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ON ESTIMATION OF THE SHAPE OF AN ISOTROPIC GRAIN IN THE BOOLEAN MODEL

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ABSTRACT

A Boolean model is considered where the typical grain ωM results from a random isotropic rotation of a deterministic convex set M of positive volume. An estimator for the set M is proposed.

Key words: Boolean model, empirical capacity functional, estimation, random closed set, typical grain.

INTRODUCTION

The Boolean model Ξ in $\mathbf{R}^d$ is defined to be the union

$$\Xi = \bigcup_{x_i \in \Pi_\Lambda} (x_i + \Xi_{0,i}) \tag{1}$$

of iid random sets $\Xi_0, \Xi_{0,1}, \ldots$ (called the grains) driven by the stationary Poisson point process Π_Λ with intensity λ. Estimators for numerical parameters of Ξ are well-known, see Serra (1982) and Stoyan, Kendall and Mecke (1987). Estimators for the mean body (the Aumann expectation) of the grain were considered in Weil (1991, 1993).

Here we consider the case where the typical grain $\Xi_0 = \omega M$ is obtained by a random isotropic rotation ω of a deterministic convex set M. This set is supposed to have a positive Lebesgue measure. Since it is convex, M coincides with the closure of its interior.

Note that because of the isotropy the mean body is not informative, since it is equal to the ball with the same surface area as M. The approach here follows to some extent the idea elaborated in Molchanov (1992, 1993) for the case of completely deterministic grain Ξ_0. It yields a set-valued estimator for the grain which is based on the examination of the tails of the covariance function (which is obtained through two-point covering probabilities). Namely, the covariance $C(r\mathrm{e}) = \mathbf{P}\{o \in \Xi, r\mathrm{e} \in \Xi\}$ decreases to the square p^2 of the volume fraction p as $r \to \infty$ and the value r providing *larger* than $p^2 - \varepsilon$ values

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of $C(r\mathbf{e})$ gives an ε-biased estimator for the width function of the grain for each direction $\mathbf{e} \in \mathbf{S}^{d-1}$. Here $\varepsilon > 0$ is a bias parameter. The width function of M is determined as $b_M(\mathbf{e}) = h_M(\mathbf{e}) + h_M(-\mathbf{e})$, where h_M is the support function of M. Let us suppose that the width function b_M takes its maximum $d_M = \sup_{\mathbf{e}} b_M(\mathbf{e})$ at a unique point, denoted by $\mathbf{e}_0$.

By definition, the covariance is given by the capacity functional of Ξ on the family of two-point sets. In this paper it will be shown that the *capacity functional of three-point sets* may be used to estimate the shape of the grain when $\Xi_0 = \omega M$. This corresponds to the fact that three-point covering probabilities determine the distribution of our Boolean model, see Lešanovský and Rataj (1990) and also Serra (1982, p.498).

FINITE POINT COVERING PROBALITIES

The capacity functional of the Boolean model (1) is given by

$$T(K) = \mathbf{P}\{K \cap \Xi \neq \emptyset\} = 1 - \exp\{-\lambda \boldsymbol{E}\mu_d(\Xi_0 \oplus \check{K})\}\,, \tag{2}$$

where μ_d is the Lebesgue measure in $\mathbf{R}^d$, $\check{K} = \{-x : x \in K\}$ and K is a compact set, see Stoyan et al. (1987). In particular, for finite-point K we define

$$p(x_1, x_2, \ldots, x_n) = T(\{-x_1, -x_2, \ldots, -x_n\})\,. \tag{3}$$

Evidently, for $n = 1$ the value $p(x) = p$ does not depend on x.

The covariance $C(x)$ is related to the function $p(x_1, x_2)$ on two-point sets, namely, $C(x) = 2p - p(o, x)$, where o is the origin. Furthermore, put

$$\psi(x_1, x_2, \ldots, x_n) = -\log(1 - p(x_1, x_2, \ldots, x_n))\,. \tag{4}$$

As above, $\psi(x) = \psi = -\log(1-p)$ for $n = 1$. It follows from (2) that

$$\begin{aligned}\psi(x_1, x_2, \ldots, x_n) &= \psi(o, x_2 - x_1, \ldots, x_n - x_1) \\ &= \lambda \boldsymbol{E}\mu_d(\Xi_0 \cup (\Xi_0 + x_2 - x_1) \cup \cdots \cup (\Xi_0 + x_n - x_1))\,.\end{aligned}$$

Furthermore,

$$2\psi - \psi(o, r\mathbf{e}) = \lambda \boldsymbol{E}\mu_d(\omega M \cap (\omega M + r\mathbf{e})) = \lambda \gamma_M(r)\,,$$

where $\mathbf{e}$ is a unit vector, $\gamma_M(r)$ is the isotropized covariance function of M, see Stoyan and Stoyan (1992, p.140). Introduce the function ϕ as follows,

$$\begin{aligned}\phi_M(o, x, y) &= \boldsymbol{E}\mu_d(\omega M \cap (\omega M + x) \cap (\omega M + y)) \\ &= \lambda^{-1}(\psi(o, x, y) + 3\psi - \psi(o, x) - \psi(o, y) - \psi(o, y - x))\,.\end{aligned}$$

In particular, $\phi_M(o, x, x) = \gamma_M(\|x\|)$.

A natural estimator for the capacity functional and, therefore, for covering probabilities, is the empirical capacity functional $\hat{T}$ considered in Molchanov (1991, 1992) and Molchanov and Stoyan (1993). The functional $\hat{T}$ is evaluated by the observation of Ξ within a certain window W. For formulating limit theorems, W is replaced by a family of expanding windows W_s, with $W_s \uparrow \mathbf{R}^d$ for $s \to \infty$. Then estimators $\hat{T}_s$, $s \geq 0$ is considered. For compact K,

$$\hat{T}_s(K) = \mu_d((\Xi \oplus \check{K}) \cap (W_s \ominus K))/\mu_d(W_s \ominus K)\,.$$

Using $\hat{T}_s$, the empirical functions $\hat{p}_s$ and $\hat{\psi}_s$ are defined analogously to (3) and (4). The Glivenko-Cantelli theorem for the capacity functional (see Molchanov, 1991) ensures that, for each compact K_0 and $n \geq 1$,

$$\nu_s(n, K_0) = \sup_{x_1,\dots,x_n \in K_0} |p(x_1,\dots,x_n) - \hat{p}_s(x_1,\dots,x_n)| \to 0 \quad \text{a.s. as} \quad s \to \infty \,. \tag{5}$$

The same is valid for $\hat{\psi}_s$ provided $T(K_0) < 1$.

ESTIMATION OF THE SHAPE

In the following we use a similar approach as in Molchanov (1992, 1993). First, we define a set $M(\varepsilon, \delta)$ which approximates M and is determined through the covering probabilities and some "bias" parameters ε and δ (these parameters determine the quality of approximation). Replacing the covering probabilities by their empirical counterparts produces a *biased estimator* of M.

For $\varepsilon \in [0,1]$ put

$$\begin{aligned} r(\varepsilon) &= \sup\{r \geq 0 \colon \boldsymbol{E}\mu_d(\omega M \cap (\omega M + r\mathrm{e})) \geq \varepsilon\mu_d(M)\} \\ &= \sup\{r \geq 0 \colon 2\psi - \psi(o, r\mathrm{e}) \geq \varepsilon\psi\} \,. \end{aligned} \tag{6}$$

Then, for $\varepsilon \downarrow 0$, we get $r(\varepsilon) \uparrow d_M = b(\mathrm{e}_0)$ (the maximum of the width function).

Furthermore, put $\Omega_\varepsilon = \{\omega \colon \omega M \cap (\omega M + r\mathrm{e}_0) \neq \emptyset\}$. Then $\Omega_\varepsilon \downarrow \{\omega_0\}$ as $\varepsilon \downarrow 0$, where ω_0 is the unit element of the *group of rotations*.

For $0 < \delta < 1$ let us define the set

$$M(\varepsilon, \delta) = \{x \colon \phi_M(o, r(\varepsilon)\mathrm{e}_0, x) \geq \delta\varepsilon\} \,. \tag{7}$$

Theorem 1. *In the Hausdorff metric $M(\varepsilon,\delta) \to M \oplus \Delta_0$ as $\varepsilon \downarrow 0$, where Δ_0 is the singleton $M \cap (M + d_M\mathrm{e}_0)$.*

PROOF. Evidently,

$$M_*(\varepsilon) = \bigcup_{\omega \in \Omega_\varepsilon} \{y \colon \omega M + y \supset \Delta_\varepsilon\} \subset M(\varepsilon, \delta) \subset \bigcup_{\omega \in \Omega_\varepsilon} \{y \colon \omega M + y \cap \Delta_\varepsilon \neq \emptyset\} = M^*(\varepsilon) \,,$$

where

$$\Delta_\varepsilon = \bigcup_{\omega \in \Omega_\varepsilon} \omega M \cap (\omega M + r(\varepsilon)\mathrm{e}) \downarrow \Delta_0 \quad \text{as} \quad \varepsilon \downarrow 0$$

(since the width function b_M has only one maximum point). Furthermore,

$$M_*(\varepsilon) = \bigcap_{\omega \in \Omega_\varepsilon} \{y \in \omega(M \ominus \Delta'_\varepsilon\} \,,$$

where $\Delta'_\varepsilon = \bigcup_{\omega \in \Omega_\varepsilon} \omega^{-1}\Delta_\varepsilon \downarrow \Delta_0$ as $\varepsilon \downarrow 0$. Since $\Omega_\varepsilon \downarrow \{\omega_0\}$, we get $M_*(\varepsilon) \uparrow M + d_M\mathrm{e}$ and $M^*(\varepsilon) \downarrow M + d_M\mathrm{e}$. Henceforth, $M(\varepsilon, \delta)$ approximates M up to a shift. □

Replacing the finite-point covering probabilities in (6) and (7) by their empirical counterparts yields the random closed set $\hat{M}_s(\varepsilon, \delta)$, which can be used as an estimator of $M(\varepsilon, \delta)$. Meanwhile, the empirical analogue of $r(\varepsilon)$, see (6) will serve as a (biased) estimator of d_M.

The following theorem establishes the strong consistency of the *set-valued* estimator $\hat{M}_s(\varepsilon, \delta)$ within an arbitrary compact set.

Theorem 2. *For each compact K_0, in the Hausdorff metric,*

$$\hat{M}_s(\varepsilon, \delta) \cap K_0 \longrightarrow M(\varepsilon, \delta) \cap K_0 \quad a.s. \ as \quad s \to \infty .$$

PROOF. First, $\hat{r}_s(\varepsilon) \to r(\varepsilon)$ a.s. as $s \to \infty$, since the function $\gamma_M(r)$ has a strictly negative derivative for each $r < d_M$. Thus, $r(\varepsilon + \zeta_s) \le \hat{r}_s(\varepsilon) \le r(\varepsilon - \zeta_s)$ for $\zeta_s \to 0$ a.s. as $s \to \infty$. Moreover, $\zeta_s = \mathcal{O}(\nu_s(3, K_0))$. Without loss of generality we can take the same ζ_s to prove as in Molchanov (1992, 1993) that

$$M(\varepsilon - \zeta_s, \delta + \zeta_s) \subset \hat{M}_s(\varepsilon, \delta) \subset M(\varepsilon + \zeta_s, \delta - \zeta_s) .$$

Now the result follows from the continuity of finite-point covering probabilities and (5). □

Since the estimator of $M(\varepsilon, \delta)$ is determined by three-point covering probabilities, the estimation method can be applied to censored observations as it has been already done in Molchanov (1992) for the case of two-point covering probabilities.

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