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ON THE COVARIANCE OF THE BOUNDARY OF A SET

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ABSTRACT
A generalization of the set covariance is considered which may be appropriate to describe second-order properties of the boundary of a set, such as anisotropies or clustering. A stereological formula is derived which requires observations on pairs of test lines and avoids the measurement of angles.

Key words: Crofton's formula, integral geomety, set covariance, stereology

QUANTITIES OF FIRST AND SECOND ORDER

Consider a set X which is embedded in a specimen B. The classical stereological formulae show how the volume, surface and curvature (resp. breadth if X is convex) of X can be estimated when the profiles on appropriate sections are observed. The mentioned parameters of the set are to be considered as "total" or "mean specific" ones, depending on the chosen approach. For example, when X is regarded as a fixed set (this implies a design-based approach) one can estimate the total volume of X. If X is considered as a (part of a) realization of a (stationary) random set then one estimates the mean specific volume (volume fraction) of this set - a model-based approach. Both the total and the mean parameters are called *first* order characteristics of a set. They bear only limited information about the "structure" or "texture" of a set, e.g. no information about anisotropy or cluster effects.

It is a usual step to investigate *second* order features in the hope of finding quantities which express various structural properties of a set (cf. Stoyan et al., 1988, Ripley, 1981). The basic idea of a second order analysis is to explore the "behaviour" of the set at two distinct points simultaneously. The set covariance (see Matern, 1960, Matheron, 1975) is closely related to the co-occurence matrices in digital image processing. For a pair of points (with a determined difference vector) one observes whether both points fall into the set X. Such a statistical analysis serves to quantify structural characteristics of the "volume measure" of X. This association to volume will be intuitively clear to stereologists since point sections allow an estimation of the volume only (and not, e.g., the surface). However, it is not yet known how much information about X covariances contain. (For the special case of planar convex polygons it is proved, that the set covariance determines X uniquely up to translation and reflection (Nagel,1989).)

For a second order analysis of the boundary of X, or for a fibre system, other sampling techniques are necessary besides observations in pairs of points. Stoyan (1981), Ambartzumjan(1981) and Schwandtke (1988) proposed (for two-dimensional sets) the use of a line as section element and require the measurement of *distances* between intersection points *and intersection angles*. This idea has been developed further by Jensen et al. (1989) and Zaehle (1989).

Another approach was chosen by Hanisch(1985) and by Nagel(1987). *Pairs of lines* are used as section elements - a straightforward analogy to pairs of points. In the present note a basic formula relating to the last mentioned method is given. It is emphasized that angles need not to be measured.

A GENERALIZED CROFTON-FORMULA FOR THE COVARIANCE OF THE BOUNDARY

For $X \subseteq B$ denote by ∂X the boundary of X. The length measure (in the case of a planar set X) resp the surface measure are denoted by S. For $B_1, B_2 \subseteq B$ we define

$$C_X(B_1, B_2) = \int dx\ S(\partial X \cap (B_1+x)) \cdot S(\partial X \cap (B_2+x)) \tag{1}$$

and call it *the boundary covariance* with respect to (B_1, B_2). The integrand is the product of the boundary contents in a pair of test sets B_1, B_2, and integration is with respect to all translations of this pair. (If X is a set of fibres one has to write X instead of ∂X).

Depending on the shapes and the mutual location of B_1 and B_2 these values $C_X(B_1,B_2)$ will express some structural features of ∂X, for example anisotropies.

The classical Crofton-formula of integral geometry permits the determination of a surface area (or boundary length resp.) by point counting on linear sections. We use it in the following form:

$$S(\partial X \cap (B_j+x)) = c^{-1} \int dg_j \sum_{x_i \in g_j \cap \partial X} 1_{B_j+x}(x_i), \qquad j=1,2.$$

Here $c=\pi$ in three-dimensional space (surfaces) and $c=2$ in two-dimensinal plane. dg denotes the element of the invariant line measure, with

$$\int_{g \cap K \neq \emptyset} dg = \begin{matrix} 2\pi^2 & \text{(space)} \\ 2\pi & \text{(plane)}, \end{matrix}$$

where K represents the unit sphere.

A twofold application of this formula to (1) yields the formula

$$C_X(B_1,B_2) = c^{-2} \int dg_1 \int dg_2 \sum_{x_1 \in g_1 \cap \partial X} \; \sum_{x_2 \in g_2 \cap \partial X} V(B_1 \cap (B_2+x_1-x_2)) \tag{2}$$

where V denotes the volume or area resp.

Note that application of formula (2) requires more than point counting: all pairs of intersection points x_1, x_2 arising *from different test lines* must be taken into account, and the *difference* x_1-x_2 *of their coordinates* is necessary for determining the weight $V(B_1 \cap (B_2+x_1-x_2))$.

If, e.g., B_1, B_2 are rectangles in the plane with edges parallel to the axes then these weights are the areas of smaller rectangles or zero. The computation is very simple in this case. An appropriate choice of such rectangles permits a cluster or an orientation analysis resp.

CONCLUDING REMARKS

Consider the problem of a statistical estimation of the boundary covariance (1) using (2). The integrations $\int dg_1$, $\int dg_2$ can be interpreted (up to a normalizing constant) as expected values w.r.t. isotropic uniform random (IUR) lines g_1, g_2 through the block B. The product form of the integrals indicates that the two lines have to be independent. Here occurs a typical sampling procedure in stereology: one has to realize a series of independent pairs of independent IUR-lines. Perhaps it may be a useful fact that any pair of lines in three-dimensional space can be embedded into a pair of parallel planes (think about Sterio's disector!) as well as into a pair of "vertical planes".

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