are strong convex sets. In other words, T^n_X behaves like the opposite of a distance function defined on a weak convex set. This remark suggests that the number of farthest points is the same, X being weakly convex or not. This is in fact true:

Theorem 2: *Let X be a simply connected compact set. There exists a set Y of cardinality less or equal to n such that $T^n_X(x) = \sup_{y \in Y} d^n_X(x, y)$.*

This theorem is not only of theoretical interest, giving a new property of the propagation function, but also of practical interest.



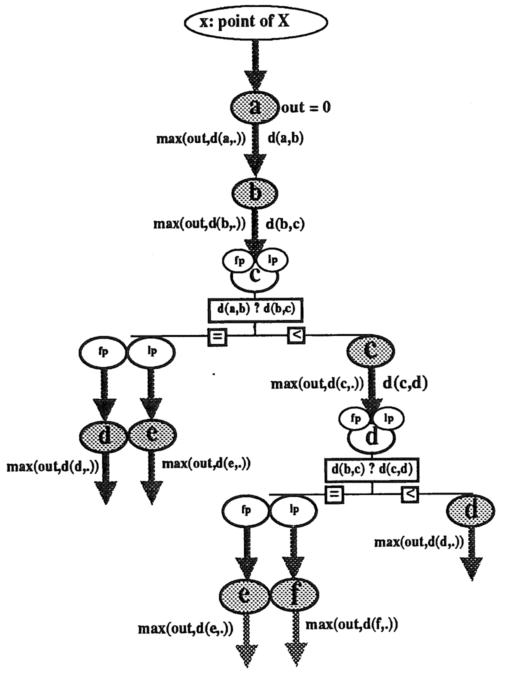
Distance function Propagation function

Fig. 5. Similarity between distance function and propagation function.

CONSTANT TIME ALGORITHM FOR T^4_X

The purpose of this section is to show how such ultimate farthest points may be computed. The principle is to chose a point x_0 of X, then to find the farthest point x_1 to x_0 in X. Then, by iteration, we find a sequence of points where x_p is the farthest point to x_{p-1} and so on. Due to the fact that we have equipped the plane with a d^n metrics, this sequence will oscillate between a couple of points and this after at most n iterations.

For the d^n metrics, we have not been able to derive an algorithm, except for the d^4 metrics. The flow chart diagram is presented in fig. 6. Starting from x_0, if the sequence x_p is composed of 4 different points (a, b, c, d), then, according to theorem 2, it can be shown that we have the four desired points. If not, we take the two oscillating points, say a and b. We compute $\max(d^4_X(x,a), d^4_X(x,b))$ for all x on X. The global minimum of this function is a segment with slope $\pm\frac{\pi}{4}$. The farthest points to the end points of this line segment (called fp and lp in the fig. 6) build then with a and b the desired set Y.



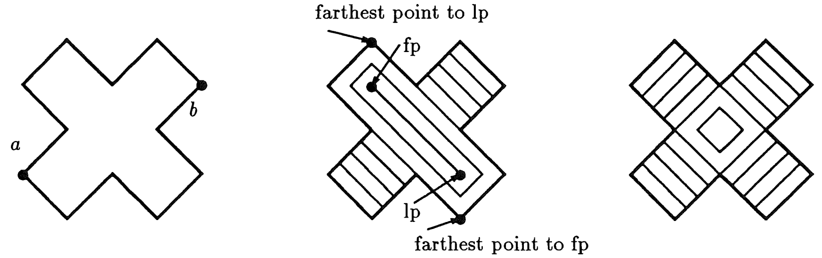
Explanations:

- Dark arrows indicate the computation of a farthest point, *i.e.* the computation of the geodesic distance function to a point.
- T^4_X is computed in the image named **out**. At each time a geodesic distance function is computed (say to point y), we take the maximum of this function with image **out**. This is expressed in the flow-chart by **max(out,d(y,.))**.
- fp and lp are the two end points of the global function computed so far. This minimum is a line segment.

Fig. 6. Constant time algorithm for the 4-connectivity square lattice.

CONCLUSION

This paper states a general structure theorem on the propagation function, showing that this function depends only on n points in the d^n metrics. We have derived a constant time algorithm for d^4 with 7 computations of d^4-geodesic ball at most. It remains an open problem to extend this algorithm to d^n metrics (at least d^6 or d^{12} metrics on the hexagonal grid), which are more isotropic and will lead to more reliable measurements on the objects.



a and *b* are a couple of farthest points | $\max(d^4_X(x,a), d^4_X(x,b))$ | Propagation function

Fig. 7. Example of successive steps of the algorithm (left branch of the algorithm presented in fig. 6). Only the level lines of the different functions are drawn.

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