ACTA STEREOL 1987; 6/SUPPL II: 101 — 103
ISS COMMEMORATIVE-MEMORIAL VOLUME
ORIGINAL SCIENTIFIC PAPER

A STEREOLOGICAL PROBLEM FOR RANDOM LINES

Joseph Mecke

Department of Mathematics, FSU Jena, DDR-6900 Jena, UHH,
German Democratic Republic

ABSTRACT

It is shown that the distribution of a random line in the plane can be determined from the probabilities of hitting segments which are directed to a fixed point. The relation of the problem to the theory of the Radon transform is explained.

1. INTRODUCTION

Let γ be a random line in the Euclidean plane. For every segment s in the plane we denote the probability that γ intersects s by $c(s)$. The results of Ambartzumian (1976) concerning line measures imply that the distribution P of γ is uniquely determined by the system of all probabilities $c(s)$. Moreover, explicit formulas are known which enable the calculation of P from the $c(s)$ (Mecke and Nagel, 1982).

In this paper it is demonstrated that for the determination of P one only needs the probabilities $c(s)$ for segments which are directed to a fixed point. The problem is very theoretical and may be regarded as belonging to the two-dimensional mathematical stereology. But it seems to be interesting because not only moments but a complete two-dimensional distribution is calculated.

In section 4 the special case is considered that the distribution P of γ has a density p. The Radon transform $\hat{p}$ of p can be represented as a function depending on the probabilities $c(s)$. A general solution without using densities is given in section 3.

The analogous problem for a finite number of random lines may be treated in a similar manner.

2. PROBLEM

In the following the Euclidean plane is identified with the space $R^2 = \{(x_1, x_2) :$

$x_1, x_2 \in R\}$, where R is the set of all real numbers. The set of lines in R^2 is denoted by G.

Let γ be a random line, i.e. a random element in G. Its distribution is a probability measure P on G. The problem described in the introduction can be formulated in the following way: Let G_0 be the set of all lines containing the origin $\mathbf{O} = (0,0)$. We suppose $P(G_0) = 0$. For every $h \in G_0$ the intersection point of the random line γ with h is denoted by $\boldsymbol{\eta}(h)$. Is it possible to determine the distribution P of γ from the distributions of all $\boldsymbol{\eta}(h)$, where h runs through the set G_0?

An affirmative answer is given in the next section.

3. SOLUTION

The inner product of $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in R^2$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$, the norm of $\mathbf{x}$ by $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$.

Let $\mathbf{a}(g)$ be the foot of the perpendicular from the origin $\mathbf{O}$ to the line $g \in G \backslash G_0$. Put

$$\mathbf{z}(g) = |\mathbf{a}(g)|^{-2} \mathbf{a}(g) \; ; \quad g \in G \backslash G_0 \; .$$

Then the random line γ is uniquely described by the random point $\boldsymbol{\zeta} = \mathbf{z}(\gamma)$. (Note $P(G_0) = 0$.) The distribution of $\boldsymbol{\zeta}$ (and hence P) is uniquely determined by the characteristic function (Fourier transform)

$$f(\mathbf{t}) = \mathrm{E} \exp[i \langle \mathbf{t}, \boldsymbol{\zeta} \rangle] \; ; \quad \mathbf{t} \in R^2 \; , \tag{1}$$

where E denotes the mathematical expectation and $i = \sqrt{-1}$ the imaginary unit.

The problem is now to calculate f from the distributions of the random points $\boldsymbol{\eta}(h)$; $h \in G_0$. Given $\mathbf{t} \neq \mathbf{O}$, let $h[\mathbf{t}]$ be the line containing $\mathbf{O}$ and $\mathbf{t}$. Then the intersection point of $h[\mathbf{t}]$ with the line $g \in G \backslash G_0$ is given by

$$h[\mathbf{t}] \cap g = \langle \mathbf{t}, \mathbf{z}(g) \rangle^{-1} \mathbf{t} \; . \tag{2}$$

If $\langle \mathbf{t}, \mathbf{z}(g) \rangle = 0$ (g parallel to $h[\mathbf{t}]$), both sides of (2) are said to equal ∞. According to (2) the random intersection point $\boldsymbol{\eta}(h[\mathbf{t}]) = \gamma \cap h[\mathbf{t}]$ satisfies

$$|\mathbf{t}|^2 / \langle \boldsymbol{\eta}(h[\mathbf{t}]) \, , \mathbf{t} \rangle = \langle \mathbf{t}, \boldsymbol{\zeta} \rangle \; . \tag{3}$$

If $\boldsymbol{\eta}(h[\mathbf{t}]) = \infty$, we say that the left-hand side of (3) equals 0.

From (1) and (3) we obtain the result

$$f(\mathbf{t}) = \mathrm{E} \exp\left[i |\mathbf{t}|^2 / \langle \boldsymbol{\eta}(h[\mathbf{t}]) \, , \mathbf{t} \rangle \right] \; ; \quad \mathbf{t} \neq \mathbf{O} \; . \tag{4}$$

This formula implies that for fixed $\mathbf{t} \neq \mathbf{O}$ the value $f(\mathbf{t})$ only depends on the distribution of the intersection point of the random line γ with the line $h[\mathbf{t}] \in G_0$.

4. PROBABILITY DENSITIES

We suppose that the distribution of $\boldsymbol{\zeta}$ has a density p on R^2. Then for given $u \in R^2$ with $|\mathbf{u}| = 1$ the random variable $\langle \mathbf{u}, \boldsymbol{\zeta} \rangle$ has a distribution density $w(\mathbf{u}; \cdot)$ with

$$w(\mathbf{u}; r) = \int_{-\infty}^{\infty} p(r\mathbf{u} + x\mathbf{u}^{\perp})\, dx \ ,$$

where $\langle \mathbf{u}, \mathbf{u}^{\perp} \rangle = 0$, $|\mathbf{u}^{\perp}| = 1$. In other words, $w(\mathbf{u}; r)$ is the Radon transform of p for the line $g = g(r\mathbf{u})$ which perpendicularly intersects $h[\mathbf{u}]$ at the point $r\mathbf{u}$, i.e. $\mathbf{z}(g) = r^{-1}\mathbf{u}$. Using the notation of Helgason (1980) we can write

$$w(\mathbf{u}; r) = \hat{p}(g(r\mathbf{u})) \ ; \ \ |\mathbf{u}| = 1 \ . \tag{5}$$

According to (3) we find

$$\langle \mathbf{u}, \boldsymbol{\zeta} \rangle = \langle \eta(h[\mathbf{u}]), \mathbf{u} \rangle^{-1} \ ; \ \ |\mathbf{u}| = 1 \ .$$

Hence

$$w(\mathbf{u}; r) = b(\mathbf{u}; 1/r)/r^2 \ , \tag{6}$$

where $b(\mathbf{u}; \cdot)$ is the distribution density of the random variable $\langle \eta(h[\mathbf{u}]), \mathbf{u} \rangle$.

Formulas (5) and (6) imply

$$\hat{p}(g) = |\mathbf{z}(g)|^2\, b\big(|\mathbf{z}(g)|^{-1}\mathbf{z}(g);\ |\mathbf{z}(g)|\big) \ ; \ \ g \in G \backslash G_0 \ .$$

Now the desired probability density p can be calculated by the inverse Radon transform.

REFERENCES

Ambartzumian RV. A note on pseudometrics in the plane. Z Wahrscheinlichkeitstheorie verw Geb 1976; 37: 145–55.

Helgason S. The Radon transform. Ch.1. Basel: Birkhäuser, 1980.

Mecke J, Nagel W. Über Schnitte von Geradenprozessen. Jena: Geobild '82. Wissenschaftliche Beiträge der FSU, 1982: 148–53.