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ESTIMATION OF ORIENTED NORMAL DIRECTION DISTRIBUTION VIA MIXED AREAS

Jan RATAJ

Mathematical Institute of the Charles University,
Sokolovská 83, 186 00 Praha 8, Czech Republic

ABSTRACT

An indirect method for the estimation of the area measure of a planar set X from the convex ring is proposed. At first, the mixed area $A(X, \vartheta K)$ of X with sufficiently enough rotations ϑ of a chosen convex test set K is estimated by means of the dilation area. Then, the area measure estimate is obtained by solving an integral equation. The estimation bias is discussed and several particular examples are presented.

Key words: direction distribution, unit outer normal, convex ring, dilation area, mixed area, area measure, estimation variance.

INTRODUCTION

Let X be a planar body from the convex ring (i.e. X can be represented as a finite union of convex compact sets with non-empty interiors). The oriented normal direction distribution is the distribution of the unit outer normal to X over the boundary ∂X. Multiplied by the total boundary length, it equals the area measure σ_X of X introduced originally for convex bodies and extended to finite unions by additivity (see Schneider, 1993). (The notion "area measure" follows from the spatial case, where $\sigma_X(B)$ is the surface area of all boundary points of X at which the unit outer normal falls within B.) When considering only the direction of the line normal to ∂X, the (non-oriented) normal direction distribution is obtained; it is an even distribution on the unit sphere S^1, depends only on ∂X and is usually estimated by the stereological method of linear intersection counts (Stoyan et al., 1987).

Linear probes are clearly not sufficient for the estimation of σ_X. The proposed method of estimation is based on the relation for mixed areas

$$A(X, K) = \frac{1}{2} \int_{S^1} h(K, u)\, \sigma_X(du) \tag{1}$$

where K is a convex body with support function $h(K, \cdot)$ ($h(K, u)$ is the distance from the origin of the support line to K perpendicular to $u \in \mathbf{S}^1$). The mixed area can be estimated by means of the relation

$$A(X, K) = \lim_{\varepsilon \searrow 0} \frac{A(X \oplus \varepsilon K) - A(X)}{2\varepsilon} \tag{2}$$

(Rataj, 1996). Note that the area difference $A(X \oplus \varepsilon K) - A(X)$ can be estimated by standard stereological methods.

When estimating σ_X by means of (1), we chose a fixed convex body K (test set) and estimate $A(X, \vartheta K)$ for a sufficiently dense net of rotations $\vartheta \in SO(2)$. Then, (1) becomes

$$A(X, \vartheta K) = \frac{1}{2} \int_{S^1} h(K, \vartheta^{-1} u)\, \sigma_X(du), \quad \vartheta \in SO(2). \tag{3}$$

An estimate of σ_X is then obtained as a solution of this integral equation, which can be obtained by using the Fourier transforms.

The proposed method can be adapted to the spatial case without great difficulties. The mixed area is replaced by mixed volume and, analogously to (2), it can be proved that

$$V(X, X, K) = \lim_{\varepsilon \searrow 0} \frac{V(X \oplus \varepsilon K) - V(X)}{3\varepsilon}.$$

The main difference is that the integral equation analogous to (3) must be solved by using the spherical harmonics.

The method is not strictly limited to sets from the convex ring. The relations (1,2,3) remain valid if X is a set of positive reach (a set system including both the convex sets and sets with C^2-smooth boundary) or a finite union of such sets. In such case, (1) is taken for definition of mixed area (volume) and (2) follows from a recent result of Rataj & Zähle (1995).

The area measure σ_X corresponds uniquely to the convex body called the convexification of X introduced by Weil (1995): in fact, the convexification is the unique (up to translations) convex body whose area function coincides with that of X. In his paper, Weil (1995) mentions two different estimation methods of the convexification (or, equivalently, of the area measure). The first one is based on the polygonal approximation of X (direct method) and the second one on the representation of X as a finite union of convex bodies and using the additivity of area measures.

ESTIMATION OF MIXED VOLUME

Let X belong to the convex ring. Relation (2) suggests us to use an estimator of

$$A_\varepsilon(X, K) = \frac{A(X \oplus \varepsilon K) - A(X)}{2\varepsilon}$$

as an estimator of the mixed area $A(X, K)$. The area $A(Z)$ of a set Z is estimated standardly by using a grid of test points of density λ (number of test points per unit area), with estimator variance $\lambda^{-1} A(Z)$ if the test points are chosen independently (for a regular point lattice, the variance is generally even smaller - see e.g. Matérn, 1989). It follows that the variance of the corresponding estimator $\hat{A}_\varepsilon(X, K)$ of $A_\varepsilon(X, K)$ fulfils in general

$$\operatorname{var} \hat{A}_\varepsilon(X, K) \leq \frac{A((X \oplus \varepsilon K) \setminus X)}{4\lambda\varepsilon^2} \cong \frac{A(X, K)}{4\lambda\varepsilon} \leq \frac{A(X, B)}{4\lambda\varepsilon},$$

assuming that K is contained in the unit ball B (we have used again (2)).

The quality of the mixed volume estimator depends critically on the bias

$$\Delta_\varepsilon(X,K) = A_\varepsilon(X,K) - A(X,K).$$

It is clear that ε must not be chosen too small, in order to keep the estimator variance of $A_\varepsilon(X,K)$ small enough. An upper bound for $\Delta_\varepsilon(X,K)$ has been found in Rataj (1996):

$$|\Delta_\varepsilon(X,K)| \leq \frac{h_\varepsilon(X)}{1-h_\varepsilon(X))} A(X,B) + \frac{\varepsilon\pi}{2(1-h_\varepsilon(X))}\chi(X),$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X and $h_\varepsilon(X)$ is a quantity defined by means of the exoskeleton $S(X^C)$ of X and with the property $\lim_{\varepsilon\to 0} h_\varepsilon(X) = 0$ and

$$h_\varepsilon(X) \cong \tilde{h}_\varepsilon(X) = 2\frac{A\big((S(X^C)\oplus\varepsilon B)\cap(\partial X\oplus\varepsilon B)\big)}{A(\partial X\oplus\varepsilon B)} \qquad (\varepsilon\to 0),$$

i.e. $h_\varepsilon(X)$ is approximately twice the area fraction of the ε-parallel set of the boundary of X covered by the ε-parallel set of the exoskeleton. This quantity can again been estimated by means of the point count method. The estimation of the mixed volume can thus procede in the following steps:

1. choose ε so that $h_\varepsilon(X)$ and, consequently, the bias $|\Delta_\varepsilon(X,K)|$ is small enough;
2. choose the density λ of the point grid so that the variance of $\hat{A}_\varepsilon(X,K)$ is small enough.

Of course we neglect the errors caused by the image delineation.

We illustrate the bias on two simple examples shown on Figure 1: choosing the interval with end points $(-1,0)$ and $(1,0)$ for the test set K, we have

$$\Delta_\varepsilon(X,K) = \begin{cases} 0, & \varepsilon\leq\rho, \\ 2a(1-\rho/\varepsilon), & \varepsilon\geq\rho \end{cases}$$

for the case a) and

$$\Delta_\varepsilon(X,K) = \begin{cases} b\varepsilon/\rho, & \varepsilon\leq\rho, \\ b(2-\rho/\varepsilon), & \varepsilon\geq\rho \end{cases}$$

for the case b).

ESTIMATION OF σ_X

As mentioned in the Introduction, the proposed method consists in the estimation of the mixed volume of X with (sufficiently enough) rotations of a chosen convex test set K. This requires the solution of the integral equation (3). Identifying S^1

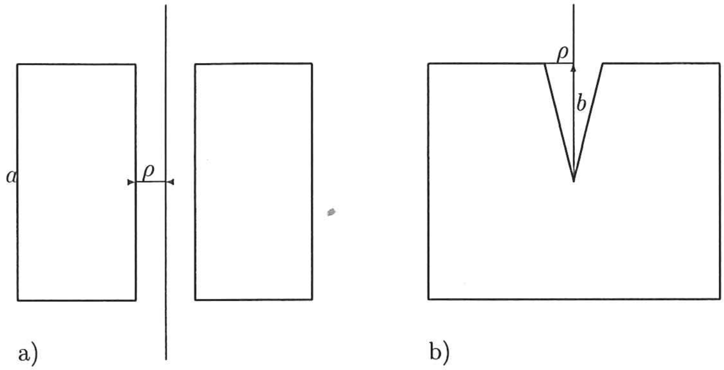

Figure 1:

with the interval $[0, 2\pi)$ and denoting by K^s the rotation of K by an angle s, we can rewrite (3) in the form

$$A(X, K^s) = \frac{1}{2} \int_{[0,2\pi)} h(K, t-s)\, \sigma_X(dt), \quad s \in [0, 2\pi). \tag{4}$$

The first question is whether (4) is theoretically solvable, i.e. whether the values $A(X, K^s)$ $(s \in [0, 2\pi))$ determine the measure σ_X uniquely. It has been shown in Rataj (1996) that (4) is solvable if and only if

$$a_k = \int_0^{2\pi} h(K, t) \mathrm{e}^{ikt} dt \neq 0 \quad \text{for } k = 0, 2, 3, 4, \ldots. \tag{5}$$

Sets fulfilling (5) cannot obviously be symmetric w.r.t. the origin. Examples of such sets are:

- sector with angle δ with either δ/π irrational or $\delta/\pi = p/2q$ with p odd,
- isosceles triangle with angle δ between the two sides of the same length which is not a rational multiple of π.

The practical solution of (4) can be carried out in different ways. We shall discuss here the approach based on the estimation of Fourier coefficients.

Denote

$$b_k = \int_{[0,2\pi)} \mathrm{e}^{ikt} \sigma_X(dt),$$

$$c_k = \int_{[0,2\pi)} \mathrm{e}^{iks} A(X, K^s)\, ds$$

the Fourier coefficients of σ_X and the function $s \mapsto A(X, K^s)$, respectively. Using (4) and (5), we find that

$$\bar{c}_k = a_k \bar{b}_k, \quad k = 0, 1, 2, \ldots.$$

Suppose that we have estimates $\hat{A}(X, K^{s_j})$ of $A(X, K^{s_j})$ at the points

$$0 \leq s_1 < s_2 < \cdots < s_N < 2\pi.$$

Then, we can estimate the coefficients c_k by

$$\hat{c}_k = \sum_{j=1}^{N}(s_j - s_{j-1})e^{iks_j}\hat{A}(X, K^{s_j}) \tag{6}$$

(we set $s_0 = s_N - 2\pi$) and the Fourier coefficients of σ_X by

$$\hat{b}_k = \hat{c}_k/\bar{a}_k, \quad k \neq 1. \tag{7}$$

The estimation bias of c_k fulfills

$$|\hat{c}_k - c_k| \leq k \max_j |s_j - s_{j-1}| + \max_j |\hat{A}(X, K^{s_j}) - A(X, K^{s_j})|$$

(we have used the fact that k is the Lipschitz constant of $s \mapsto e^{iks}$). The method described in Section 2 enables us to obtain estimates of mixed area with estimation error sufficiently small in probability. Here we need, however, these errors to be bounded uniformly for all directions s_j. This can be achieved only after additional smoothing applied to $\hat{A}(X, K^{s_j})$, so that the resulting estimators of b_k are consistent.

In practice we estimate, of course, only a few coefficients b_k, $k \leq k_0$ (the variance of $\hat{b}_k$ increases with k). A question is, what is the quality of the approximation of σ_X by the density function in the form of the trigonometrical polynomial

$$g_{k_0}(s) = b_0 + \sum_{k=0}^{k_0} b_k e^{iks}.$$

The example of X being a rectangular triangle is shown on Figure 2.

APPLICABILITY OF THE METHOD

The proposed method of estimation of oriented normal direction distribution can be used for planar sets with "enough smooth" boundary (for such sets, the bias upper bound $h_\varepsilon(X)$ tends to 0 quickly enough). For the test set K, any of the examples mentioned in the previous section can be used. In fact, we need not fulfil strictly the condition on K being convex, since it can be shown that

$$|A(X \oplus \varepsilon K) - A(X \oplus \varepsilon \mathrm{conv}(K))| = o(\varepsilon).$$

Thus, instead of a triangle we can use a there-point test set K, in which case the dilation can be obtained then as a union of the object with its two translates (third-order analysis). The mixed areas must be estimated with a high accuracy and, for the estimation of the Fourier coefficients of σ_X, a preliminary smoothing of the estimated mixed areas should be done.

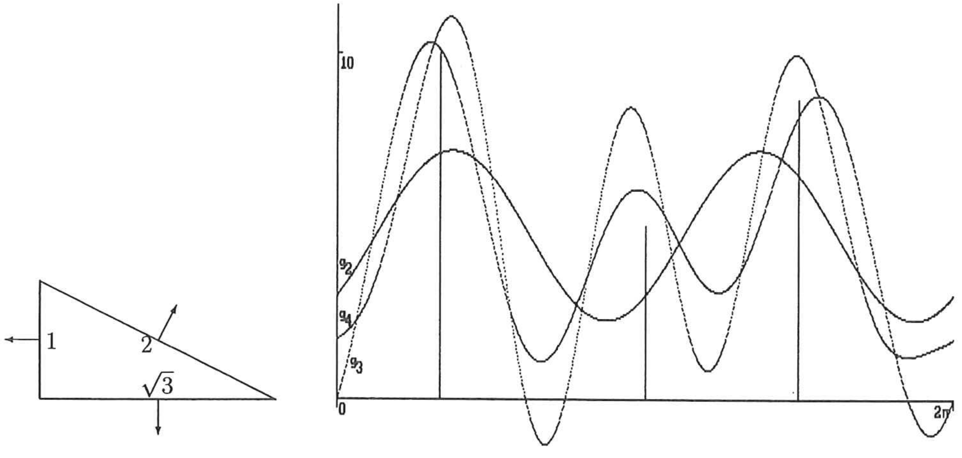

Figure 2: The rectangular triangle X and the corresponding approximations g_2, g_3 and g_4 of its area measure (the exact discrete area measure is indicated by the vertical segments of lengths proportional to the edge lengths).

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