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ON THE PRECISION OF SOME STEREOLOGICAL ESTIMATORS FOR THE MODEL PARAMETER OF THE SPATIAL POISSON-VORONOI TESSELLATION

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ABSTRACT

Four different stereological estimators for the model parameter τ of the spatial Poisson-Voronoi tessellation are compared with respect to their bias and variance by means of a Monte-Carlo study. Formulae are given for variance prediction. An estimator based on vertex counting is found to be the best one. Robustness is investigated by applying the estimators to Voronoi tessellations with respect to other generating point processes.

Key words: parameter estimation, spatial Poisson-Voronoi tessellation, stereology.

INTRODUCTION

Spatial cellular or crystalline structures which result from a growth process can successfully be described using Voronoi tessellations, for examples see e.g. Okabe et al. (1992) or Stoyan et al. (1987). A random spatial Voronoi tessellation is a division of space into convex regions – the cells – defined with respect to a generating point process of so-called germs. Each cell consists of those points in space which are closer to the generating germ than to all other germs.

If the germs constitute a homogeneous Poisson point process, then the tessellation is said to be a Poisson-Voronoi tessellation (PVT). The only parameter of this model is the intensity τ of the generating germ process, the mean number of points per unit volume.

For a variety of mean values and higher moments of geometric characteristics of the spatial cells as well as of the planar section cells of the PVT, formulae in terms of τ are known, see Stoyan et al. (1987) and Brakke (1985) for summaries. For example, the mean surface area is given by $\mathbb{E}(S_3) = 5.821\tau^{-2/3}$, and the mean average breadth by $\mathbb{E}(\bar{b}_3) = 1.458\tau^{-1/3}$. Thus, knowledge of τ implies complete information about the properties of the PVT.

In practice, often only a single plane section of an investigated structure is available. Therefore, the present paper gives estimators for τ based on information from one plane section. They are compared with respect to their bias and variance by means of Monte-Carlo simulations.

Independent of the generating point process of a random Voronoi tessellation, the mean volume of the cells is equal to τ^{-1}. Hence, τ is a parameter of high practical relevance. To get an impression on the robustness of the different estimators for τ, they are exemplarily applied to Voronoi tessellations with respect to germ processes which are in some sense more regular and more irregular than a Poisson point process of the same intensity, respectively.

THE ESTIMATORS

The four estimators studied in this paper are based on the following relations between stereological mean values and the parameter τ of the PVT, cf. Stoyan et al. (1987):

$$P_A = \frac{1}{2} L_V = \frac{8}{15} \cdot \left(\frac{3}{4}\right)^{1/3} \cdot \pi^{5/3}\, \Gamma\!\left(\frac{4}{3}\right) \cdot \tau^{2/3} , \tag{1}$$

$$N_A = N_V\, \mathbb{E}(\bar{b}_3) = \frac{4}{15} \cdot \left(\frac{3}{4}\right)^{1/3} \cdot \pi^{5/3}\, \Gamma\!\left(\frac{4}{3}\right) \cdot \tau^{2/3} , \tag{2}$$

and

$$L_A = \frac{\pi}{4} S_V = \pi \cdot \left(\frac{\pi}{6}\right)^{1/3} \cdot \Gamma\!\left(\frac{5}{3}\right) \cdot \tau^{1/3} , \tag{3}$$

where P_A denotes the mean number of vertices of the planar section tessellation per unit area, N_A the mean number of cell profiles per unit area, and L_A the mean total edge length per unit area. A simple relation between N_A and $\mathbb{E}(a)$, the mean area of the cell profiles, can be used to construct a fourth estimator, namely

$$N_A = \frac{1}{\mathbb{E}(a)} . \tag{4}$$

From these formulae, four estimators of τ are derived as follows:

$$\hat{\tau}_P = \frac{15\sqrt{5}}{8\sqrt{2}} \cdot \pi^{-5/2}\, \Gamma\!\left(\frac{4}{3}\right)^{-3/2} \cdot \hat{P}_A{}^{3/2} \approx 0.2008 \cdot \hat{P}_A^{3/2} , \tag{5}$$

$$\hat{\tau}_N = \frac{15\sqrt{5}}{4} \cdot \pi^{-5/2} \Gamma(\tfrac{4}{3})^{-3/2} \cdot \hat{N}_A{}^{3/2} \approx 0.5680 \cdot \hat{N}_A^{3/2} \,, \tag{6}$$

$$\hat{\tau}_L = \frac{6}{\pi^4} \cdot \Gamma(\tfrac{5}{3})^{-3} \cdot \hat{L}_A{}^3 \approx 0.0837 \cdot \hat{L}_A^3 \,, \tag{7}$$

and

$$\hat{\tau}_a \approx 0.5680 \cdot \bar{a}^{-3/2} \,. \tag{8}$$

While $\hat{\tau}_L$ and $\hat{\tau}_a$ require measurements of edge lengths and areas, respectively, $\hat{\tau}_P$ and $\hat{\tau}_N$ use simple point or cell counting. Given an observation window W with area $A(W)$,

$$\hat{P}_A = \frac{P(W)}{A(W)} \tag{9}$$

and

$$\hat{L}_A = \frac{L(W)}{A(W)} \tag{10}$$

are unbiased estimators for P_A and L_A, respectively. The number of vertices $P(W)$ and the total edge length $L(W)$ inside W can be evaluated without taking care of edge effects. To determine

$$\hat{N}_A = \frac{N(W)}{A(W)} \,, \tag{11}$$

where $N(W)$ denotes the number of cells 'inside' W, a counting procedure has to be used which considers edge effects, see Schwandtke et al. (1987) or Gundersen (1978). In the present study, a unique 'sampling' point, the leftmost vertex, was assigned to each cell, and the number of these points in W was counted. Correspondingly, the mean area $\bar{a}$ of the section cells was obtained by averaging over all the cells whose sampling point was inside W.

SIMULATION

The four estimators were investigated by means of a computer simulation. To this end, aggregates of cells of spatial PVT with $\tau = 1$ were generated. For details concerning the simulation procedure see Lorz and Hahn (1993) and Møller et al. (1989). Isotropic random planar sections were taken from the aggregates and square observation windows were drawn in the section planes. The aggregates were made large enough to ensure that the observation windows were completely filled with cells. The edge lengths of the observation windows were calculated using Eq. 2 such that the expected number of cells inside was 50, 60, ... , 200. Thus, the results can be related to the sample size. About 7200 samples were generated for each window size.

Additionally, about 5500 samples were generated each for Voronoi tessellations with respect to other point processes of same intensity $\tau = 1$, to get information about the robustness of the estimators. As examples, Voronoi tessellations were chosen, which are constructed with respect to a Matern hard-core point process (HVT), to a simple sequential inhibition point process (SVT), and to a Matern cluster point process (CVT). SVT and HVT are, in some sense, more regular than PVT whereas CVT is more irregular. For a mathematical definition of these point processes see Diggle (1983) and Stoyan et al. (1987).

The HVT as well as the SVT model can be characterized by the scale parameter λ_{hc}, the mean number of points of the generating point process per unit volume, and the shape parameter $p_{hc} = \lambda_{hc} \frac{4}{3}\pi\, R_{hc}^3$, the mean volume fraction of the hard cores with radius R_{hc} (Lorz and Hahn, 1993). For the HVT the parameter p_{hc} has to be taken from the interval $[0, \frac{1}{8})$ whereas for the SVT p_{hc} can be chosen between 0 and approximately 0.4.

The model parameters of the CVT are the scale parameter λ_{cl}, the mean number of points of the generating cluster point process per unit volume, and the shape parameters N_{cl}, the mean number of points per cluster, and R_{cl}, the cluster radius (Stoyan et al., 1987). Instead of R_{cl},

$$p_{cl} = 1 - \exp\left\{-\frac{\lambda_{cl}}{N_{cl}} \frac{32}{3}\pi\, R_{cl}^3\right\} \tag{12}$$

is used as third model parameter. It is scale invariant and can be interpreted as (approximately) the probability that neighbouring clusters 'overlap'.

As in Hahn and Lorz (1993) and Krawietz and Lorz (1991), the parameters $p_{hc} = 0.1$ (HVT), $p_{hc} = 0.2$ (SVT), and $N_{cl} = 10$ and $p_{cl} = 0.7$ (CVT) were chosen for the investigation of the robustness of the estimators. The intensities λ_{hc} and λ_{cl} were set to unity.

RESULTS AND DISCUSSION

Empirical biases were calculated for every window. The results are summarized in Tbl. 1, which also contains the empirical estimation variances. In practical applications, it is desirable to have a relation between the size of the observation window, which is expressed here in terms of the expected number n of section cells, and the coefficient of variation, $\mathrm{cv} = \sqrt{\mathrm{variance}}/\mathrm{mean}$. In the case of regular experiments, $\mathrm{cv} = \mathrm{const}/\sqrt{n}$. Therefore, $\sqrt{n}\cdot\mathrm{cv}$ is also contained in Tbl. 1. For all the four estimators, the biases are less than 1% for a sample size of $n = 50$, and they decrease rapidly with increasing sample size. This suggests an asymptotic unbiasedness. The quantity $\sqrt{n}\cdot\mathrm{cv}$ seems to be independent of n, so that it appears admissible to give the following rules of thumb:

$\mathrm{cv}(\hat{\tau}_P) \approx 1.04/\sqrt{n}$, $\mathrm{cv}(\hat{\tau}_N) \approx 1.04/\sqrt{n}$, $\mathrm{cv}(\hat{\tau}_L) \approx 1.09/\sqrt{n}$, and $\mathrm{cv}(\hat{\tau}_a) \approx 1.05/\sqrt{n}$.

The four estimators show hardly any difference concerning bias and variance. In practical applications, one has to expect that the variance of both $\hat{\tau}_L$ and $\hat{\tau}_a$ would be increased by additional measuring errors. Thus, the estimators $\hat{\tau}_P$ and $\hat{\tau}_N$, which are based on counting, should be preferred. Among these two methods, $\hat{\tau}_P$ is easier to manage: there are no problems with edge effects.

Table 1. Empirical bias, variance and $\sqrt{n} \cdot$cv of the four estimators.

n	$\hat{\tau}_P$			$\hat{\tau}_N$			$\hat{\tau}_L$			$\hat{\tau}_a$		
	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv
50	0.0033	0.0213	1.0329	0.0041	0.0212	1.0298	0.0079	0.0235	1.0842	0.0056	0.0218	1.0440
60	0.0040	0.0179	1.0370	0.0036	0.0179	1.0371	0.0068	0.0199	1.0927	0.0044	0.0184	1.0502
70	0.0033	0.0156	1.0444	0.0025	0.0156	1.0460	0.0060	0.0171	1.0949	0.0039	0.0158	1.0516
80	0.0033	0.0137	1.0458	0.0026	0.0137	1.0481	0.0056	0.0149	1.0935	0.0028	0.0137	1.0471
90	0.0022	0.0122	1.0485	0.0026	0.0123	1.0503	0.0045	0.0133	1.0948	0.0036	0.0124	1.0584
100	0.0021	0.0110	1.0510	0.0016	0.0110	1.0501	0.0042	0.0121	1.1018	0.0024	0.0112	1.0598
110	0.0019	0.0101	1.0524	0.0012	0.0100	1.0469	0.0037	0.0111	1.1027	0.0017	0.0102	1.0574
120	0.0015	0.0092	1.0489	0.0014	0.0092	1.0495	0.0034	0.0101	1.0993	0.0017	0.0093	1.0541
130	0.0011	0.0085	1.0497	0.0014	0.0085	1.0502	0.0032	0.0093	1.0976	0.0015	0.0086	1.0550
140	0.0012	0.0078	1.0461	0.0010	0.0077	1.0394	0.0029	0.0086	1.0974	0.0015	0.0079	1.0546
150	0.0013	0.0073	1.0435	0.0009	0.0072	1.0401	0.0029	0.0080	1.0945	0.0014	0.0073	1.0478
160	0.0014	0.0067	1.0379	0.0010	0.0067	1.0369	0.0027	0.0074	1.0853	0.0012	0.0068	1.0406
170	0.0015	0.0063	1.0356	0.0009	0.0064	1.0421	0.0028	0.0069	1.0857	0.0012	0.0063	1.0365
180	0.0013	0.0059	1.0304	0.0012	0.0059	1.0320	0.0028	0.0065	1.0827	0.0014	0.0059	1.0341
190	0.0013	0.0056	1.0315	0.0013	0.0057	1.0367	0.0026	0.0062	1.0835	0.0013	0.0057	1.0399
200	0.0009	0.0054	1.0363	0.0013	0.0054	1.0390	0.0023	0.0059	1.0884	0.0015	0.0054	1.0416

Table 2. Empirical bias, variance and $\sqrt{n} \cdot$cv of the four estimators for tessellations which are more regular (SVT, HVT) and more irregular (CVT) than the PVT, respectively.

SVT	$\hat{\tau}_P$			$\hat{\tau}_N$			$\hat{\tau}_L$			$\hat{\tau}_a$		
n	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv
50	-0.0353	0.0086	0.6561	-0.0355	0.0086	0.6550	-0.0814	0.0076	0.6149	-0.0342	0.0080	0.6340
100	-0.0371	0.0042	0.6486	-0.0372	0.0042	0.6475	-0.0843	0.0037	0.6114	-0.0371	0.0040	0.6363
150	-0.0382	0.0027	0.6343	-0.0380	0.0027	0.6420	-0.0855	0.0025	0.6074	-0.0385	0.0027	0.6325
200	-0.0386	0.0021	0.6438	-0.0390	0.0021	0.6458	-0.0857	0.0018	0.6053	-0.0387	0.0020	0.6353
HVT	$\hat{\tau}_P$			$\hat{\tau}_N$			$\hat{\tau}_L$			$\hat{\tau}_a$		
n	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv
50	-0.0211	0.0121	0.7789	-0.0205	0.0122	0.7824	-0.0459	0.0116	0.7623	-0.0198	0.0119	0.7727
100	-0.0219	0.0060	0.7722	-0.0219	0.0059	0.7659	-0.0474	0.0058	0.7629	-0.0211	0.0059	0.7690
150	-0.0217	0.0040	0.7768	-0.0219	0.0040	0.7750	-0.0480	0.0039	0.7618	-0.0217	0.0040	0.7699
200	-0.0221	0.0029	0.7638	-0.0218	0.0029	0.7635	-0.0477	0.0028	0.7495	-0.0220	0.0028	0.7537
CVT	$\hat{\tau}_P$			$\hat{\tau}_N$			$\hat{\tau}_L$			$\hat{\tau}_a$		
n	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv	bias	var	$\sqrt{n} \cdot$cv
50	-0.0419	0.0928	2.1542	-0.0410	0.0940	2.1684	-0.0646	0.0908	2.1303	-0.0336	0.1008	2.2446
100	-0.0503	0.0492	2.2189	-0.0499	0.0495	2.2242	-0.0822	0.0472	2.1737	-0.0489	0.0512	2.2627
150	-0.0531	0.0338	2.2525	-0.0536	0.0339	2.2549	-0.0874	0.0325	2.2068	-0.0527	0.0348	2.2859
200	-0.0545	0.0252	2.2444	-0.0544	0.0253	2.2476	-0.0894	0.0243	2.2040	-0.0548	0.0259	2.2740

Tbl. 2 gives an excerpt of the corresponding results for the three other tessellation models, SVT, HVT and CVT. Even in the case of more regular and more irregular tessellations, the simulation study revealed only small biases (far less than 10 %). The intensity τ was underestimated there.

On the whole, $\hat{\tau}_P$ seems to be very appropriate for the estimation of the parameter τ of a spatial Voronoi tessellation, even if the generating germ process is unknown and could slightly differ from a Poisson point process.

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REFERENCES

Brakke KA. Statistics of three dimensional random Voronoi tessellations. *unpublished manuscript*, Susquehanna Univ., Dept. of Math. Sciences, Selinsgrove, 1985.

Brakke KA. 200,000,000 random Voronoi polygons. *unpublished manuscript*, Susquehanna Univ., Dept. of Math. Sciences, Selinsgrove, 1985.

Diggle PJ. *Statistical Analysis of Spatial Point Processes.* London: Academic Press, 1983.

Gundersen HJG. Estimators of the numbers of objects per unit area unbiased by edge effects. *Micros Acta* 1978; **81**: 107–117.

Hahn U, Lorz U. Stereological model tests for the spatial Poisson-Voronoi tessellation II. submitted to *Acta Stereol* (1993).

Krawietz R, Lorz U. Stereological model tests for the spatial Poisson-Voronoi tessellation I. *Acta Stereol* 1991; **10**: 203–212.

Lorz U, Hahn U. Geometric characteristics of spatial Voronoi tessellations and planar sections. submitted to *J Visual Comm and Image Repres* (1993).

Møller J, Jensen EB, Petersen JS, Gundersen HJG. Modelling an aggregate of space-filling cells from sectional data. Research Report No. 182, Department of Theoretical Statistics, University of Aarhus, 1989.

Okabe A, Boots BN, Sugihara K. *Spatial Tessellations.* Chichester: Wiley, 1992.

Schwandtke A, Ohser J, Stoyan D. Improved estimation in planar sampling. *Acta Stereol* 1987; **6**: 325–334.

Stoyan D, Kendall WS, Mecke J. *Stochastic Geometry and Its Applications.* Chichester: Wiley, 1987.

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