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ISOPERIMETRIC INEQUALITIES AND SHAPE PARAMETERS

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ABSTRACT

This article proposes new planar shape parameters :
Circularity parameters issued from classical or new isoperimetric inequalities.
Regularity parameters evaluating the proximity of a given shape to a regular polygon or a circumscrible polygon.

Keywords : Isoperimetric inequalities, shape parameters, Image Processing, classification, mathematical Morphology.

I - INTRODUCTION

We call "shape" a simply connected compact set of $\mathbb{R}^2$. Furthermore, it will be restricted to a planar shape with a non empty interior, and such that the perimeter exists; such a shape will be denoted by A in what follows.

Image Processing uses shape parameters in order to give a shape classification or, more simply, a proximity degree of the studied shape to a reference one.

If the reference shape is a disk these parameters are circularity parameters and if the reference shape is a regular polygon they are regularity parameters.

Let us recall that a positive real valued function f defined on the set of planar shapes is a shape parameter provided f is scale invariant.

II - CIRCULARITY PARAMETERS

Let A denote a planar shape .

1) The most classical circularity parameter is defined by :

$$\boxed{I_0(A) = \frac{P^2(A)}{4\pi\mu(A)}}$$

where P denotes the perimeter and μ the area.

This well known parameter derives from the isoperimetric inequality :

$$P^2(A) - 4\pi\mu(A) \geq 0 \quad (1)$$

If A is convex, the equality holds if and only if A is a disk.

This inequality can be deduced from Bonnesen's inequality [Bonnesen (1929)], which has been proved for a convex compact set A:

$$P^2(A) - 4\pi\mu(A) \geq \pi^2 (R(A) - r(A))^2 \quad (2)$$

where R(A) and r(A) denote respectively the circumradius and the inradius of A.

Then : $I_0(A) \geq 1$. If A is convex $I_0(A) = 1 \Leftrightarrow$ A is a disk.

For the implementation, if we denote by ε an arbitrary allowance ($\varepsilon > 0$), the nearness of the shape A to a disk will be expressed by : $I_0(A) - 1 \leq \varepsilon$

In the following, shape parameters will be always employed in this way.

Notes :

- It is better to apply the parameter I_0 if A is convex (because if there is a concavity in the boundary , P(A) increases and $\mu(A)$ decreases).
- We need a precise computation of the perimeter on the grid.

2) New circularity parameters

Let A be a convex body. With previous notations, we can define :

$$I_1(A) = \frac{P(A)\,R(A)}{2\mu(A)} \qquad I_2(A) = \frac{P(A)\,r(A)}{\mu(A) + \pi r^2(A)}$$

These parameters are clearly scale invariant. They are derived from the following inequalities [see Bandle (1980) and Bonnesen (1929)] :

$$P(A) \,.\, R(A) \geq 2\,\mu(A) \quad (3)$$

$$P(A) \,.\, r(A) \geq \mu(A) + \pi r^2(A) \quad (4)$$

The equalities hold if and only if A is a disk. So :

$\forall\, i \in [1,2] \quad I_i(A) \geq 1$

$I_1(A) = 1 \Leftrightarrow$ A is a disk. If A has a unique inscribed disk $I_2(A) = 1 \Leftrightarrow$ A is a disk.

Notes :

The implementation of these coefficients necessitates efficient algorithms for the determination of R(A) and r(A). The ultimate eroded set from which r(A) is derived can be given by a distance map (see Danielsson (1980).

The "circumscribed window" algorithm gives R(A) [Jourlin, Laget (1984)].

3) Circularity parameters derived from Brunn-Minkowski inequality

Let us recall that the Brunn-Minkowski inequality for two planar shapes A and B, [Brunn (1928), Berger (1977)] says :

$$\forall \lambda \in [0, 1] \quad \sqrt{\mu(\lambda A \oplus (1 - \lambda) B)} \geq \lambda \sqrt{\mu(A)} + (1 - \lambda) \sqrt{\mu(B)} \tag{5}$$

where $\oplus$ denotes Minkowski addition [Matheron (1975)]

The equality holds if and only if A and B are homothetic convex compact sets.

If $\lambda = \frac{1}{2}$ (5) becomes :

$$\sqrt{\mu(A \oplus B)} \geq \sqrt{\mu(A)} + \sqrt{\mu(B)} \tag{6}$$

We denote the equivalent radius by :

$$r_e(A) = \sqrt{\frac{\mu(A)}{\pi}} \tag{7}$$

which is not more than the radius of a disk of area $\mu(A)$.

If B_ρ denotes the disk of radius ρ ($\rho > 0$) centred at the origin, we can deduce :

$$r_e(A \oplus B_\rho) \geq r_e(A) + \rho$$

with equality if A is a disk.

Thus, we can obtain a new circularity parameter :

$$\boxed{I_3(\lambda, A) = \frac{r_e(A \oplus B_{\lambda r_e(A)}) - r_e(A)}{\lambda r_e(A)}} \quad \text{with } \lambda > 0$$

Using (7) it can be reexpressed as:

$$I_3(\lambda, A) = \frac{\sqrt{\mu(A \oplus B_{\lambda r_e(A)})} - \sqrt{\mu(A)}}{\lambda \sqrt{\mu(A)}}$$

Thus $\qquad I_3(\lambda, A) \geq 1$

If A is convex $\qquad I_3(\lambda, A) = 1 \Leftrightarrow$ A is a disk.

III - REGULARITY PARAMETERS AND "CIRCUMSCRIBILITY" PARAMETERS

On a grid , a shape is always polygonal. So it seems interesting to compare a polygonal shape to a reference one (a regular polygon for example).

In the following, **A denotes a convex polygonal shape.**

1) If A is n-sided $\qquad$ (n ≥ 3)

The following inequalities hold [Fejes Toth (1953)] :

$$n \tan \frac{\pi}{n} r^2(A) \leq \mu(A) \leq \frac{1}{2} n \sin \frac{2\pi}{n} R^2(A) \tag{8}$$

$$2\,n \tan\frac{\pi}{n}\, r(A) \leq P(A) \leq 2\,n \sin\frac{\pi}{n}\, R(A) \tag{9}$$

$$P^2(A) \geq 4\,n\,\mu(A) \tan\frac{\pi}{n} \tag{10}$$

These inequalities become equalities if and only if A is regular [Fejes Toth (1953), Blaschke (1916)] . They mean that, among the n-sided polygonal shapes of given perimeter (respectively of given area), the regular ones have a maximal area (respectively a minimal perimeter). (Inequality (10) is similar to (1) : the limit of $n \tan\frac{\pi}{n}$ is π when n tends to infinity , and we obtain (1) from (10)).

Thus, the following coefficients are regularity parameters :

$$I_4(A) = \frac{2\,\mu(A)}{n \sin\frac{2\pi}{n} R^2(A)} \qquad I_5(A) = \frac{\mu(A)}{n \tan\frac{\pi}{n} r^2(A)}$$

$$I_6(A) = \frac{P(A)}{2\,n \sin\frac{\pi}{n} R(A)} \qquad I_7(A) = \frac{P(A)}{2\,n \tan\frac{\pi}{n} r(A)}$$

$$I_8(A) = \frac{P^2(A)}{4\,n \tan\frac{\pi}{n} \mu(A)}$$

$I_4(A), I_6(A) \leq 1 \qquad I_5(A), I_7(A), I_8(A) \geq 1$

$\forall\, j \in [4, 8] \quad I_j(A) = 1 \Leftrightarrow A$ is regular

<u>Note</u> : The use of these coefficients rests on the choice of an adequate convex polygonal approximation (edge vectorization) to obtain the number n of sides of A.

2) If the number of sides of A is unknown

a) The Lhuillier inequality [Fejes Toth (1953)] can be expressed by

$$P^2(A) \geq 4\,\mu(A)\,\mu(A') \tag{11}$$

where A' is the convex polygon whose sides are parallel to the sides of A, taken in the same order, and all tangent to the **unit disk**. Therefore $\mu(A')$ is dimensionless.

<u>Notes</u> :

Since $\mu(A') > \pi$, we derive from (11) the classical isoperimetric inequality (1). The equality case in (11) is realized if the sides of A are all tangent to a disk. Such a polygon will be called a circumscrible polygon (see figure 1) (a regular polygon, a triangle are particular circumscrible polygons).

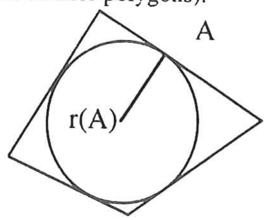


fig1:a circumscrible polygon

Thus we deduce a new shape parameter :

$$\boxed{I_9(A) = \frac{P^2(A)}{4\,\mu(A)\,\mu(A')}}$$

$I_9(A) \geq 1$ $\qquad$ $I_9(A) = 1 \Leftrightarrow$ A is circumscrible

b) From the inequality :

$$P(A)\,r(A) \leq 2\,\mu(A) \tag{12}$$

where the equality holds if and only if A is circumscrible we derive another shape parameter:

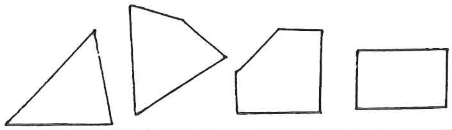
$$\boxed{I_{10}(A) = \frac{P(A)\,r(A)}{2\,\mu(A)}}$$

$I_{10}(A) \leq 1$ $\qquad$ $I_{10}(A) = 1 \Leftrightarrow$ A is circumscrible

IV. IMPLEMENTATION - RESULTS - CONCLUSION

Here are computed results for simple shapes :

I_0	1.07	1.13	1.59	1.17	1.04
I_1	1.16	1.21	2.26	1.52	1.20
I_2	1.03	1.06	1.06	1.02	1.00
I_3 (λ=1)	1.02	1.03	1.13	1.04	1.01



I_4	0.87	0.72	0.74	0.92
I_5	1.11	1.19	1.26	1.50
I_6	0.98	0.93	0.89	0.98
I_7	1.11	1.19	1.17	1.25
I_8	1.11	1.19	1.08	1.04
I_9	1.00	1.00	4.29	1.04
I_{10}	1.00	1.00	0.93	0.83

Using I_1 allows to differentiate easily the three ellipses. But usually, for a better estimation of the circularity several parameters should be used.The discrepancy with theoretical values is less than 1%; it is due to the difficulty to compute on a grid accurate values of perimeter, circumradius and inradius.

For a circumscrible polygon $I_5 = I_7 = I_8$, $I_9 = I_{10} = 1$. The indices I_4, I_5, I_6, I_7, I_8 (respectively I_9, I_{10}) get apart from the value 1 when the shape gets apart from a regular polygon (respectively a circumscrible polygon). For a best evaluation of polygonal regularity or circumscribility all these last parameters should be used.

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