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PAPER

GRAIN SIZE DISTRIBUTIONS IN STATIONARY GRAIN MODELS

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ABSTRACT

The so-called systematic point counting is proposed in order to determine the grain size distribution in stationary grain models. This procedure does not depend on the samples of the grain model, which is advantageous in automatic data analysis. The theoretical background is a general point process approach, where in contrast to the classical Boolean model, no assumptions of stochastic independence are made.

INTRODUCTION

Consider a stationary grain model in the plane R^2, i.e. the union $\bigcup A_i$ of a sequence of random grains $\{A_i\}$ with randomly positioned germs G_i and with random linear size factor S_i. Thereby, the assumption of stationarity means that the marked point process $\Phi = \{[G_i, A_i]\}_i$ is stationary, i.e. its probability distribution is invariant with respect to translations. Assume moreover that the grains are non-overlapping, identically shaped and identically orientated convex random subsets of R^2. For example, let the grains be equilateral triangles with the lateral length as size factor (fig. 1).

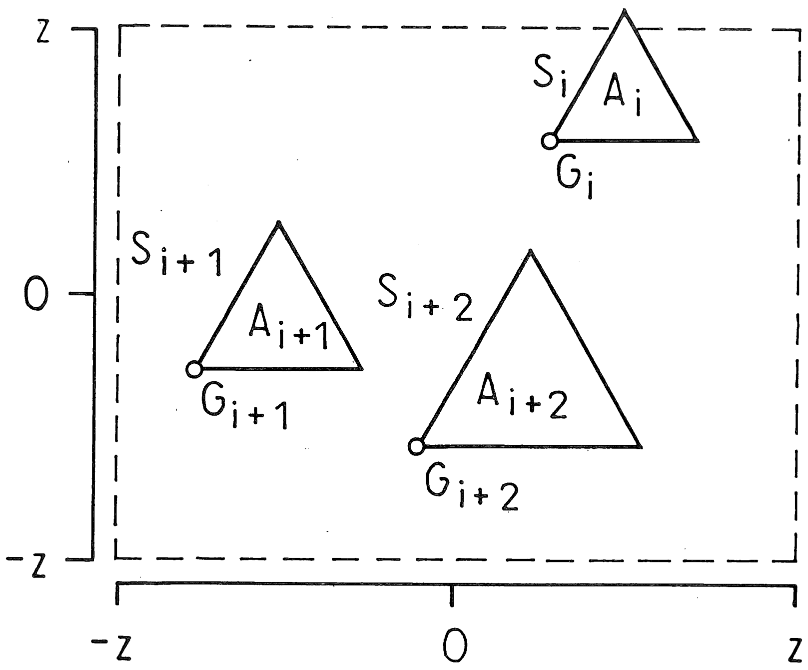

Fig. 1. Random triangles

We are interested in the distribution function $F(x) = P(S > x)$, $0 \leqq x < \infty$, of the size S of a typical grain. This grain size distribution can be estimated by the ratio

$$\frac{\text{Number of grains } A_i \text{ in } [-z,z]^2 \text{ with } S_i > x}{\text{Total number of grains } A_i \text{ in } [-z,z]^2}$$

within a sample window $[-z,z]^2$. More precisely, under some ergodicity conditions we have

$$\frac{\# \; i \; : G_i \in [-z,z]^2, \; S_i > x}{\# \; i \; : G_i \in [-z,z]^2} \xrightarrow[z \to \infty]{} P(S > x)$$

with probability one.

However, the direct counting of the quantities appearing in this ratio may be not convenient in some cases of automatic data analysis, when the computer gets the data from a large population. Thus, a method is proposed how the grain size probabilities $P(S > x)$ can be obtained from systematic point counting, a speech used by Underwood (1970).

METHOD

Let be given a fixed deterministic lattice of nodes N_n and a (for the given grain structure) suitably chosen direction $\vec{g}$ (fig. 2).

In the following this lattice is used in order to get an estimation for some area-weighted characteristic of the grain model.

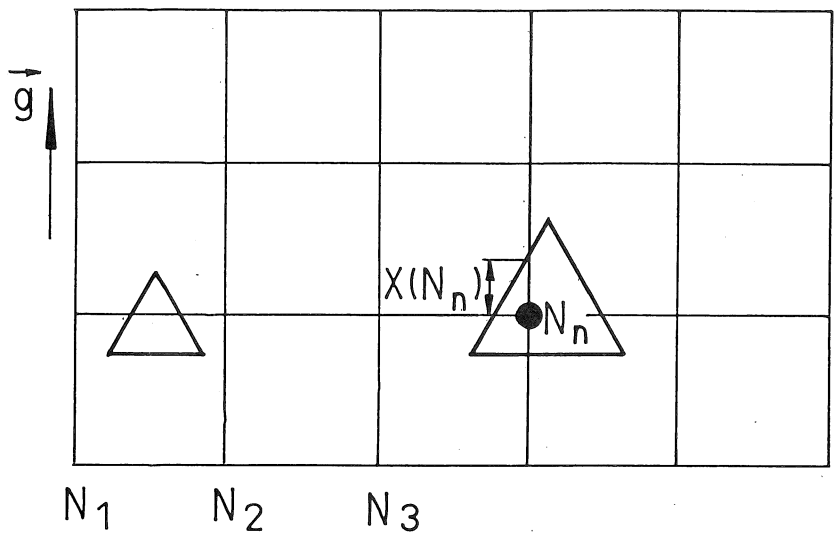

Fig. 2. Systematic point counting

We define $X(N_n)$ as the distance from N_n to the boundary of the covering grain in the direction $\vec{g}$ if the node N_n is covered by a grain of the grain model. In the other case, we put $X(N_n) = 0$.

Next, in the square $[-z,z]^2$ we count the number of nodes N_n with $X(N_n) > x$, which can be easily realized by automatic data analysis because the measurement is a fixed procedure not depending on the samples of the grain model. Thereby, for fixed x the measuring technique of grain erosion can be used (Serra, 1982).

We consider the ratio

$$\frac{\text{Number of nodes } N_n \text{ in } [-z,z]^2 \text{ with } X(N_n) > x}{\text{Total number of nodes in } [-z,z]^2},$$

which is an estimator for the distribution $P(X > x)$ of the random distance from a node of the lattice to the boundary of the covering grain in direction $\vec{g}$, since in the ergodic case

$$\frac{\# n : N_n \in [-z,z]^2,\ X(N_n) > x}{\# n : N_n \in [-z,z]^2} \xrightarrow[z \to \infty]{} P(X > x)$$

with probability one.

Now, the problem arises how the probabilities $P(X > x)$ estimated by systematic point counting are related to the desired grain size distribution $P(S > x)$.

For the above considered example of equilateral triangles the following relationship holds:

$$P(S > x) = \frac{\frac{d^{2+}}{dx^2} P(X > x)}{\frac{d^{2+}}{dx^2} P(X > x)\Big|_{x=0}} \qquad (1)$$

where d^{2+}/dx^2 denotes the second-order right-hand derivative with respect to x.

Thereby, in the strong mathematical proof of (1) given in König and Schmidt (1983), a general point process approach is used (König, Matthes and Nawrotzki (1967), Franken, König, Arndt and Schmidt (1981)), where $P(S > x)$ appears as number-weighted (Palm-type) law with respect to the number of grains, whereas $P(X > x)$ as area-weighted (stationary) law. The second-order derivative in (1) arises from the fact that the transition from $P(X > x)$ to $P(S > x)$ is mediated by a number-weighted law of triangles intersected by some line, where in each step the transition is realized by differentiation.

DISCUSSION

We remark that in König (1982) and in König and Schmidt (1983), besides the theoretical background, further examples of stationary grain models have been investigated. In particular, it has been shown that analogous results can be obtained for stationary grain models in the space R^3. Thereby, the case of identically orientated regular tetrahedrons is calculated in detail.

In general, it is complicated to express $P(S > x)$ explicitly by $P(X > x)$ for an arbitrary grain shape. Namely, already in the cases of random circles in R^2 or spheres in R^3, a non-trivial Abel integral equation must be solved. However, conversely, $P(X > x)$ can be always expressed by $P(S > x)$ using a general stereological argument (formulas (4.1) and (4.4) in König and Schmidt (1983)).

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