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SPHERICAL CONTACT DISTANCES IN NEYMAN-SCOTT PROCESS OF REGULAR CLUSTERS

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ABSTRACT

The Neyman-Scott cluster process of regular 2^k-tuples - vertices of a k-cube of random edge length in $\mathbb{R}^d$, $k=0,\dots,d$, is considered. The attention is focused on the properties of the spherical contact distribution function $H(\ell)$. It is shown that the corresponding probability density function $h(\ell)$ is in certain sense intermediate between $h_p(\ell)$ of the parent process and $h_{cl}(\ell)$ of the Poisson point process of the daughter process intensity λ_{cl}. Particular cases of point pairs and 2^d-tuples of constant size in $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$ as well as the effect of the edge length distribution are treated in detail and the results are presented in a graphical form.

Key words: Neyman-Scott cluster process, regular 2^k-tuples, spherical contact distribution function

INTRODUCTION

In the present contribution, an attractive combination of randomness and regularity in the arrangement of a spatial point pattern is considered, namely the Boolean model Ξ (in $\mathbb{R}^d$) of regular clusters C_k formed by the vertices of a k-dimensional cube, $0\le k\le d$, In the full generality, we consider a cluster process of the Neyman-Scott type with a typical cluster N_0 being either a void set or a regular 2^k-tuple C_k of points - vertices of a k-cube centred in the origin O, $k=0,\dots,d$, with the probabilities $p_{-1}, p_0, p_1, \dots, p_d$ fulfilling the condition $\Sigma_{-1}^{d} p_j=1$. The size of the cube can be a random variable with the distribution $F_k(\xi)=\Pr(a_k\le\xi)$, where $2a_k$ is the cube edge. Finally, the orientation of the cluster can be either random (preferably isotropic) or arbitrary fixed. Such a special type of the Neyman-Scott process generalizes the Gauss-Poisson process (Milne and Westcott, 1972; Stoyan *et al.*, 1987).

SPHERICAL CONTACT DISTANCES

The spherical contact distribution function $H(\ell)$ of Ξ can be derived using the well known formula valid for the Boolean model (Stoyan *et al.*, 1987)

$$H(\ell)=1-\exp[-\lambda_p \mathrm{E}\nu_d(N_0\oplus B(O,\ell))], \tag{1}$$

where $\nu_d(A)$ is the d-dimensional volume of a set A, $B(O,r)$ is a d-ball of radius r centred in the origin O. Further, the volume of the unit d-ball will be denoted by κ_d and the surface area of the unit d-sphere by O_d.

Introducing the mean volume of dilation $G_k(\ell)=\mathrm{E}\nu_d[C_k(\xi)\oplus B(O,\ell)]$ and using Lemma 1 from the Appendix, we can write for 2^k-tuples C_k with a size distribution $F_k(\xi)$

$$G_k(\ell) = 2^k \sum_{i=0}^{k} (-1)^i \binom{m}{i} \int_0^{\infty} \xi^d \, H_i^d(\ell/\xi)\mathrm{d}F_k(\xi), \quad k>0, \tag{2}$$

and $G_0(\ell)=\kappa_d\ell^d$. $H_i^d(\ell/\xi)$ are the intersection volumes defined in the Appendix. Summing the contributions of clusters of different types, we obtain the spherical contact distribution function of the considered generalized Gauss-Poisson process

$$H(\ell)=1-\exp(-\lambda_p G(\ell)), \tag{3}$$

where $G(\ell)=\sum_0^d p_k G_k(\ell)$. The probability density function $h(\ell)=\mathrm{d}H(\ell)/\mathrm{d}\ell$ is the main object of the present study, because it more or less sensitively reflects the spatial arrangement of the points of the process. By comparison of its shape with that ones of the PPP of either parent (λ_p) or daughter (λ_{cl}) intensity, the effect of the cluster size and and of its distribution can be examined. The effect of the cluster size can be seen at best if ξ is constant, say ξ=a. Then for k>0

$$G_k(\ell)=2^k \sum_{i=0}^{k} (-1)^i \binom{m}{i} \, a^d \, H_i^d(\ell/a). \tag{4}$$

The general forms of $G_1(\ell)$ are given in the Appendix (Eq. (A2)), the formulae for $H_i^d(\eta)$, $d=1,2,3$, $i=1,.,d$ are in Tbl. 1.

Table 1. Intersection volumes $H_i^d(\eta)$ $(u = \eta^2-1)$

$d \backslash i$	0	1	2
1	2η	$\eta-1$	———
2	$\pi\eta^2$	$-\sqrt{u}+\eta^2\text{arc cos } \eta^{-1}$	$1 - \sqrt{u} + \eta^2\left(\frac{\pi}{4} - \text{arc sin } \eta^{-1}\right)$
3	$\frac{4}{3}\pi\eta^3$	$\frac{\pi}{3}(2\eta^3-3\eta^2+1)$	$\frac{2}{3}\left[\sqrt{u-1}-(3u+2)\text{arc cos } u^{-1/2}+\eta^3\text{arc cos } u^{-1}\right]$

RESULTS

Simplified versions of the generalized GP process have been examined in detail, namely "pure" processes of clusters of selected type. Consequently, p_m=1 for m=k and zero otherwise and the corresponding process Ξ_k of regular 2^k-tuples C_k in $\mathbb{R}^d$ will be denoted by the symbol $\{2^k,d\}_t$, where t=a for clusters of constant size a and $t=[a,b]$ for clusters of variable size distributed uniformly on $[a,b]$. The parent Poisson point process intensity was set λ_p=1 in all cases.

The results are presented in a graphical form as a sequence of plots of the p.d.f. $h(\ell, a\text{=const.})$ for $a=0,\Delta a,2\Delta a,...$ (clusters of constant size), or, as

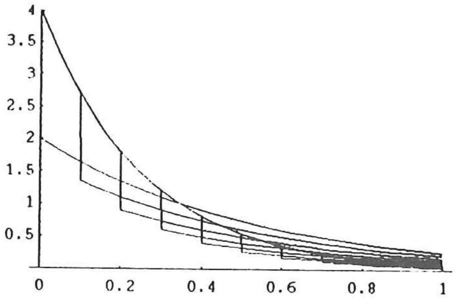

Fig. 1. Case $\{2^1,1\}_a$ - pairs of fixed size on a line. P.d.f. $h(\ell, a=\text{const.})$ for $a=0, 0.1, \ldots, 0.9$. The scale units are $1/\lambda_p$ for ℓ and λ_p for $h(\ell,a)$.

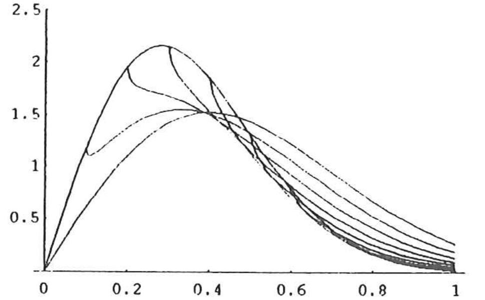

Fig. 2. Case $\{2^1,2\}_a$ - pairs of fixed size in a plane. P.d.f. $h(\ell, a=\text{const.})$ for $a=0, 0.1, \ldots, 0.9$. The scale units are $1/\sqrt{\lambda_p}$ for ℓ and $\sqrt{\lambda_p}$ for $h(\ell,a)$.

a sequence of plots of p.d.f. $h(\ell, a=\text{const.}, b=a+s)$ for selected values of a and $s=0, \Delta s, 2\Delta s, \ldots$ (clusters of size uniformly distributed on $[a,b]$). The following cases have been examined: point pairs $\{2^1,1\}_a$, $\{2^1,2\}_a$, $\{2^1,2\}_{[a,a+s]}$ and $\{2^1,3\}_a$, quadruples $\{2^2,2\}_a$ and 8-tuples $\{2^3,3\}_a$ - see Fig's 1 - 6.

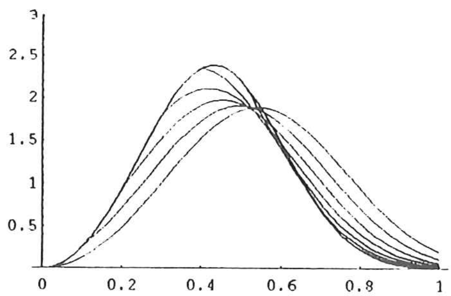

Fig. 3. Case $\{2^1,3\}_a$ - pairs of fixed size in a 3D space. P.d.f. $h(\ell, a=\text{const.})$ for $a=0, 0.1, \ldots, 0.9$. The scale units are $\lambda_p^{-1/3}$ for ℓ, a and $\lambda_p^{1/3}$ for $h(\ell,a)$.

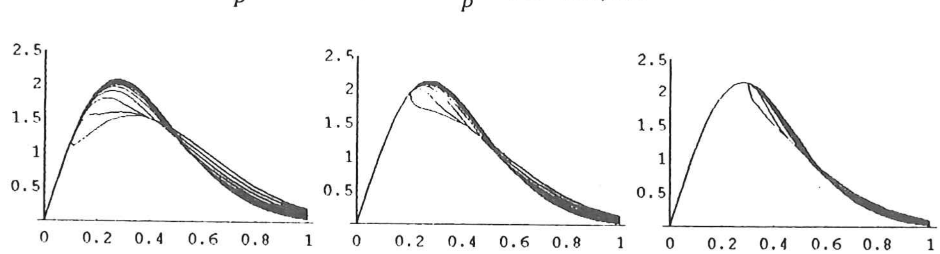

Fig. 4. Case $\{2^1,2\}_{[a,a+s]}$ - pairs of the uniform random size $\xi \in [a, a+s]$ in a plane. P.d.f. $h(\ell, a+s=\text{const.})$ for $a=0.1$, 0.2 and 0.3, $s=0, 0.1, \ldots, 0.9-a$. The scale units as in Fig. 2.

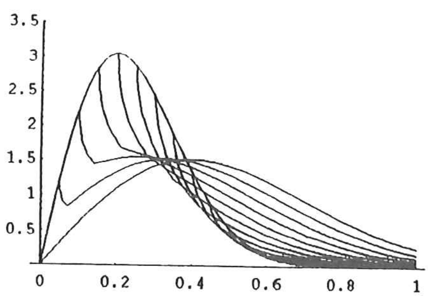

Fig. 5. Case $\{2^2, 2\}$ - quadruples of fixed size in a plane. P.d.f. $h(\ell, a=\text{const.})$ for $a=0, 0.05, \ldots, 0.9$. The scale units as in Fig. 2.

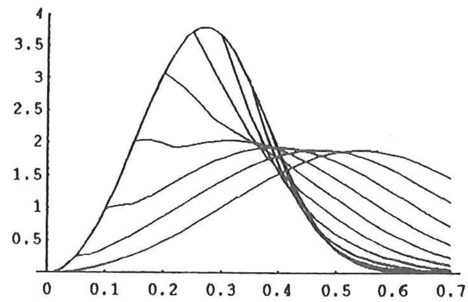

Fig. 6. Case $\{2^3, 3\}$ - 8-tuples of fixed size in a 3D space. P.d.f. $h(\ell, a=\text{const.})$ for $a=0, 0.05, \ldots, 0.6$. The scale units as in Fig. 3.

DISCUSSION

The p.d.f. of spherical contact distances $h(\ell)$ describes the relative frequency of the populations of points $x \in \mathbb{R}^d \setminus \Xi$ that "see" the point pattern Ξ in the same manner, which means that their distance to the closest point of the pattern is just ℓ. The points lying very far from any point of the pattern see even the closest cluster as a whole, namely as one multi-point placed at a Poissonian position. Their $h(\ell)$ is that one of the parent PPP of intensity λ_p, $h_p(\ell)$ say. On the other hand, there is a community of points x lying so close to some point of the cluster, that no other point of the same cluster can lie in the contacting ball $B(x, \ell)$. Such points see in their neighbourhood only one point of the pattern in a Poissonian position and their $h(\ell)$, $\ell < \min[\xi]$, is that one of the PPP of intensity λ_{cl}, $h_{cl}(\ell)$ say. All other points are in an intermediate situation and their frequency is influenced by the both features of the pattern - Poissonian positions of cluster centres and regularity of daughter positions around it.
This situation is clearly seen from Eq. (A2), which is valid for point pairs but can be suitably generalized even for clusters of higher order. For any k, we can write $H(\ell)=H_{cl}(\ell)$ for $\ell<\min[\xi]$ and $H(\ell)=H_p(\ell).Q(\ell)$ for $\ell\geq\min[\xi]$, where $\lim_{\ell\to\infty} Q(\ell)=1$. Similar relations hold also for $h(\ell)$. Hence $h_{cl}(\ell)$, $h_p(\ell)$ are "master curves" of $h(\ell)$, which accomplishes a continuous passage between them depending on the distribution $F(\xi)$ and on the values of k and d. It is well known that $H_p(\ell)$, $H_{cl}(\ell)$ follow the Weibull distribution $F(\ell)=1-\exp[-(\ell/\alpha)^d]$ (Johnson and Kotz, 1970), where $\alpha=(\lambda\kappa_d)^{-1/d}$, i.e. $\alpha_{cl}=(\lambda_{cl}\kappa_d)^{-1/d}=2^{-k/d}\alpha_p$. The Weibull distribution is unimodal with the mode $\ell^*=\alpha(1-1/d)^{1/d}$ and the mean $\sigma=\alpha\Gamma(1+1/d)$. Hence, we can write $\ell^*=\gamma\sigma$, where $\gamma=(1-1/d)^{1/d}/\Gamma(1+1/d)$ approaches 1 with increasing d and only weakly depends on d ($\gamma=0.80$, 0.98, 1.03 and 1.04 for $d=2,3,4$ and 10, resp.). With increasing dimension d, ℓ^* very slowly increases from zero to infinity (for $\lambda=1$, $\ell^*=0, 0.40, 0,54$ and 0.90 for $d=1,2,3$ and 10, resp.). Denoting by h^* the modal frequency $h(\ell^*)$, we can simply prove the relation $h^*_{cl}/h^*_p=\ell^*_p/\ell^*_{cl}=2^{k/d}$, $d>1$. For given d, the distance between the modes ℓ^*_p, ℓ^*_{cl} increases with increasing k (see Fig. 3 and 6), for fixed k, it decreases with increasing d

(see Fig. 2 and 4).

i) Point pairs $\{2^1,d\}$. The cases d=1,2,3 have already been solved in a slightly different setting of the problem by Coleman (1974).

The case $\{2^1,1\}$ with $h(\ell)$ being a combination of two simple exponentials is a unique one; note that only in this case is $h(\ell)$ discontinuous (in the point $\ell=a$ - Fig. 1). The remaining two cases are typical for d even (complicated shape of $h(\ell)$ with cusps - Fig. 2) and odd (a combination of two smooth curves of similar shapes, the first of which is the Weibull p.d.f. of the degree of d and the other is $h(\ell)\sim P'_d(\ell)\exp[-P_d(\ell)]$ with the polynomial $P_d(\ell)$ of degree d given by Eq. (A2) - Fig. 3).

It can be seen by inspections of Fig's 1, 2, 3 that the greatest difference between $h(\ell)$ and $h_{cl}(\ell)$ occurs if the cluster size parameter a is smaller than or comparable with the mode ℓ^*_{cl}. Recalling the above given relation between the mode and the mean of a Weibull distribution, we can conclude that if the inter-daughter distance $2a$ exceeds $4\sigma_{cl}=2^{2-1/d}\sigma_p$, then the corresponding GPP of point pairs is not much different from the PPP with λ_{cl}, at least from the point of view of the spherical contact distance (here σ_p, σ_{cl} are clearly the mean spherical contact distances in the PPP of parent and daughter process intensity, respectively).
If the cluster size ξ is a random variable on an interval $[a,b]$, the above described situation can be seriously modified. Such a situation is shown in Fig. 4 assuming the uniform distribution of ξ on $[a,a+s]$ and examining $h(\ell)$ for selected fixed values of a and gradually increasing s. The cusps at $\ell=a$ quickly vanish and $h(\ell)$ approaches $h_{cl}(\ell)$. Consequently, a cluster process with a considerable dispersion of sizes would be little different from the PPP of the same intensity.
ii) Higher order clusters $(2^k,d), 1<k\le d$. The situation now is quite analogical to the case of point pairs, only the effects are more pronounced. If $\ell<a$ (i.e. only one point of a cluster can be included in a ball of radius ℓ), $h(\ell)\equiv h_{cl}(\ell)$ of the PPP with the intensity $2^k\lambda_p$. Critical points of $h(\ell)$ (an abrupt change of slope) can occur at the values $\ell=a\sqrt{i}$, $i=1,2,\ldots,k$, at which a higher number, namely 2^i, of points of the cluster can be embedded in the test ball of radius ℓ.
With increasing ℓ, $h(\ell)$ again approaches $h_p(\ell)$ of the parent PPP. The above mentioned relation between modes and modal frequencies explains certain similarity between the cases $(2^d,d)$ -compare Fig. 5 and 6.
The graphical representation of $h(\ell)$ clearly shows the advantages as well as shortcomings of this function in the description of spatial arrangements in point patterns based on the PPP. It reflects sensitively the differences in a close neighbourhood of individual points, namely in a ball of radius smaller than $4\sigma_{cl}=2^{2-k/d}\sigma_p$, or, taking into account that σ approximately equals $0.5\lambda^{1/d}$ for d=1,2,3, smaller than $2^{1-k/d}.\lambda_p^{1/d}$. Within this range of ℓ, the shape of $h(\ell)$ quantitatively characterizes the examined clusters, i.e. their size as well as the number of daughter points. A similar result has been found also for regular clusters implanted in a translation lattice of points (Saxl and Rataj, 1990).
As $H(\ell)$ is in the considered case the capacity functional (Choquet capacity)

$T_{\Xi}(B(0,\ell))=\Pr(B\cap\Xi\neq\emptyset)$ of the cluster process Ξ on the class of d-balls, the methods proposed by Molchanov (1991) for the estimation of T, λ, k and a can be applied. In this paper, the above results concerning the process of C_k-clusters are obtained by considering a more general case of a Boolean model with the primary grain being a non-random set A_m of m points characterized by the minimum inter-daughter distance a and the diameter $D=\mathrm{Diam}(A_m)$. Even in this case $H(\ell)=H_{cl}(\ell)$ for $\ell<a$ and $H(\ell)\longrightarrow H_p(\ell)$ for $\ell\gg D$, which can be used to estimate m and λ. Further also, the first deviation of $H(\ell)$ from the Weibull distribution $H_{cl}(\ell)$ occurs at $\ell=a$ (compare Fig's 1÷6), which can be used to estimate a. Unfortunately, these methods break down whenever clusters are random with $a\longrightarrow 0$.
The estimation of parameters of the Matern cluster process (daughters are uniformly distributed in a circle of radius R and their number has a Poisson distribution with the mean m) has been recently discussed and tested by Stoyan (1992) and the estimation based on $H(\ell)$ was not recommended because of its weak dependence on model parameters. Better results have been obtained by estimation based on the K-function or L-function. It remains an open question, whether such a conclusion is unavoidable also in the present case. It can be expected that the analysis based on $H(\ell)$ could give satisfactory results in the case of clusters of fixed size. Otherwise, the results presented in Fig. 4 for clusters of variable size confirm the weak dependence of $H(\ell)$ on model parameters in more complicated cases. Promising can be also another approach. The Voronoi mosaics generated by the planar Gauss-Poisson process of the above considered type (pairs and quadruples of points) have been tested and compared with the Poisson-Voronoi mosaic of the density λ_{cl} by Kohútek and Saxl (1993); higher moments of cell area and perimeter distributions differ considerably within the whole range $0<a\leq 0.5/\sqrt{\lambda_p}$ and can be detected up to $a\simeq 1/\sqrt{\lambda_p}$.

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APPENDIX

We will prove the following Lemma:

Lemma 1. *Let $C_m(\xi)=\{0,2\xi\}^m$ be the 2^m-tuple of vertices of an m-cube embedded in $\mathbb{R}^d$ and $B(0,\ell)$ the d-ball of radius ℓ centred in the origin 0. Then the volume of the spherical dilation $C_m(\xi)\oplus B(0,\ell)$ is*

$$\nu_d(C_m(\xi)\oplus B(0,\ell))=2^m\sum_{i=0}^{m}(-1)^i\binom{m}{i}\xi^d H_i^d(\ell/\xi), \qquad \text{(A1)}$$

where $H_0^d(\eta)=\kappa_d\eta^d$ is the volume of a d-ball of radius η, $H_i^d(\eta)=0$ for $0\le\eta^2\le i$ and $H_i^d(\eta)=\int_1^{\eta_i}H_{i-1}^{d-1}\left[\sqrt{\eta^2-\zeta^2}\right]d\zeta$ for $\eta^2>i\ge 1$, where $\eta_i=\sqrt{\eta^2-(i-1)}$.

Proof: Let $C_m\equiv C_m(1)$. Then $C_m\oplus B(0,\eta)=\bigcup_{\zeta\in C_m}B(\zeta,\eta)$ and by symmetry we can write

$$\nu_d\Big(\bigcup_{\zeta\in C_m}B(\zeta,\ell)\Big)=2^m\,\nu_d\left[\bigcup_{\zeta\in C_m}B(\zeta,\eta)\cap\left[(-\infty,1]^m\times(-\infty,\infty)^{d-m}\right]\right]=$$

$$=2^m\nu_d\left[B(0,\eta)\cap\left[(-\infty,1]^m\times(-\infty,\infty)^{d-m}\right]\right]=$$

$$=2^m\left\{\nu_d(B(0,\eta))-\nu_d\left[B(0,\eta)\cap\left[\bigcup_{i=1}^{m}\{x\in\mathbb{R}^d;x_i\ge 1\}\right]\right]\right\}=$$

$$=2^m\left\{\nu_d(B(0,\eta))-\sum_{i=1}^{m}(-1)^m\binom{m}{i}\nu_d\left[B(0,\eta)\cap\{x\in\mathbb{R}^d;x_1,\dots,x_i\ge 1\}\right]\right\}.$$

Denoting the volume $\nu_d B(0,\eta)$ by $H_0^d(\eta)$ and $\nu_d[B(0,\eta)\cap\{x\in\mathbb{R}^d;x_1,\dots,x_i\ge 1\}]$ by $H_i^d(\eta)$, inserting $\ell=\xi\eta$ and using the homogeneity of degree d of the volume ν_d, we obtain (A1). Further

$$\nu_d[B(0,\eta)\cap\{x\in\mathbb{R}^d;x_1,\dots,x_i\ge 1\}]=\int_1^{\infty}dx_1\dots\int_1^{\infty}dx_i\int_{-\infty}^{\infty}dx_{i+1}\dots\int_{-\infty}^{\infty}\chi_B(x)dx_d\,.$$

The integration of the characteristic function $\chi_B(x)$ of $B(0,\eta)$ over $\mathbb{R}^{d-i}$ (the last $(d-i)$ integrals) gives the volume of a $(d-i)$-ball of radius $\left[\eta^2-\sum_{j=1}^{i}x_j^2\right]^{1/2}$, hence

$$H_i^d(\eta)=\kappa_{d-i}\int_1^{y_1}dx_1\dots\int_1^{y_i}\left[\eta^2-\sum_{j=1}^{i}x_j^2\right]^{\frac{d-i}{2}}dx_i\,,$$

where $y_1=\left[\eta^2-(i-1)\right]^{1/2}$ and $y_j=\left[\eta^2-(i-j)-\sum_{j=1}^{i-1}x_j^2\right]^{1/2}$, $1\le j<i$. The recurrence relation then follows immediately.

It follows from the definition of $H_i^d(\eta)$, $i>0$, that their meaning is the $2^{-i}\times$ volume of the intersection of 2^i balls of radius η centred in the points forming C_i. An important property of $H_i^d(\eta)$, $i=0,\dots,d$, is their linear dependence. It can be shown that $H_d^d(\eta)=(-1)^d\left[1-\sum_{i=0}^{d-1}(-1)^i 2^{i-d}\binom{d}{i}H_i^d(\eta)\right]$ for $\eta\geq\sqrt{d}$, , which can be used to calculate $H_3^3(\eta)$ missing in Tbl. 1.

An explicit solution can be given for the spherical dilation $\nu_d(C_1(a)\oplus B(O,\ell)) = 2\kappa_d\ell^d-2a^dH_1^d(\ell/a)$ of a point pair of constant size a (the point spacing $2a$) in $\mathbb{R}^d$. We obtain

$$\nu_d(C_1(a)\oplus B(O,\ell))=\begin{cases}2\kappa_d\ell^d \text{ for } \ell\leq a,\\ \kappa_d\ell^d\ [1+h_d(a/\ell)] \text{ for } \ell\geq a,\end{cases} \tag{A2}$$

where $h_d(\tau)=\begin{cases}\tau\tilde{\mathcal{P}}_{d-1}^{(d)}(\tau) \text{ for } d \text{ odd},\\ \frac{2}{\pi}\arcsin\tau+\sqrt{1-\tau^2}\ \mathcal{P}_{d-1}^{(d)}(\tau) \text{for } d \text{ even}\end{cases}$. Here $\tilde{\mathcal{P}}_{d-1}^{(d)}(\tau)$ and $\mathcal{P}_{d-1}^{(d)}(\tau)$ are the polynomials of the degree $(d-1)$ with alternating coefficients

$$\alpha_{2j+1}^{(d)} = \frac{\kappa_{2j}}{\pi}\sum_{i=j}^{\frac{d-2}{2}}\frac{O_{2i+2}\,\kappa_{2i-2j}}{O_{2i+1}\,\kappa_{2i}}(-1)^j,\quad \alpha_{2j}=0,\ j=0,1,\dots,\frac{d-2}{2} \text{ for } d \text{ even and}$$

$$\tilde{\alpha}_{2j}^{(d)} = \frac{2O_{d-2j}\,\kappa_{2j}}{O_{d+1}(2j+1)}(-1)^j,\quad \tilde{\alpha}_{2j+1}=0,\ j=0,1,2,\dots,\frac{d-1}{2} \text{ for } d \text{ odd}.$$

Then we obtain

Table 2. Polynomials $\tilde{\mathcal{P}}_{d-1}^{(d)}(\tau)$ and $\mathcal{P}_{d-1}^{(d)}(\tau)$.

$\tilde{\mathcal{P}}_0^{(1)}(\tau)$	1	$\mathcal{P}_1^{(2)}(\tau)$	$\frac{2\tau}{\pi}$
$\tilde{\mathcal{P}}_2^{(3)}(\tau)$	$\frac{1}{2}(3-\tau^2)$	$\mathcal{P}_3^{(4)}(\tau)$	$\frac{2}{3\pi}(5\tau-2\tau^3)$
$\tilde{\mathcal{P}}_4^{(5)}(\tau)$	$\frac{1}{8}(15-10\tau^2+3\tau^4)$	$\mathcal{P}_5^{(6)}(\tau)$	$\frac{2}{15\pi}(33\tau-26\tau^3+8\tau^5)$

Note that $h_d(\tau)=1-2\tau^dH_1^d(\tau^{-1})/\kappa_d$ is defined and increasing on $[0,1]$ and $h_d(0)=0$, $h_d(1)=1$.