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Testing and Modeling of Friction and Slip in Mechanical Interfaces: State of the Art and Perspectives for the Next Decade

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Abstract

Experiments and physics-based modeling efforts both show that the features within a jointed interface can have an outsized influence on the nonlinear dynamics of a large-scale structure. The interfacial features, including asperities and meso-scale topology, are often six to ten orders of magnitude smaller in scale than the structure itself, yet can significantly change the natural frequencies and damping of a structure and can lead to the premature failure due to wear if not properly designed. A significant amount of recent research has been invested in understanding and predicting the nonlinear dynamics of structures with jointed interfaces; however, there are many challenges that still remain before accurate predictions of a jointed structure's nonlinear dynamics and wear properties becomes accessible to design engineers. This article is a reflection of the outcomes of the 2023 International Workshop on the Mechanics of Jointed Structures in which the state of the art of joints modeling was assessed and future directions for research on jointed structures were identified. As such, this paper makes several recommendations for new research thrusts to improve the understanding of jointed structures in addition to highlighting the current state of the art and recent advances in modeling and experimentally characterizing jointed structures.

Keywords: jointed structures; hysteresis; tribomechadynamics; nonlinear dynamics; interfacial mechanics

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"Joints are so ubiquitous that a tiny improvement can have far reaching ramifications." — David J. Ewins (1942–2023)

1 Introduction and Context: The Mechanics of Jointed Structures

Mechanical interfaces - joints - are commonplace in modern engineering. Even with the advent of additive manufacturing, it is neither desirable nor practical to create purely monolithic structures, which necessitates mechanical means of joining distinct components to create assemblies. Despite the use of joints for millennia and research into their mechanics spanning centuries, the physics governing how mechanical interfaces dissipate and transmit mechanical energy is still poorly understood [1]. As a result, predictive modeling frameworks do not yet exist.

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The influence of joints on the mechanical response of structures began to be recognized in the middle of the twentieth century. For example, in the 1950's Lazan performed detailed studies of dissipation due to mechanical joints, noting that the damping was nonlinear and increased in a power law fashion with vibration amplitude [2]. Even then, test data allowed for calibrated modeling to suffice [3, 4], and so, throughout the twentieth century, joints were designed to support a static or dynamic load, and only in rare cases were efforts made to characterize their energy dissipation or long-term wear behavior.

By the 1990s, multiple industries were beginning to recognize that the lack of understanding of mechanical joints was problematic for predictive design [3, 4]. Exacerbating this capability gap, in recent decades pressure has mounted to reduce the time to market for various products (e.g., automobiles, aerospace vehicles), which has led to fabrication and testing of initial designs often being eliminated in favor of virtual (computational) prototyping and analysis. This cross-industry decision has resulted in a lack of calibration data for tuned model approaches. Second, there is increasing interest in making structures (particularly engines, vehicles, and aerospace structures) lighter in order to improve performance or increase efficiency. Light-weighting structures has led to an increased need for joints to dissipate energy in otherwise underdamped assemblies, resulting in the joints becoming more dynamically active and thus having more pronounced nonlinear behavior. An unintended consequence of relying on joints to dissipate energy and having them become more dynamically active is that this can significantly increase their wear (as energy is usually dissipated by damaging the interface at larger excitation amplitudes, compared to by heat generation at small excitation amplitudes[1]), which can eventually lead to mechanical failure.

As engineers look to address the existential threats facing humanity in the twenty-first century, they must help society consume energy more efficiently to lessen our impact on our environment. Approximately one quarter of the world's energy consumption is due to friction and wear (with 86.5% of that energy being used to overcome friction and 13.5% of that energy being used to replace worn parts) [6, 7]. Mechanical joints contribute to a significant portion of these lost resources. From an economic point of view, this parasitic loss of energy leads to trillions of dollars of lost profit per year, a sizeable percentage of the gross world product of ≈$90 trillion in 2019 [1].

The goal of the present article is to articulate the challenges for joints mechanics research and to identify the necessary steps for significant advancement within the field. As such, the present article is not intended as a review paper, but rather an assessment of the current capabilities and challenges. For more thorough background covering the mechanics of jointed structures, the reader is referred to the reviews of [8, 9, 10, 1] for modeling friction and contact in joints (or even the review of [11] for modeling friction and wear in general), and the books [12, 13] for assessments of the state-of-the-art in the joint mechanics community in the early 2010s. In what follows, the experimental understanding of the physics of jointed structures is described in §3. Following from these experiments, the consequences and evolution of recent modeling efforts is detailed in §4. Taking together the recent experimental and computational advances, §5 identifies the short-comings in the existing approaches for modeling jointed structures, and §6 presents the major challenges and path forward for joints research.

This article is one outcome of the Fifth International Workshop on the Mechanics of Jointed Structures. Like the previous workshops, [4, 14, 15, 16, 17], the Fifth International Workshop on the Mechanics of Jointed Structures convened a group of 30 researchers from across academia, industry, and government, and charged them with charting a path forward for the community over the next decade. The ultimate goal of this community is to enable more efficient, lighter weight, and more wear-resistant designs of structures and engines through an advanced understanding of the physics and nonlinear dynamics of joints in large assemblies.

2 Industrial Context for Joints Research

To contextualize the role of joints in industrial applications at present, consider this anecdote from the automotive industry[2]: Prior to the pandemic, designing a new car was an 18 month process that involved the design, fabrication, and testing of multiple prototypes. In the post-pandemic era, this process has been reduced to ***nine*** months from conceptualization through to production with ***zero*** prototypes. This new design philosophy requires more accurate and more accessible models of jointed interfaces in order to virtually design a vehicle and have high confidence that it will perform as intended. As such, the structural joints are, as best as possible, over-designed to ensure a linear, more predictable response. This avoidance of nonlinearities likely has resulted in missed opportunities to reduce weight (improving fuel economy) and optimize the performance of new vehicle designs. Aside from the structural joints (including both welds and bolted connections) [18], other prominent joints within an automobile that have significant

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[1] It should be noted that rigorous analysis of this statement is still needed, including the assessment of when the transition from heat generation to damage occurs for typical interfaces. Preliminary work indicates that this can occur at moderate excitation levels for lap joints [5].

[2] This is intentionally left vague due to the sensitivity of associating this information with a specific manufacturer.

influences on the ride quality include the disk brakes [19, 20, 21, 22] (particularly with regards to squeal and noise vibration harshness) and leaf springs [23, 24]. These pressures to reduce design time and prototyping, and the increased influence of joints on structures being optimized for efficiency, are also felt across the defense, aerospace, naval, and other industries.

A wide variety of joints are typically encountered in engineering practice, with several typical joints illustrated in Fig. 1. Of course, there are many other kinds of joints that may be encountered including tape joints [25], dovetail joints [26], spline joints [27], riveted joints [28], pin joints [29], etc. The important factors to consider in designing a joint are the preload direction and the contact forces that it induces, the direction(s) in which the joint is loaded, the shape of the interface(s), and the proximity to other joints. Much of the research to date has focused on lap joints, and much of that when the joints were loaded only in shear as shown in Fig. 1.

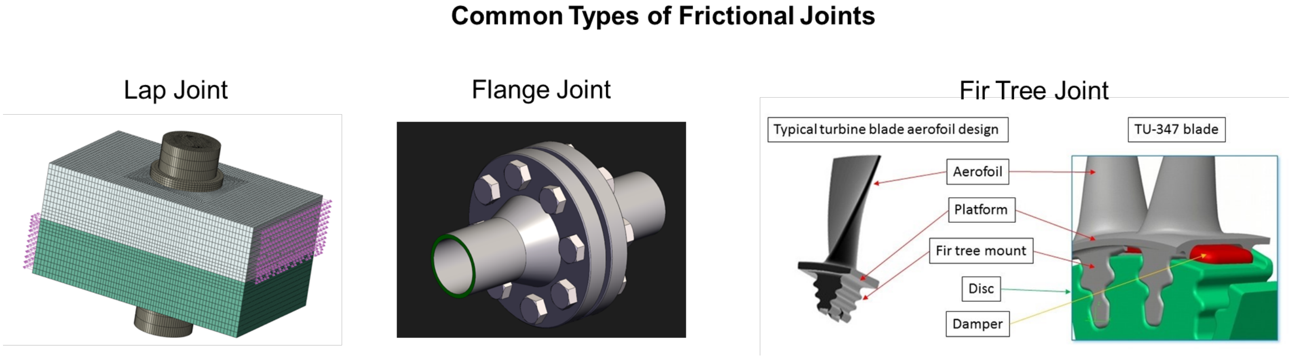

Fig. 1: Typical types of joints encountered in engineered structures. The image of the fir tree joint is reproduced from the accident report by the Australian Government's Australian Transport Safety Bureau on the "In-Flight Engine Failure Involving Sikorsky S-76C Helicopter, VH-EXU".

Industry has shown varied levels of interest in joints, ranging from those who are actively seeking to exploit them to improve structural performance to those who simply view them as a nuisance. One of the industries that has been most actively engaged is the turbine engine industry, because turbines contain joints that provide damping that is critical to the life of the turbine. Specifically, under-platform dampers have been included in turbine engines for several decades to increase damping [30] without incurring wear in the primary load path of the structure. As aeroturbines have been redesigned from an assembly with many blades to a single turbine disk (or blisk [31]), the dynamics and dissipative characteristics of the turbine have become significant issues for the fatigue life of the turbine blades. In the defense industry, joints are not commonly exploited but are seen as a major source of variability and uncertainty. Consequently, the defense industry has also been active in studying and contributing to research on joints in order to develop predictive models of various defense applications. Other industries are uninterested in exploiting the damping or stiffness effects, but highly motivated to avoid failures due to fretting, bolt loosening, or loss of preload [32]. In some industries, such as earthquake protection of buildings, frictional damping is used quite commonly and joints play a critical role; however, the approaches taken are often experimental and empirical model development. Consequently, the civil engineering industry (e.g., [33, 34]) is unfortunately not very well represented in this perspective paper. Similarly, the bolts used as pins in heavy machinery [29], which are designed to carry the load in the transverse direction of the pin, are not included either.

The industry participants that were represented at the workshop, either in person or through their collaborators, expressed interest in expanded research into mechanical joints to address several key areas:

- Shock and random vibration
- Large articulation/motion across joints
- Coupled loads analysis (often referred to as CLA)
- Multi-physics/combined environments
- Predictable joint designs

Mechanical Shock: During a shock event, nothing behaves rigidly; bolts can loosen and slip, and it is challenging to estimate how subcomponents will respond when subjected to very high g loading [35]. The majority of our understanding of how a joint behaves is based on studies of low amplitude regimes (both to avoid wear and due to the complexities of expanding into high amplitude regimes). For example, many studies have collected slowly decaying free vibration measurements [36, 37, 38, 39, 40] or steady-state vibration measurements [41, 42, 43, 44] and compared them with Segalman's four parameter Iwan model [45] (or similar) showing that the micro-slip behavior is captured well by that model. The four parameter Iwan model also includes a term to capture macro-slip, yet no studies to date have thoroughly validated it in the macro-slip regime (although macro-slip was observed in [41, 46], it is rarely studied due to the exacerbation of wear in the joint [47, 43], which results in the nonlinear properties changing over time). In other disciplines, various friction models are used and many of them treat the transition to sliding differently than is done in the Coulomb model (upon which the four parameter Iwan model is based, as well as most finite element studies, see, e.g. [48]) to capture the initiation of sliding [49]. Models such as the LuGre model (e.g., see [50, 8, 10]) consider that the tangential force is large right before sliding is initiated, and then decreases sharply after sliding begins; however, the LuGre and other similar models are less accurate at capturing the micro-slip regime [51].

For shock-specific applications, several concerns exist. First, how much is a shock attenuated across a joint? The rule of thumb from handbooks and standards is that each joint can be expected to attenuate vibration levels by 40% of the shock response spectrum (for up to three joints) during a shock event [52]. Second, given a structural geometry, how will a joint cause a shock to be transmitted, attenuated, reflected, and shunted? Lastly, how much preload will a joint lose during a shock event? Much of the present intuition of a joint's behavior during shock comes from an extensive test campaign throughout the 1950s and 1960s [52]. As the fidelity of high performance modeling capabilities has improved, modeling approaches are just now becoming mature enough to investigate shock response across joints computationally instead of the traditional experiment-based studies [53, 35].

Multi-physics/combined environments: Industrial partners have expressed interest in effects that couple mechanical, thermal and material effects. For example, jointed, geometrically nonlinear panels can show sensitivity due to temperature, so this must be considered as well as the joint nonlinearity. While much data exists on the influence of elevated temperatures on an interface's hysteretic properties (e.g., [54]), there is little to no data on lowered temperature responses, which is germane for both terrestrial applications such as liquid natural gas plants and naval vessels as well as space-based applications such as in satellites and the space frame of the international space station.

Similarly, the studies to date on joints much less frequently address their behavior in random vibration environments, or the behavior of joints that are capable of large articulation such as robot arms or deployable satellites. Large articulation, in particular, violates many of the assumptions present in the state-of-the-art joint modeling frameworks [55, 56]. Furthermore, research is needed regarding how to implement recent technologies into workflows such as coupled loads analysis [57], where thousands of simulations are typically performed – necessitating that the models are reduced and linear.

To facilitate prioritization of needs and tasks related to research on jointed structures and the integration of their nonlinearities into industrial applications, the relevancy-complexity chart is introduced in Fig. 2. In the upper left-hand corner of the chart is Zone 1: low relevancy, high complexity. Research in Zone 1 neither has an immediate impact nor fills an immediate need. Further, research in this zone is also extremely complex and will require extraordinary effort to solve. By contrast, the lower right-hand corner of the chart is Zone 9: high relevancy, low complexity. Research in Zone 9 is extremely useful, requires little effort or new tool development, and is often observed as maturing a technology from research to application. The ultimate goal of research, in the context of the relevancy-complexity chart, is to take highly complex concepts (such as in Zone 3: high complexity and high relevancy) and to lower the complexity over time through proper research and development and tool development. Very often, to accomplish this, a number of projects in other zones will be necessary as stepping stones. Consequently, most research projects fall in Zone 5: medium relevancy, medium complexity. Even some projects in Zones 7 (low relevancy, low complexity) and 8 (medium relevancy, low complexity) are necessary at times to make progress towards a larger challenge.

The relevancy-complexity chart can be a useful mechanism for engaging industry for research. Often the relevancy of task or discovery is determined by industrial partners or application use; the complexity axis, conversely, is controlled by the research community. As an example, the field of nonlinear structural dynamics was initially seen as a high complexity, low relevancy problem. Eventually, industrial applications demonstrated evidence of the significance of nonlinear response in bolted joints, resulting in them becoming a larger priority (and thus high relevancy). In turn, researchers focused on lowering the complexity to model the dynamics of jointed structures, which eventually led several industries to see investment in further research as a low-risk, high reward activity. Thus, since 2000, research on nonlinear structural dynamics evolved from Zone 1 to Zone 6.

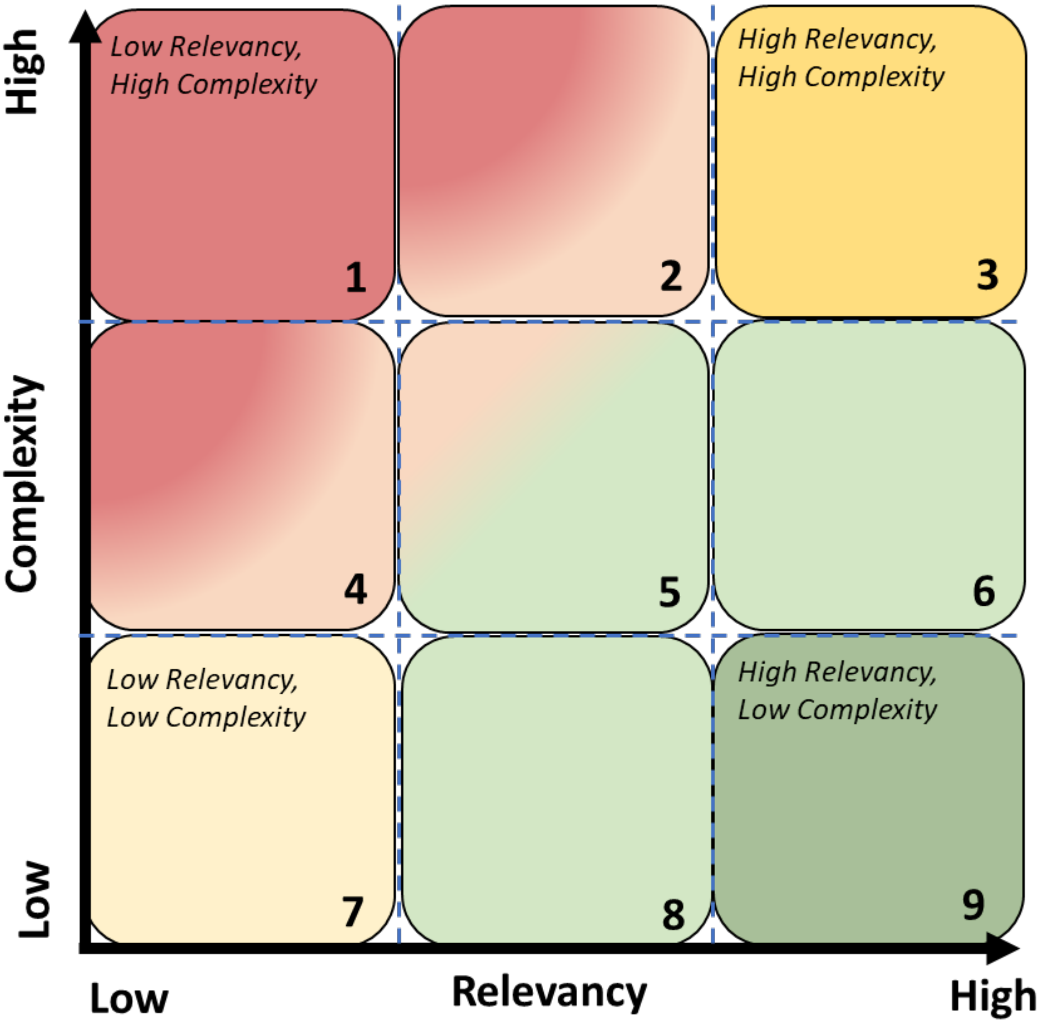

Fig. 2: The relevancy-complexity chart for prioritizing research needs and tasks.

3 Experimental Investigations of Joint Physics and Nonlinearity

One barrier to understanding the physics of jointed interfaces is that it is not possible to see inside of the interface without changing it. Measurements of structures with joints, though, can provide insights into how the presence of an interface changes a structure. In particular, most measurements of jointed structures focus on *in situ* characterization of a structure's nonlinear properties. Further insights into the physics of interfaces have come from *ex situ* measurements of the tribological properties of interfaces [1]. Thus, there are two major categories of experiments to understand the physics that govern jointed structures: *ex situ* measurements and *in situ* measurements of bolted joints. Due to the large-scale size of the systems, *in situ* measurements of component interfaces are impractical and insights are instead derived from custom-designed rigs that provide meaningful *ex situ* measurements of the physics associated with component interfaces. The term 'component interfaces' is chosen here to denote a joint held together without bolts, such as a dovetail joint, firtree joint, or tape joint, or a frictional interface that is used to affect the dynamics of a structure, such as an underplatform damper or disk brake.

3.1 Characteristics of a Jointed Interface

The mechanics for each type of jointed interface can vary significantly depending on the amount of relative motion present across the interface, generally categorized as **micro-slip** and **macro-slip**. For an interface undergoing micro-slip, typically only a small part of the overall interface exhibits relative tangential motion, while the remaining area is stuck. Such a situation often occurs in bolted joints where the outer edges of the joint slip and are accompanied by wear. In contrast, macro-slip refers to the gross sliding of the entire interface, seen, for example, in underplatform dampers. In properly designed joints under nominal operating conditions, micro-slip is often encountered in joints that are designed to carry static loads, while joints that are designed to dissipate energy typically undergo macro-slip.

Commonalities across all jointed interfaces are that they dissipate energy through friction/wear and they provide stiffness in both the normal direction (contact or normal stiffness) and the sliding direction (tangential stiffness). The frictional properties of these interfaces are not Coulombic [12, 58] (i.e., it does not conform to the conventional model of friction f being equal to a constant coefficient μ multiplied into the normal force N, $f = \mu N$). This is partly due to the properties of the joints evolving with time (wear) [58, 47] and partly due to the Coulombic model being a heuristic model that is valid for describing macro-scale phenomena, but not smaller scale phenomena.

The evolution of a joint with wear is typically associated with a process known as **fretting** [59, 47]. Fretting wear is a process that combines multiple wear mechanisms: first, adhesive wear pulls material off of one surface, creating a new surface in which the substrate has not previously been exposed to oxygen. This newly exposed surface then oxidizes (corrosive wear), creating an oxide that is typically much harder than the original material. The oxides trapped in the interface then begin ploughing more material off of the surface (abrasive wear), which, in turn, creates more new surfaces and perpetuates this cycle of oxide formation leading to increased abrasive wear [1]. Each of these individual mechanisms are often investigated as part of *ex situ* experiments. The interaction of all of the mechanisms simultaneously to produce fretting, though, is dependent upon the unique tribosystem found in the *in situ* applications. Thus, the influence of the different wear mechanisms on the nonlinear dynamics of a structure can be viewed as an emergent process [60, 61] – i.e., study of the nano-scale interactions alone is not enough to understand the macro-scale response, but rather the behavior observed at the macro-scale emerges from the complex interaction of the many mechanisms across multiple scales.

3.2 *Ex Situ* Measurements

Broadly, there are two categories of *ex situ* measurements: classical tribological measurements that often involve tribometers or pin-on-disk wear tests, and custom designed fretting rigs that measure the hysteretic behavior of a given material pair or interface. In particular, a tribological concept pervasive throughout experiments is the **tribosystem**: the unique confluence of materials, geometry, loading, lubrication, environment (temperature, humidity, atmospheric gas, etc.) that give rise to a specific set of wear mechanisms within an application.

Classical experiments using tribometers (or similar) are used to characterize the coefficient of friction and wear properties for a material under gross slip conditions. While much of the existing literature on tribology experiments is specialized to specific alloys and lubrication conditions, the mechanistic origins of the frictional and wear properties are fundamentally the same [62]. Many classical experiments investigating the wear and friction properties of a given material pair (with or without lubrication) often do not recreate the tribosystem germane to a jointed structure. For instance, bolted joints do not exhibit gross slip; however, the fundamental loading in tribometers and other classical experiments assume gross slip. These experiments can still be useful in providing estimates of frictional properties in a micro-slip regime; but, for the most part, they are less relevant than measurements from custom designed fretting rigs, such as shown in Fig. 3.

The physics governing component interfaces are often investigated through custom designed fretting rigs, such as [63, 64, 54, 65], which recreate many of the aspects observed in the tribosystems of interest: materials, normal contact pressures, temperature, and environment; however, the fundamental wear mechanisms of fretting rigs are sometimes disconnected from the application as most fretting rigs study gross sliding, whereas many applications are in the micro-slip regime. Additionally, the contact geometries of the fretting rigs are significantly different both between each other and with respect to the motivating application. A recent comparison across multiple fretting rigs showed that this difference in contact geometry can lead to discrepancies of up to 70% for contact stiffness and 15% for the coefficient of friction [66]. Nonetheless, these rigs have immense utility in investigating the physics governing interfacial contact. In order to further investigate the contact properties of an interface during dynamic excitation, ultrasound techniques have been applied to measure the interfacial stiffness and real contact area during dynamic excitation [67, 68] and have shown that tangential stiffness depends significantly on both normal load and real contact area, which is also influenced by local plasticity. Second, these studies have demonstrated that tribochemistry is important to consider as a sliding surface with the same normal load and real contact area as a static surface has a lower tangential stiffness due to the asperities having less time to interact [69].

To provide more representative and insightful data, more recent test rigs look to recreate as much of the application's tribosystem as possible. One such example is the underplatform damper rig at Politecnico di Torino

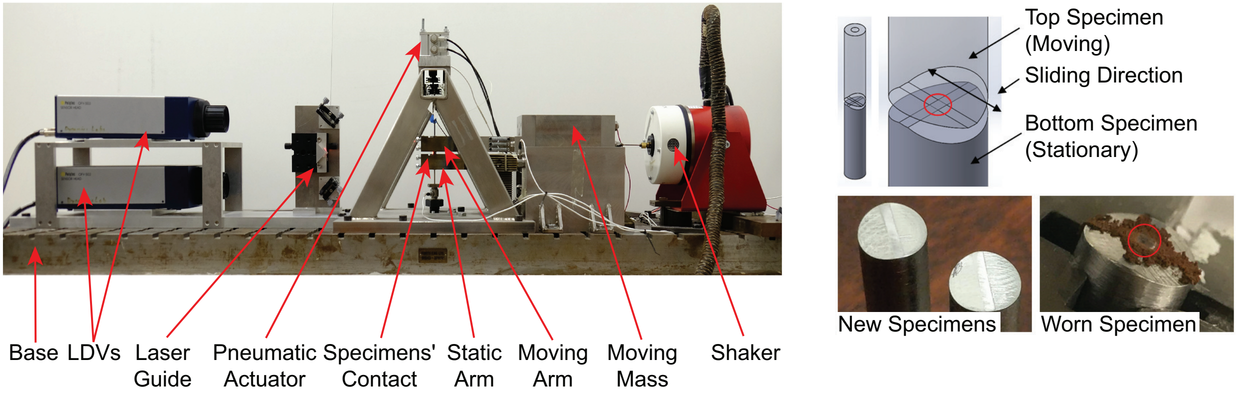

Fig. 3: (Left) The fretting test rig developed at Imperial College London for measuring frictional properties of materials [47]. (Right) Detail of the test specimen geometry, including before and after testing images.

[70], in which underplatform dampers are able to be directly tested in the *ex situ* rig under loading and environmental conditions that closely resemble the *in situ* target application. This type of approach can lead to a greater understanding of the physics of the *in situ* application [71] by allowing for methods to interrogate and study the interface that are impractical in the actual application.

3.3 *In Situ* Measurements of Bolted Joints

In situ studies of jointed structures have focused on two categories of experiments: characterizing the nonlinear properties of a joint and discovering the underlying physics. Essential data for calibrated modeling approaches (e.g., [42]) is information describing how a joint modifies a structure's stiffness and damping behavior as the excitation amplitude of the structure changes. Structures with joints often exhibit weak stiffness nonlinearities (i.e., showing relatively small changes in natural frequency as a function of excitation amplitude) and strong damping nonlinearities (i.e., often increasing the energy dissipation capacity of a structure by over an order of magnitude as the joint transitions from sticking to slipping). This data is often summarized in a backbone plot that details how the natural frequency and damping ratio change as a function of vibration amplitude [38, 72]. As an example, Fig. 4 shows a typical response for a Brake-Reuß beam (see [38, 13] for details of the Brake-Reuß beam), in which a decrease in frequency of 1-2% is observed over normal ranges of excitation amplitudes studied. At the same time, the damping ratio is typically observed to increase between 5x and 10x over the low amplitude damping ratio. Because the stiffness change is relatively small, assemblies with joints are often approximated as weakly nonlinear, or even quasi-linear, structures [36].

The backbone curves are indicators for how the hysteretic properties of a joint change as the excitation amplitude changes. The frequently observed loss in stiffness for a joint as excitation amplitudes increase is indicative of the secant stiffness of the joint's hysteretic behavior decreasing. As well, the increase in damping for a joint comes from the area enclosed by its hysteretic behavior increasing, as described in Fig. 5. Other features of the true spatially-varying hysteretic behavior are not captured by these metrics. Hence, these measurements of stiffness and damping are only a summary of the behavior within a joint. To understand the physical mechanisms within the joint, more probing experiments are needed, which can be designed to quantify the local kinematics, contact behavior, and other properties.

3.3.1 Characterizing Joint Nonlinearity

In situ experiments typically reveal the dynamics of the structure as a whole rather than the properties of an individual joint. Lap and flange joints are known to exhibit weak stiffness nonlinearities and significant energy dissipation [38, 74, 75]. However, because the stiffness nonlinearities are weak, in many cases the joints simply alter the effective modal stiffness and damping characteristics of the structure and cause these quantities to be amplitude dependent. Hence, most test methods resemble those for modal testing of linear structures. Typical measurement methods are based on either free or forced response measurements. Critically, in such tests it is the properties of the entire

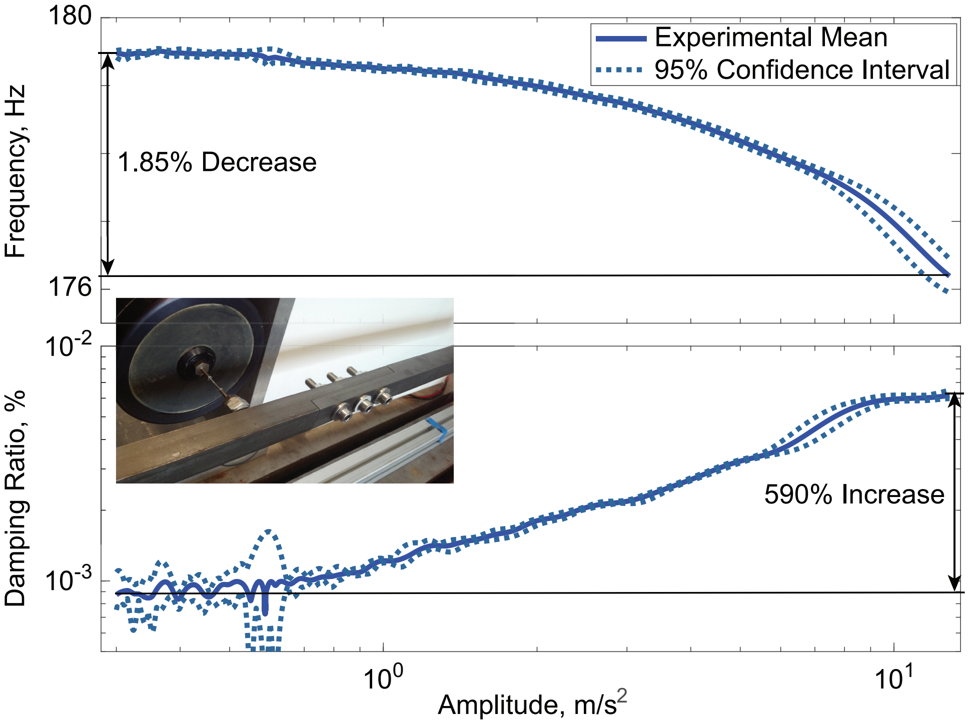

Fig. 4: Amplitude-dependent natural frequency (top) and damping ratio (bottom) for a typical Brake-Reuß beam (inset photo), which contains a three bolt lap joint, measured via a shaker ringdown test. Data from [56].

structure that are measured, which includes the influence of all of the joints; the joint characteristics are inferred from the dynamics of the structure in which they are embedded.

Free response measurements (e.g., from impact hammer tests) are typically filtered in the frequency domain [76, 72] and then analyzed via a Hilbert Transform [77, 37], Peak Finding and Fitting [72], or some other system identification technique [78, 79] to quantify the nonlinear change in frequency and damping as a function of amplitude[3]. This method has several significant limitations though. First, the joints are spatially distributed nonlinearities, rather than isolated to a specific mode (as assumed in modeling approaches for weakly nonlinear systems [36]). As a result, structures that are excited via impact hammer testing exhibit multi-modal responses in which the expression of the nonlinearity in an interface is governed by the total response (i.e., the sum of all of the modal responses). In filtering this response, the amplitude-dependent frequency and damping coefficient for a given mode changes as a function of how much the other modes in the system are excited [74, 80, 81]. Thus, the identification of nonlinear characteristics from impact hammer-driven free response measurements convolutes the nonlinear characteristics of the system with the modal content even when a filter is applied to the data. This is particularly noticeable in studies of chains of bolted joints where increasing the number of joints seemingly *decreases* the damping capacity of a given mode [82, 83] (as each joint is less excited and thus appears more linear), and is also observable in shaker tests. As an example of why modal amplitude is inappropriate for characterizing joint nonlinearities, Fig. 6 shows the modal interactions between the first and third mode of a system when either just the third mode is excited or both the third and first modes are simultaneously excited; significant coupling is observed between the two modes due to the nonlinearity of the joint in the system studied [80, 84].

In order to avoid this, the second approach is to use a shaker to provide single harmonic excitations. There are many different forced response measurement approaches, which are primarily differentiated by the control scheme employed for the shaker. Open loop methods (in which a shaker input voltage is controlled) can extract the nonlinear characteristics via the Nyquist plot [85, 43], which result in piecewise response curves that are often not contiguous, or via random excitation [86, 87, 88], which yield separate black-box models of the system for distinct excitation amplitudes. Of the many closed loop methods employed to characterize the nonlinear characteristics of a jointed structure, the most common ones are phase-locked loop testing [89, 90, 88, 91] and control-based continuation

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[3] The specific measure of amplitude (i.e., acceleration amplitude, modal amplitude, etc.) chosen varies quite dramatically from the measured acceleration at a specific location [38] to a modal acceleration or displacement amplitude [37].

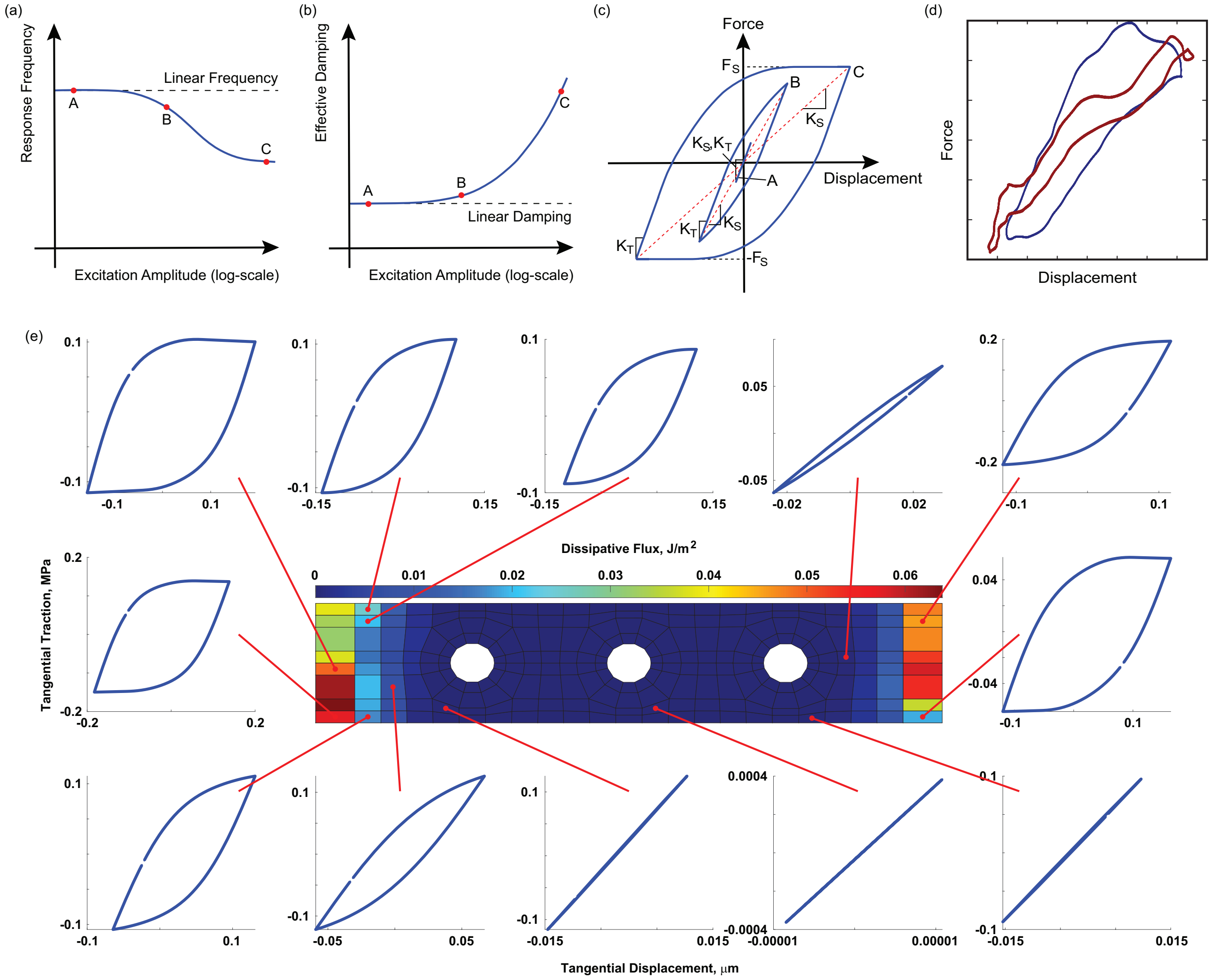

Fig. 5: Illustration of (a) amplitude dependent natural frequency, (b) amplitude dependent damping ratio, and (c) hysteresis curves for the identified points (A, B, and C) in parts (a) and (b). For each hysteresis curve, the tangential stiffness K_T and secant stiffness K_S are shown (for curve A, $K_T = K_S$), as well as the macro-slip limits $\pm F_S$. In (d) measured hysteresis loops for a jointed structure with an interface similar to that shown in (e); blue is a measured hysteresis curve from the initial testing, and red is a measured hysteresis curve from 12 hours of testing later. The presence of multiple loops within the hysteresis loops indicate the presence of higher harmonics of the response despite monotonal excitation. Data courtesy of [5] for the Brake-Reuß beam. In (e), the spatially varying hysteresis is shown, as predicted by the model of [56, 73] for the half Brake-Reuß beam; for each subplot, the vertical axes are the tangential tractions (friction force divided by area of the element, in MPa), and the horizontal axes are the relative motion (in µm).

[92, 93, 91] (which has not been widely adopted due to the challenges in implementation). In particular, phase-locked loop testing can be implemented in a number of different manners - such as by controlling for a constant response amplitude or for a constant excitation force. Of these two methods, amplitude control is seen as preferable as it maintains a constant excitation of the interface [43, 94, 40]. Two challenges in the use of shaker-based methods are that they change the system dynamics when the shaker is attached [95] and, more importantly, they can induce significant wear due to the longer-term excitation of the system compared to hammer testing [38, 47, 5]. Shaker-structure interactions often lead to higher harmonic responses even when the shaker is providing a monotone signal [95], and thus an active area of research is in developing new control schemes that eliminate these responses via a multi-harmonic shaker input [96]. The wear of jointed interfaces, on the other hand, cannot be avoided during extended shaker testing [38, 47, 5]; contrary to previously held assumptions, wear can manifest even at low load levels and high bolt torques for less than 15 minutes of cumulative testing [5].

A hybrid approach that avoids the negative consequences associated with shaker testing (namely the wear and

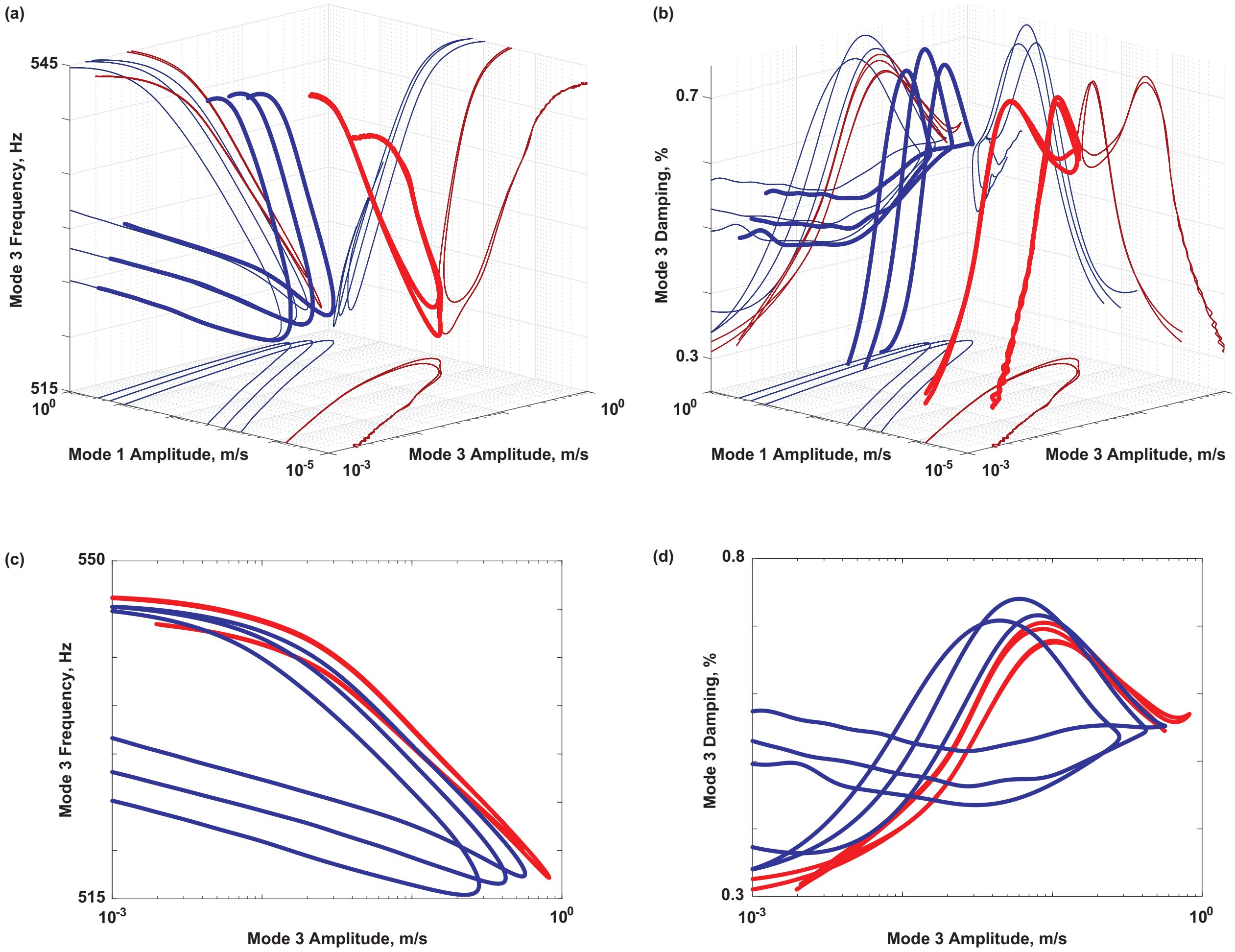

Fig. 6: (a) Amplitude dependent natural frequency and (b) amplitude dependent damping ratio for a system that is exhibiting modal coupling during multi-frequency shaker excitation. Red curves show the properties observed when exciting the system in only the third mode, while blue curves are those obtained when exciting the system in both the third and first modes. Shown are the results of five experiments for each condition. The thin lines are projections of the three-dimensional curves onto the corresponding planes. The projection onto the (c) mode 3 amplitude-mode 3 frequency plane and (d) mode 3 amplitude-mode 3 damping plane are shown to highlight that the response of the third mode is multi-valued in the presence of other modes. Data courtesy of [80, 84].

introduction of higher harmonic responses) is the shaker ring-down test [94, 40]. The experimental setup for a shaker ring-down test requires two changes from a hammer impact test (in which a test specimen is usually suspended in 'free-free' boundary conditions): first, an additional set of constraints are needed to preload the test specimen against the shaker, and second, an assembly consisting of the shaker, a stinger, a force transducer, and an impact hammer tip is aligned such that it is in contact with the test specimen at a point that would be normally used for impact hammer hits, as shown in Fig. 7. For the shaker's control scheme, a DC offset is used to both preload the shaker against the structure and to decouple the structure and shaker assembly so that they are no longer in contact. A typical experiment involves preloading the shaker assembly against the structure via a DC offset, using a phase-locked loop controller to drive the system to resonance at a fixed amplitude, providing a negative DC offset to decouple the system at a prescribed phase (typically $\approx 180^\circ$, when the system is moving towards the shaker and has just reached zero displacement), and recording the resulting free decay. The resulting free decay data contains only a single harmonic if the system is aligned correctly, resulting in time history data that does not require filtering before being analyzed to extract the nonlinear amplitude dependent properties (see Fig. 7).

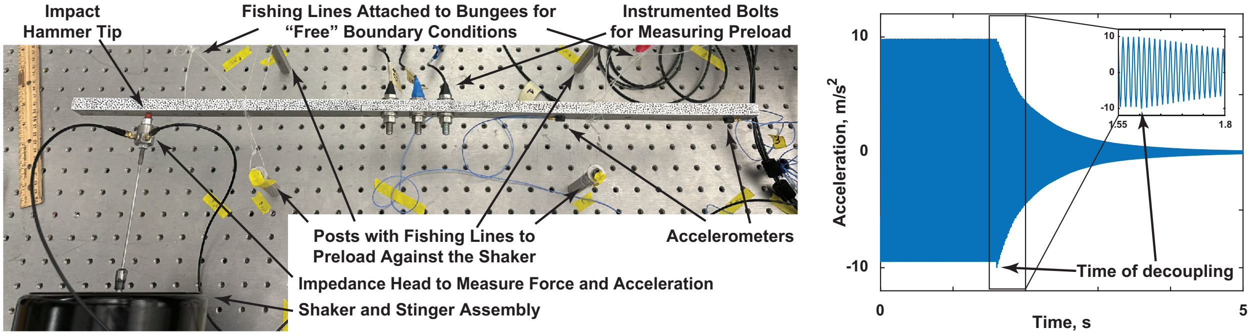

Fig. 7: (Left) Figure of the experimental setup for a shaker ringdown test. (Right) Typical time history measured from a shaker ringdown test; inset shows a magnification of the moment of decoupling.

3.3.2 Discovering the Physics of Jointed Interfaces

Response data gathered from free or forced characterization of the nonlinear dynamics of structures with bolted joints is useful for validating physics-based models of joints; however, it provides little insight by itself into the nonlinear mechanisms within the joints. Since the assessment of the state-of-the-art presented in [12, 13], there have been several extensive experimental campaigns focused on understanding the physics internal to bolted joints.

High speed videography and digital image correlation (DIC) have yielded an improved understanding of the interface kinematics [41, 97, 40, 94]. Before these studies, modeling approaches regularized the interface [12, 39, 98]; i.e., they assumed that the interface could be rigidly modeled with no substantial local kinematics. DIC studies, however, showed that the actual behavior of the interface can be far more complicated.

- Interfaces can exhibit significant clapping at their edges [41, 97], which is indicative of the contact pressures changing significantly (going to zero at the edges).
- Higher bolt preloads lead to lower contact areas due to Poisson effects; this can be explained via contact mechanics with the understanding that these interfaces exhibit receding contact [99, 1].
- The pressure cone/frustum of bolts is not 30° as commonly assumed, but significantly dependent upon the meso-scale topography[4] [97].
- Interfaces can exhibit asymmetrical behavior, which is also attributable to the meso-scale topography (specifically, in the study of [97], it was shown that one side of the interface had ten times greater separation than the other side).
- Lastly, even at very low load levels, it is possible for the entire joint to be in macro-slip [41].

With a new understanding of the interface kinematics from the DIC studies, a second set of studies sought to understand the distribution of contact pressure within the interface. Previous experiments had focused on using static pressure film to measure the interfacial contact pressure [12, 13]; however, these pressure films serve as integral measurements of the contact pressure - a record of the maximum pressure throughout the entire experiment. As such, they would indicate higher contact pressures at the edges of the contact (due to the initial assembly) than was experienced during dynamic excitation (once the bolt preload had been increased and the edges of the interface peeled apart). To measure the contact pressure during dynamic excitation in real time, an electronic pressure film was used [100]; while this pressure film changed the interfacial contact conditions, high fidelity finite element analysis verified that the trends observed are still applicable to understanding the contact pressure during dynamic excitation [101]. In particular, the electronic pressure film measurements showed that the contact pressure varied considerably across the interface throughout dynamic testing. Near the edges of the interface, the contact pressure cyclically alternated between zero and a maximum pressure (approximately 20% of the pressure near the bolts) every period of excitation. Near the bolts, previous models had assumed that the contact pressure would be constant throughout

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[4] Here, meso-scale is used to refer to features of ≈10 to 250 μm in size, which corresponds to machining features and local curvature that require polishing to remove.

dynamic excitation, but the study showed that the contact pressure could vary by up to 25% throughout a period of vibration near the outer bolts of the interface, and by 5% near the inner bolts [100].

Paramount to modeling the interface kinematics and forces is having an in-depth understanding of the role of the bolt preload on the interface's contact pressure and the overall system's response. Numerous analytical models have been used over the past century to model the relationship between the applied bolt torque and resulting bolt preload [102]. In order to better understand the variation of contact pressure around the bolts, instrumented strain gauge bolts were developed to measure the bolt forces during testing [103, 55]. The first consequence of these measurements was an improved understanding of the relationship between the bolt torque and preload. These studies, however, found that the actual bolt loads varied between 10% and 102% of the expected preload value. As illustrated in Fig. 8, with the bolt torque set to 20 Nm (equivalent to an expected value of 1800 μ-strain) the measured population density of measured bolt strains shows a median near only 35% of the expected value. Once experiments began to control for the strain in bolts rather than the torque applied via a torque wrench, the observed experimental variability dramatically decreased [38, 55, 104]. Additionally, bolt forces were only observed to change by ±1% during dynamic excitation, and did not appreciably change due to interfacial wear [5].

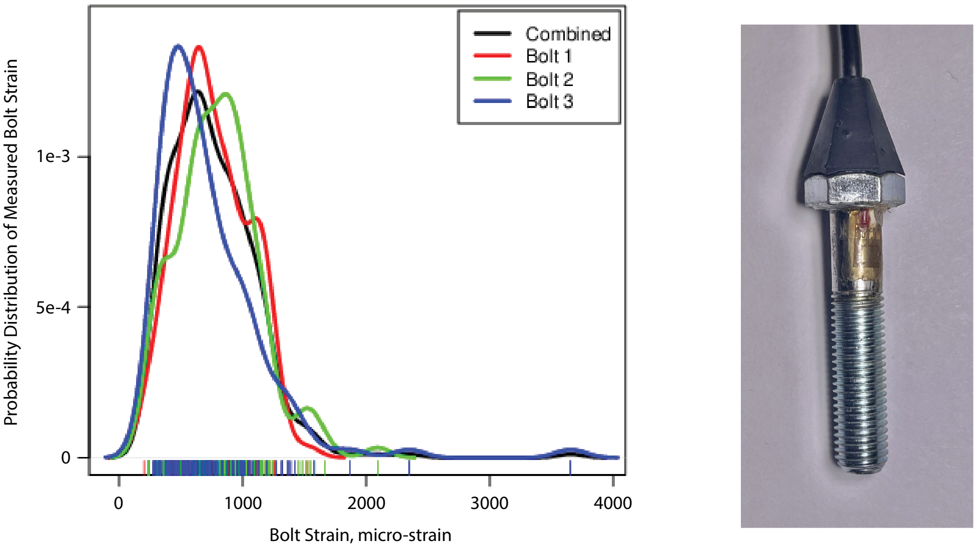

Fig. 8: (Left) Distribution of measured bolt strains for 100 different assemblies of three different instrumented bolts tightened to 20 Nm (equivalent to 1800 μ-strain) [103]. (Right) Image of an instrumented strain bolt, with fabrication instructions in [105].

To summarize, experiments over the last decade have demonstrated several cases that violate a number of previously held assumptions (see, for instance, the modeling approaches presented in [12]):

- The contact patch can vary significantly with time, including near the bolts [41, 97, 100].
- There can be measurable motions across the interface, both interface opening (clapping) at the edges of the interface, and even macro-slip across the entire interface at low load levels [41, 97].
- Energy dissipated by the interface can be due to more than just micro-slip: both clapping and plastic deformations (for more information about the plastic deformations, see [106, 56]) occur [41, 97].
- Higher bolt torques can lead to lower contact areas due to Poisson effects [41, 97].
- The meso-scale topography of the interface significantly influences the dynamics of the structure [41, 97, 74, 104].
- There can be significant mode coupling [74, 80, 81].
- Fretting wear can set in within a few minutes of shaker testing, even at moderate load levels [38, 5].

- Bolt torque is one of the greatest sources of experimental uncertainty/error, and instead bolt strain should be controlled for in experiments of jointed structures [103, 55, 74].

In light of these findings, there are two primary insights to guide high fidelity modeling efforts. First, the commonly used approach of using spider elements in order to tie all of the degrees of freedom on one surface to a single node overly rigidizes the joint surfaces and neglects the joint kinematics (e.g., clapping), which have been shown to contribute approximately the same amount of energy dissipation as frictional interactions [73]. Second, a minimum complexity model to account for the observed physics would be an elastic dry friction model (i.e., a Jenkins element that allows the normal force to vary [56]) with a penalty stiffness contact model to allow for separation. Of course, practitioners are likely to seek more simplified, calibrated modeling approaches than these. This can be a valid modeling strategy; however, care must be taken to understand the ramifications of the simplifications introduced in a lower fidelity model. Further research is needed to understand when the dominant physics of an assembly can and cannot be captured with these simplified models, and multi-scale modeling approaches are needed to relate the interface dynamics to the changes in stiffness and damping that are observed in the modes of the entire assembly.

4 Evolution of Joints Modeling Over the Last 10 Years

Building on the new experimental insights into the physics of jointed structures, two distinct and necessary modeling approaches have emerged: physics-based models and computationally efficient calibrated models. The previous assessment of the state-of-the-art, documented in [12, 13], determined that models of jointed structures typically had errors in predictions of stiffness of 25% and errors in prediction of damping capacity of two orders of magnitude. A recent community challenge [107, 108] allowed this to be reassessed for a structure that is relevant to many applications. Participants were asked to make a blind prediction of the nonlinear dynamics of a yet-to-be-fabricated structure that contained both geometric and interfacial nonlinearities. Some of the predictions obtained were within the range of experimental uncertainty for stiffness and while the damping predictions showed much more scatter, one method gave predictions that overlaid the experimentally measured damping [107] (albeit using the measured linear damping as a starting point), as shown in Fig. 9, and further highlights conclusions from [56] that methods to predict linear damping are a significant limiting factor for predicting the nonlinear dynamics of a jointed structure. This is not to say that blind predictions of jointed structures is now solved/trivial, but rather that the best modeling approaches have the potential to be predictive with continued investment. Challenges still exist, though, for extending this result to real, large-scale structures.

4.1 Friction Modeling for Jointed Structures

One topic that is paramount for understanding the evolution of joints modeling over the last decade is in how models of interfacial friction have changed during this time. The breadth of models used to describe hysteresis in joints is summarized in the recent reviews [10, 1]. In particular, there are two phenomenological categories: point-wise implementation of friction models and patch-wise implementation of friction models. Point-wise implementations (e.g., elastic dry friction or Jenkins elements [109, 110, 64, 111, 112, 113]) are often implemented in a node-to-node interfacial model. As such, many friction elements are used and a computationally efficient formulation is needed. By contrast, patch-wise friction models typically use hysteretic models that are governed by an underlying differential equation (e.g., Bouc-Wen [47]) or integral equation (e.g., Iwan [45]). All of these approaches, though, remain heuristic in nature; that is, they are not derived from a more fundamental description of the physics, and therefore introduce epistemic uncertainty to models [51].

While there have been several approaches to reduce the model form error (such as suggesting the addition of a post-slip stiffness, which is common in a Valanis model [51], or accounting for the difference between static and dynamic friction with a five-parameter Iwan model [114, 115, 116]), these heuristic models are still predicated on underlying assumptions for their development and formulation. One set of assumptions in particular, termed the Masing conditions, provide a computationally efficient framework that is often exploited. While many systems obey the Masing conditions, recent experiments have shown common cases in which they are violated. The assumptions of the Masing conditions postulate that the hysteretic behavior of a system can be completely described by the force-displacement relationship generated by the initial loading curve, as shown in Fig. 10. These assumptions are in violation with the recent experimental observations summarized in §3 - particularly that the frictional forces in an interface are not symmetric and are heavily dependent upon the time-varying normal pressure within the interface. A new modeling framework termed hysteretic manifold, which is described in [117], proposed that Masing models can

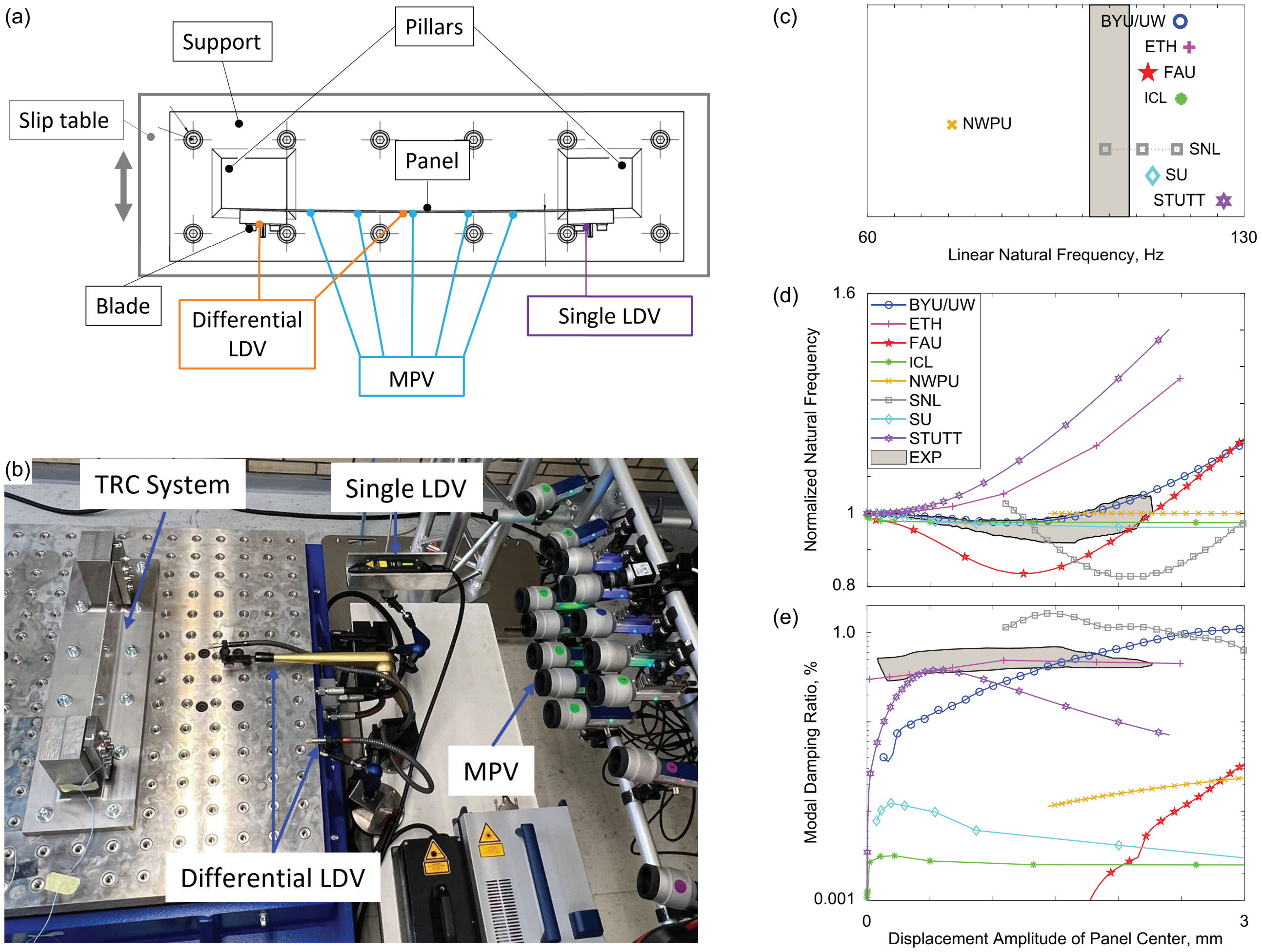

Fig. 9: Overview of the results of the Tribomechadynamics Research Challenge, in which researchers were tasked with making blind predictions on the dynamics of a novel jointed structure. (a) Schematic of the structure showing curved panel and bolted connections to the larger assembly. (b) Photo of the experimental setup with the assembly mounted on a slip table. (c) Predictions of the linear natural frequencies compared against the experimental measurements (grey box). (d) The normalized nonlinear frequency properties for the first mode as a function of displacement amplitude, and (e) the corresponding damping properties. Both (c), (d), and (e) share the same legend to indicate the different research groups that participated: Brigham Young University/University of Wisconsin Madison (BYU/UW), ETH Zurich (ETH), Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), Imperial College London (ICL), Northwestern Polytechnical University (NWPU), Sandia National Laboratories (SNL), Swansea University (SU), and the University of Stuttgart (STUTT), with the experimental results indicated by EXP. The figure was adopted from [107].

be reconciled with experiments by treating them as a multi-dimensional manifold that includes the influence of normal load on the hysteretic frictional forces.

As the assumptions underlying existing interfacial friction models are challenged, more predictive models will result. As an example, [56] presented a rough contact model that accounted for interface plasticity and meso-scale topography with only material damping and the coefficient of friction being unknown (tunable) parameters, and was able to predict the amplitude-dependent frequency and damping of a jointed structure with significantly less error than previous approaches. Much work is still needed, though, to predict the material damping of a structure and to predict both the coefficient of friction and the wear rates of an interface [1, 56].

4.2 Physics-Based Models for Jointed Structures

At the outset of the experiments described in §3, the standard modeling approach sought to calibrate models of jointed structures to the responses of their individual modes. This approach proved to require a herculean effort to match the response of more than two modes simultaneously, even for a simple structure [42]. Further, these calibrated models

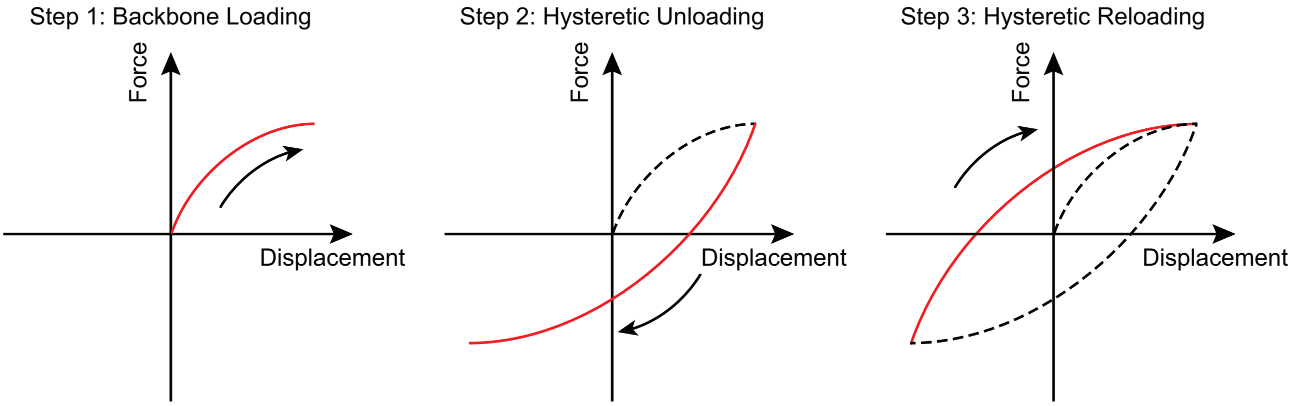

Fig. 10: Illustration of constructing a hysteresis loop using the Masing conditions. The curves in steps two and three are scaled (and rotated) versions of the curve in step one. Figure from [117].

had no generalizability to other structures with identical joints. The **surrogate system hypothesis** [118] upended this modeling mentality by demonstrating that by decoupling the dynamics of the structure from the dynamics of a joint, one could make blind predictions for the dynamics of a new structure given calibration data from a different structure with an identical joint. In order to decouple the dynamics of the structure from the dynamics of a joint, a spatially discrete joint model is required (as opposed to a modal implementation such as in [119, 120]); however, calibrative modeling frameworks are permissible.

The key to recent successes in making accurate *predictions*[5] has been in the incorporation of findings from recent experiments into new modeling frameworks. In particular, the key modeling advances to enable predictability have been:

- Initializing an analysis with a nonlinear contact mechanics step to calculate the static loads in an interface prior to dynamic loading;
- Modeling the kinematics and thus normal contact pressure throughout the entire interface;
- Incorporation of meso-scale curvature deterministically into models; and
- Incorporation of micro-scale features, and below, statistically into models.

Pioneering this modeling approach has been the multi-scale, multi-physics modeling framework referred to as tribomechadynamics [118, 55, 112, 121, 51, 56], which uses a hierarchical multi-scale modeling approach to introduce small-scale tribological effects (that span from the micro-scale to smaller features) into large-scale models for the structural dynamics. The tribomechadynamic models of [118, 55, 121, 51, 56] introduce the nano- and micro-scale (up to 10 μm) effects into the model via a one-way coupling via zero thickness elements [122, 55, 121, 56]. A mesh of zero thickness elements is superimposed on top of an interface's mesh, and the tractions within the interface are related back to the structural degrees of freedom via a hyper-reduction that allows for direct calculation of interfacial tractions without any intermediate transformation [55]. As the zero-thickness elements integrate over each surface's local deformations (i.e., kinematic behavior), the zero-thickness elements allow for nodes on each interface in contact to be non-coincident. These elements allow for both introducing the tribological effects into the model and deterministically varying the meso-scale topography of an otherwise flat interface.

With this more faithful representation of experimental observations in place, predictions on the response of a structure due to modifications of the interface can now be made with acceptable levels of accuracy. As meso-scale parameters (e.g., machining features, curvature, undulations on the order of 10 μm) are incorporated deterministically in tribomechadynamic modeling frameworks, the influence of different features and patterns in the interface topography can be systematically varied to ascertain the sensitivity of a structure's response to manufacturing uncertainty/tolerances. The consensus from a range of studies [55, 101, 123, 104] is that the meso-scale topology (i.e., features 10 μm in size and greater) significantly affects the distribution of interfacial contact pressure, which, in turn, changes the stiffness and damping properties of the entire structure (such as experimentally observed in [41, 74]).

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[5] As opposed to matching results via calibration.

A remaining open question is: what level of accuracy is necessary to capture the meso-scale topography? Experiments [38] and tribomechadynamics models [55, 56] have shown that there is significant dependency of the frequency and dissipation response of a structure on the meso-scale curvature of its interface (e.g., see Fig. 11); however, these findings have not been consistent across all of the benchmark systems, such as shown in [104], which compared both spherical and flat interfaces for the S4 beam [124].

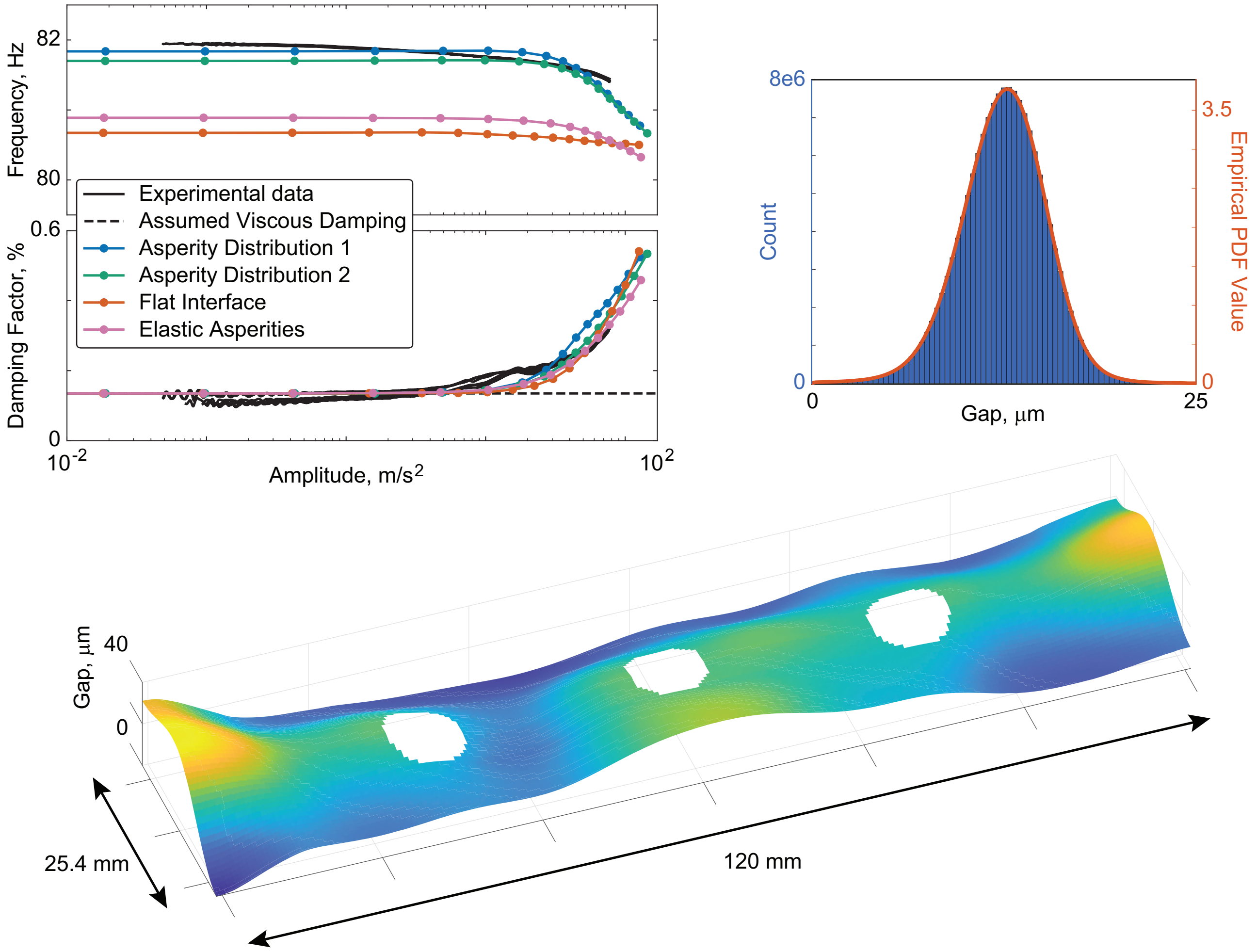

Fig. 11: (Left) The predictions for the dynamics of the half Brake-Reuß beam using a measured distribution of asperities (asperity distribution 1), the measured distribution from a second, nominally identical beam (asperity distribution 2), a flat interface with no meso-scale curvature but asperities described by asperity distribution 1, and an interface identical to the first model, but with asperities modeled as perfectly elastic. (Right) The measured asperity distribution for the half Brake-Reuß beam used in the (Left) simulations. (Bottom) The measured meso-scale topology of the same beam. Figure based on [73].

One prominent application of this research has been in the design of components for aeroturbines, particularly underplatform dampers [125, 126, 71, 127]. A challenge in predictive design for aeroturbine components is that they contain non-unique residual tractions within the frictional interfaces [128, 71, 113]. Due to the non-uniqueness of the residual tractions (as the stuck regime bounds the frictional force to be between $\pm\mu N$), this can lead to an order of magnitude difference in the accelerations at which slip initiates [113]. Predictive design, though, has enabled researchers to move beyond the iterative-based design techniques to a more computationally heavy framework [30] (which is necessitated by industry's reduction in prototyping), enabling the proposal and optimization of new underplatform damper geometries [125].

As wear is known to significantly change the contact stiffness and frictional behavior of an interface [38, 47], physics-based models have sought to investigate the influence of wear on the dynamics of assembled structures [129, 112]. As a predictive model of wear is a grand challenge within the tribology community [1], empirical methods are still needed for estimating the evolution of wear properties over time [47, 112]. An additional requirement for investigating wear is a simulation framework that is amenable to simulating many (potentially millions) of cycles of loading to assess the long term influence of wear and surface damage on the contact properties of an interface.

4.3 Computationally Efficient Modeling Approaches for Jointed Structures

By the late 1990s, finite element packages began to incorporate nonlinear solvers capable of solving frictional contact problems, as well as plasticity and the other physics that are observed at joints. Furthermore, parallel computing and powerful computer clusters can speed up these simulations. Hence, software vendors give the impression that it has long been possible to simulate the mechanics of joints. Such a view ignores a few important facts. First, it is still prohibitively computationally expensive to simulate a structural dynamic response of these structures over a suitable time window. The fine detail that is needed to capture the contact in a joint [48] (much less rough contact in general [130]), coupled with the expense of iterative nonlinear solvers and the fine time steps needed to resolve motions at the interface makes this approach intractable for most applications of interest. Second, even if these models could be solved, there is considerable uncertainty regarding the various physics to model, such as friction [12], plasticity, and scale of geometry, not to mention potential effects of wear, corrosion, heat transfer, etc. Considering all of this, research has focused on accelerating computation and determining which of all the possible physics are most important to model.

Jointed structures are most commonly simulated in the frequency domain (due to both the high computational cost of transient simulations and the heavy prevalence of rotating machinery, which often requires frequency response functions to characterize a machine's performance, in the motivating applications of research on jointed structures). To provide an accurate estimate of a system's response, multiple harmonics of the response are often needed, which drives many analysts to using harmonic balance-based simulation methods [131, 132]. In particular, the extended periodic motion concept (EPMC) [133, 134] has found much success in studying jointed structures [135, 123, 56]. The underlying advance of EPMC is that a negative damping term is introduced to the system to balance out the dissipative characteristics of the system, allowing for periodic solutions of non-conservative systems. This advance, together with the single nonlinear mode theory [134], allows for a more efficient computational method than a traditional harmonic balance method; however, in the presence of significant modal interactions, EPMC's limiting assumptions necessitate the use of harmonic balance instead. A recent alternative, termed variable phase resonance nonlinear modes [136, 73], uses EPMC to construct a reduced order model (ROM) of a system that exhibits internal resonances.

As a computationally cheaper alternative to frequency domain solvers, one major advancement in the past decade is the advent of quasi-static simulation techniques, which have proven to be efficient methodologies for calculating the hysteretic properties of a jointed structure responding in a single mode [39, 137]. The most common quasi-static solver is quasi-static modal analysis (QSMA), which combines the ideas of implicit condensation [138] with the Masing conditions [139] to calculate the amplitude-dependent stiffness and damping of a structure containing jointed interfaces [140, 39]. QSMA works by recognizing that, when a hysteretic structure is vibrating freely in a single mode of vibration, the modal force-displacement behavior can be constructed from a single hysteretic force-displacement curve. Consequently, only the initial loading of the structure from static equilibrium to some maximum displacement needs to be calculated. The unloading and reloading of the structure can then be constructed from the initial loading curve. These results can also be used to derive the effective damping (from the energy dissipated per cycle) and quasi-linear natural frequency (i.e. the average natural frequency as the stiffness varies over the vibration cycle). Hence, these two properties, which characterize the mode of interest over a range of vibration amplitude, can both be extracted from a single nonlinear quasi-static simulation. Various works [39, 104] have correlated the natural frequency and damping estimated using QSMA with those obtained from measurements, finding good agreement.

QSMA, however, does have several limitations: First, it assumes that mode shapes do not change with amplitude. While this assumption is valid for weak nonlinearities, it can introduce errors into numerical analysis that increase with amplitude. Second, its efficiency comes from assuming that the Masing conditions apply, but often jointed interfaces exhibit violations of the Masing conditions [45, 117]. To address these issues, a Rayleigh quotient-based nonlinear modal analysis (RQNMA) [137] approach was formulated based on a generalization of the work-stationarity principle in which the problem of stationarity of virtual work is posed along with an amplitude constraint (interpreted as a modal amplitude). In the linear limit, the Lagrange multiplier of the constraint is the Rayleigh quotient corresponding to the potential energy of the system. Since mode-shape variation is part of the formulation of RQNMA, it is applicable to a broader class of nonlinearities. Alternative implementations of both RQNMA and QSMA exist which do not rely on the the Masing assumptions, yet this increases the computational cost by approximately five-fold for a computation of the natural frequency and damping at a single amplitude as the first unloading curve and full reloading curve both need to be calculated in addition to the initial loading curve[6]. However, QSMA traditionally estimates these quantities at all

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[6] In cases where there is settling of the interface, this cycle sometimes needs to be repeated multiple times before a steady-state solution is

lower amplitudes as well, or a total of 50-100 points, so the actual increase in cost for QSMA can be hundreds of times larger. As an example of the errors that can be incurred by neglecting the change in the mode shapes with vibration amplitude, Fig. 12 compares both QSMA and RQNMA (which allows mode shapes to change) to EPMC [133], which is taken as a truth solution for this problem. The quasi-static approaches are unable to capture internal resonances. Further, as the mode shape of the system studied changes substantially, QSMA exhibits significant error in the large amplitude regimes. That is to be expected, because the two DOF system in question exhibits extreme mode localization, where the mode shapes change dramatically as vibration amplitude increases. Similarly, other works have shown that, when a jointed structure is excited in a clapping mode simultaneously with a shearing mode, the Masing assumptions are violated, and in that case QSMA should once again be replaced with RQNMA or some other approximation [137, 73].

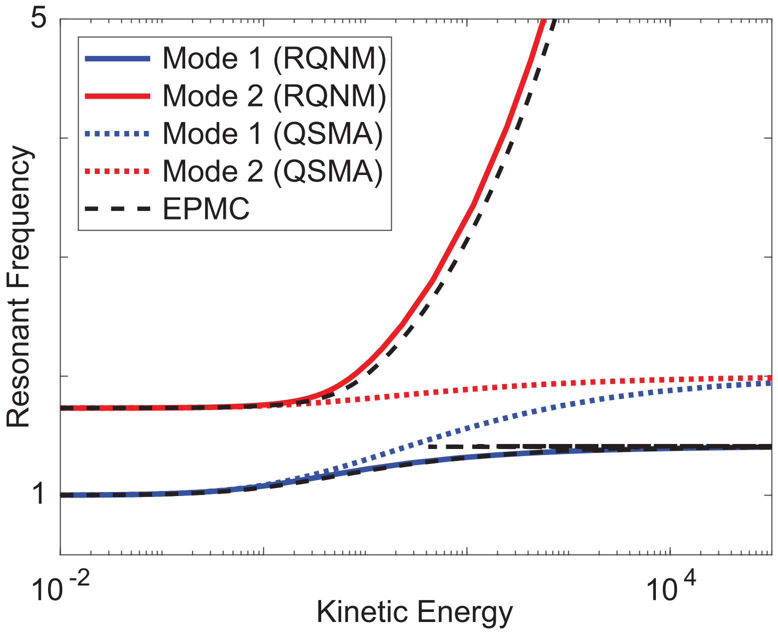

Fig. 12: Comparison of RQNMA (solid lines), QSMA (dotted lines), and EPMC (dashed lines) for predicting the resonant frequencies of a two degrees of freedom system with a cubic spring; adapted from [137].

One limitation of frequency-domain and quasi-static solvers is that they are unable to study transient (shock) or random vibration. For these categories of problems, time-domain simulation techniques are necessary. Furthermore, whereas prior research focused on capturing the average dissipation and stiffness loss over a cycle of vibration, the physics may be different under shock loading as the event of interest is one cycle of rapid deformation of the joint. There has been little research on improving existing time-domain methods, but one recent advance has demonstrated that using contact modes in place of nodal discretizations of an interface can result in a computationally efficient framework to study jointed structures [141, 142]. Contact modes use a flat projection with two groups of trial vectors: one from a standard reduced order modeling framework such as Hurty/Craig-Bampton in which the degrees of freedom of the interface are fixed [143], and a second (the contact mode itself) to add flexibility to the jointed region. Similar to [121, 51], much of the efficiency in the use of contact modes [141, 142] comes from the use of a hyper-reduction framework so that friction forces can be evaluated directly in the ROM. An alternative approach seeks to isolate the region containing the joint, capturing its influence through an appropriately defined force acting on the boundary of this nonlinear region [144, 101, 145]. With this, model order reductions can be applied separately to the structure from the joint itself to increase the computational efficiency of the simulation.

The methods mentioned above have the potential to be predictive, which can be critical in some applications where experimental data is not available. In applications where data is available, one may be able to use the surrogate system hypothesis [118]. This has the potential to produce more efficient models, or physics based models that are calibrated to measurements, and may be a more practical solution for design studies than entirely relying upon physics-based models [146].

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found [51].

5 Shortcomings in Knowledge of Jointed Structures and the State-of-the-Art Methodologies Used to Study Them

The goal of joints research is to enable more efficient, lighter weight, and more wear-resistant designs of structures and engines. Reaching this objective requires that the designer reduce the uncertainty in predictions that is currently associated with the presence of mechanical joints. Better predictive modeling is one potential method for addressing this goal, but not the only one. For instance, one might instead be able to experimentally discover an interfacial geometry that behaves linearly and reduces the variability present in current joints, making them robust to manufacturing tolerances. This would eliminate the need for predictive modeling, yet experience has shown that there would be trade-offs with such an approach; linear joints typically provide far less damping and hence increase the stresses in the structure. Hence, to truly advance the science of mechanical interfaces, predictive modeling will be an important aspect. To that end, the barriers that currently impede progress towards designing more efficient, lighter weight, and more wear-resistant structures can be divided into three categories: gaps in understanding of the physics internal to an interface, gaps in understanding how a mechanical interface interacts with the larger structure that it is a part of, and the computational techniques necessary to scale up solutions to large-scale problems.

5.1 Interfacial Physics - Knowledge Gaps and Opportunities

While significant progress has been made in understanding the physics internal to jointed interfaces over the past decade, as detailed in §3, there are still many open research questions:

- How do frictional forces evolve with load, velocity, contact area, tribochemistry, temperature, environment, and other factors?
- How does wear progress in a new tribosystem, and how does it influence the observed dynamics of the structure?
- What is the relationship between nano/micro-scale properties (e.g., the contact stiffness between two asperities in an interface), meso-scale properties (e.g., the contact stiffness across a patch containing many asperities), and macro-scale properties (e.g., the amplitude-dependent stiffness of a structure)?
- Can quantities inside of a joint be measured without significantly changing the joint itself?
- How do the physics of jointed interactions differ in the shock regime, which is typically characterized as high energy and short duration?

5.1.1 Predictive Tribological Models

An open challenge within the tribology community is the discovery of predictive friction and wear models [1]. Existing models of friction and wear are largely empirical, and often represent wear as an Archard type model, which approximates wear as being proportional to sliding distance, due to its simplicity and ease of implementation. Other common wear models include thermodynamic/entropic models [147, 148, 149], in which wear is modeled as an irreversible, dissipative process with degradation described by the generation of entropy, and the energy wear approach [150, 151, 112], which equates dissipated energy to wear volume. Like the friction models described in §4 and [10], these are heuristic approaches that simplify the phenomena of wear to be described by a small set of (or just one) mechanisms. Tribological systems, though, are multi-scale (spanning from femto-seconds to years, and from atoms to deca-meter and larger structures), multi-physical systems spanning mechanical (solid and fluid), thermal, electromagnetic, chemical, quantum, and many other mechanisms [11].

Even the most common of friction laws, namely the Amontons-Coulomb law of friction, is typically applied without regard to scale. While experimental measurements are typically performed for single-point contacts (or some limited point-like contact area) the resulting friction law is often applied for contacts with appreciable contact area. If, for example, the loading, slip velocity, tribochemistry, etc. vary over the contact area and/or evolve over time, then these factors would be expected to impact the effective coefficient of friction that would be implemented in simulation [20, 21, 152, 153, 154]. However, at the moment there is no general understanding of how to connect the small to large scale contact problems.

5.1.2 Local Versus Global Properties

As contact properties are measured in fretting and tribological test rigs, it is clear that there is a scale- and geometry-dependence of the contact properties [66]. This dependency of contact parameters could be due to the distribution of contact pressure and the real contact area, which is determined by meso-scale curvature (e.g., from machining features), roughness, and macroscopic loading of the contact patch. The prediction of the real contact area, though, has been a significant challenge within the contact mechanics community [155]. To circumvent this challenge, rough contact models have been proposed to statistically capture the sub-micro-scale effects for a meso- or macro-scale model [156, 157, 56]. However, the relationship between geometry and scale is still unknown (for instance, see [1] for the challenges of measuring hardness across scales), which gives rise to the question of how can contact properties measured at one scale (with a specific geometry and set of loading conditions) be used to predict the contact properties aggregated at a larger scale?

Further complicating the relationship between local and global properties is the manner in which measurements are made at both scales. For instance, in the Brake-Reuß beam, contact stiffness (as evidenced by the natural frequency of the structure) decreases with wear (Fig. 13(a)); however, in the fretting rig of Fig. 3, the same material exhibits a stiffening with wear (Fig. 13(b)). Thus, in order to have comparable measurements across scale, not only is reproducing the material and normal load important, but also the excitation, boundary conditions, and stress state of the interface – i.e., the entire tribosystem of interest.

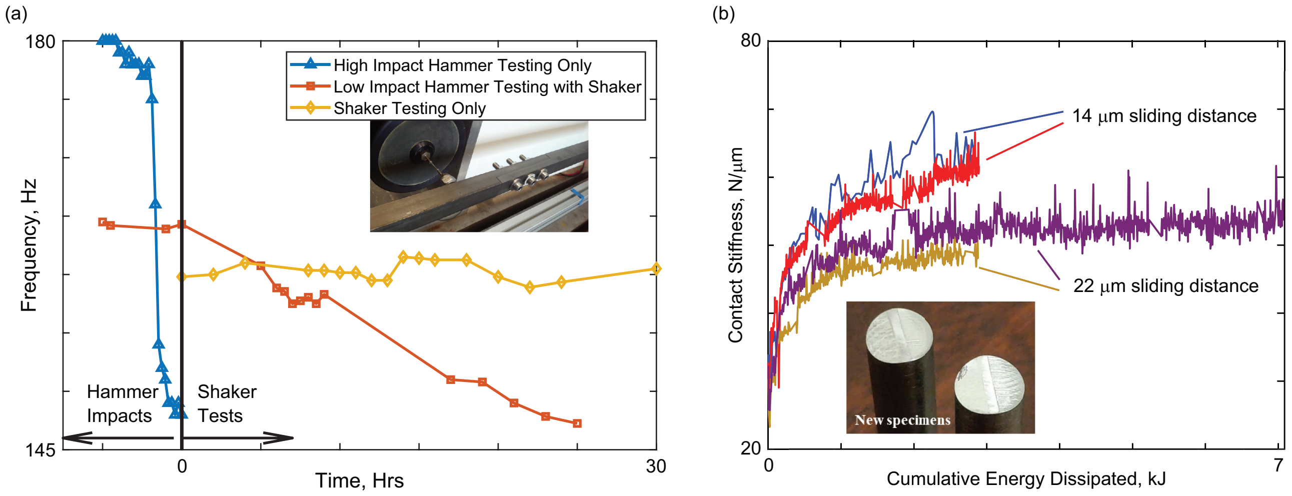


Fig. 13: (a) The evolution of wear as manifested in the natural frequency (structural stiffness) of the Brake-Reuß beam [43], (b) and the evolution of wear in the contact stiffness of two fretting specimens made from the same lot of material as the Brake-Reuß beams shown in part (a); the measurements from (b) come from [47] (Fig. 3).

5.1.3 Discovering the Physics of Jointed Interfaces

Most research on jointed structures focus on metallic joints. By their very nature, it is impossible to see inside of the joint, relegating the physics of the interface to be deduced either by *ex situ* approaches, indirect inferences (such as measuring a frequency-dependent amplitude [38]), or by fundamentally changing the interface (such as by switching materials to a clear polymer [158] or inserting an electronic pressure film [100]). Recent approaches have focused on using ultrasonic methods to study the real contact area of metallic interfaces *without* modifying the contact interface itself [68]. While digital image correlation [97, 41] has proven extremely useful in understanding the interface's kinematics (under the restriction of exciting and measuring the response in only one plane), efforts to use advanced techniques such as x-ray-based digital image correlation [159, 160] have proven unsuccessful as the techniques need further maturation before being useful. Thus, opportunities exist in pioneering new techniques to measure inside of the joint - whether through advanced imaging methods or switching to categories of metals and ceramics that are transparent at measurable wavelengths (such as the infrared).

In contrast to the Coulombic view of friction, frictional forces are highly rate-dependent. In a study of high frequency (kHz) vibration, new phenomena appeared in measurements that were not observed at the lower frequencies that most frictional characterization tests occur [161]. Further, in the context of shock loading of joints, it is presently unclear how relevant the existing body of knowledge is for modeling energy dissipation and shock transmission within a joint.

Existing models are predicated on studies of joints under low amplitude excitation (keeping the joints themselves within the micro-slip regime). Shock, conversely, is a loading event that can take a joint well into the macro-slip regime. This change in regimes from micro-slip to macro-slip both goes substantially beyond the limits of existing modeling frameworks and experimental data, and engages different mechanisms for energy dissipation. Beyond the three joint rule (in which 40% of the shock response spectrum is dissipated by a joint for up to three successive joints) [52], what are the salient physics of energy dissipation (and transmission/reflection/shunting) during shock? Most high rate experiments studying the response of joints under shock loading utilize split Hopkinson bar setups [162]; however, these experiments typically study bonded joints [163, 164]. Even for bonded joints, the mechanisms of energy dissipation and failure during quasi-static loading are significantly different than the energy dissipation and failure mechanisms during high strain rate loading [164]. It is clear that communication between the high strain rate testing community and the joint mechanics community would be beneficial for exploring and understanding the salient physics of jointed interfaces during shock events.

5.2 Jointed Structures - Knowledge Gaps and Opportunities

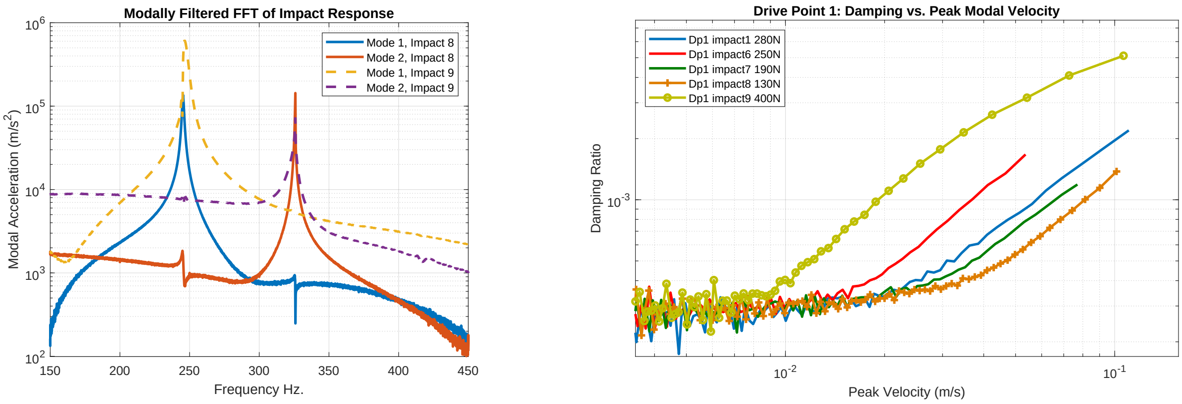
Independent of discovering the physics of interfacial contact, a parallel research effort focuses on understanding how a localized nonlinearity, such as a joint, couples with and modifies a larger structure. Prior works have relied on quasi-linear models, so that measurements and modeling efforts could be focused on the effect of the joints on the responses of different modes of the structure. However, a joint is a spatially localized nonlinearity (and not a modal nonlinearity), so these approaches are known to be limited to special cases or weak nonlinearities. Experimental methods are far simpler when a structure can be approximated as quasi-linear, and simulation methods such as QSMA rely on similar assumptions. As such, there are both modeling and experimental gaps in understanding the effect that joints have on the response of a structure in the most general case:

- What is the most effective way to experimentally characterize a nonlinearity when multiple modes are likely present in a structure's response?
- Can a building block approach be formulated for accurately incorporating jointed subcomponents into a larger structure?
- How does a structure influence the expression of damage within an interface, and how does that damage couple with the structure's response?
- How can we predict material (low amplitude) damping?

5.2.1 Characterization and Analysis of Nonlinearities in the Presence of Multiple Modes

In general, the experimental and computational techniques for studying the response of nonlinear systems are well suited for investigating single mode responses (e.g., QSMA, EPMC, nonlinear system identification techniques [78], etc.). The current methods, however, all break down in the presence of multiple modes. As highlighted by Fig. 6, the relative participation of different modes in a system can significantly change the measured expression of a nonlinearity. This is further illustrated in Fig. 14. In general the presence of nonlinearities couples together the structural modes of the system, so that the energy distribution across the modes alters the observed frequency and damping within an individual mode [165]. In addition, resonances between modal frequencies can lead to direct energy exchange between modes. Because joint nonlinearities are spatially localized, their contribution to the dynamics of a structure depend on all of the modal responses, and filtering the response about a single mode will lead to mischaracterizations.

To approach this challenge, it is worthwhile to specify what the characterization of nonlinearities in an experiment is to be used for. For model validation, it is paramount that the precise loading of the experiment (and thus modal content) is replicated. Experiments that are single frequency initially (e.g., shaker ring down experiments [40, 94]) significantly simplify this task. On the other hand, if the goal of an experimental campaign is to understand how an interface transmits, dissipates, and reflects energy within a structure, then a novel approach such as a wave-based characterization of the joint [166, 167] could be utilized instead. A few works have begun to seek to characterize the amplitude-dependent stiffness and damping of a structure with multiple modes simultaneously participating [168, 81]. The efficacy of this approach will depend on whether a structure only has a few modes participating in the response (in which these approaches are highly promising), or if it has many modes participating in a response (in which case, these approaches will be infeasible). For applications in which many modes participate simultaneously, rendering the modal approach intractable, a more fundamental question can be posed: 'what does it mean to characterize a localized nonlinearity for a structure that has many modes active at the location of that nonlinearity?'



(a) **(b)**

Fig. 14: Evidence of mode coupling in measurements of (a) frequency response functions and (b) damping ratio from the S4 Beam from [74]. Mode 2 of this beam shows vastly different damping in Impacts 8 and 9. In the latter, Mode 1 is excited more strongly than Mode 2, and this causes Mode 2 to exhibit much stronger damping than it does when it is excited in isolation.

5.2.2 Nonlinear Substructuring Approaches to Model Many-Jointed Structures

Once a joint has been characterized in a simple structure (or in an *ex situ* experiment), the next challenge is understanding how a structure with *many* nominally identical joints will behave. One avenue to address this challenge is a nonlinear substructuring approach that faithfully accounts for the joint forces at each union of components. The surrogate system hypothesis [118] has been proposed for faithfully incorporating spatially discrete joint models in a larger structure. Once the response of a joint is decoupled from the structure that it is in, then that joint specific model can be applied to other structures with the same joint. The basic concepts needed to assemble a model comprised of flexible structures and nonlinear joints are well established [169]. In essence, one need only enforce displacement compatibility (i.e. equal displacements) at the interfaces between the components and balance the forces at the interface. The primary challenges with regard to jointed structures are:

- What mathematical models can be used to capture, both accurately and efficiently, the hysteretic force-displacement behavior of realistic bolted joints?
- What are viable methods for deriving those models from experiments or from first principles?
- What level of fidelity is needed to describe the interfaces between the components, and where should those interfaces be defined? Are the rigid (i.e., RBE2) or averaging-type (i.e., RBE3) spiders that are frequently employed adequate for this purpose? Or do more advanced interface reduction methods [143] need to be employed?

The first two questions of this list are related to both having a better understanding of the physics of jointed interfaces (e.g., §5.1) and how knowledge of those physics can be used to derive a computationally efficient and *sufficiently* accurate model of the interactions of jointed surfaces. The word “sufficiently” is emphasized here as it is likely that not all physics observed within a jointed interface need to be reproduced to predict the nonlinear dynamics of a large structure. For instance, the evolution of asperities with wear or the reaction kinematics involved in rough contact at small scales are likely both able to be described in statistical representations or with ROMs. Thus, the balance between introducing epistemic uncertainty (i.e., model form error) [170] and having a computationally tractable solution must be explored.

With regards to the third question, physics-based models of jointed structures often have orders of magnitude more elements representing a jointed interface compared to the rest of the structure [56]. Methods are needed to reduce the computational burden of the nonlinear interface elements used to represent joints. Recent research on improving methods to connect jointed structures have postulated non-stiffening RBE spider elements and interface hyper-reduction/mesh coarsening techniques [121]; however, these alone each have drawbacks. To date, spider elements have not been shown to reproduce all of the salient physics of a joint when only a few nonlinear degrees of freedom are used [42, 121]. Hyper-reduction techniques, on the other hand, still require hundreds of interface degrees of freedom for reproducing the observed physics. Thus, the question of “what is a sufficiently accurate model?” persists.

It is interesting to view these questions in light of recent progress in applying substructuring to systems with geometric nonlinearities [171, 172, 173, 174]. In that application, there are well established model forms that have been shown to work well for geometrically nonlinear structures, and well established methods for deriving those models using finite element analysis. Hence, for geometrically nonlinear structures the third item constitutes the primary unanswered question. In contrast, all three items need considerable research for jointed structures. For example, while Segalman's Iwan element [45] has been shown to work well for joints that are loaded in shear in the micro-slip regime, it has not been extended to multi-axial loading and it is somewhat computationally expensive. Other joint models fail to capture the response over a wide range of amplitudes [175].

One additional consideration for structures with many joints is that the dynamics will likely have some properties of emergent behavior [60, 61]. As observed in Goyder's experiments on chains of joints [82, 83], increasing the number of joints can lead to decreases in the damping capacity of a mode. One possible explanation for this behavior is that when multiple joints are being excited, the strain energy can be well-distributed over them resulting in all/most of the joints remaining in the linear regime. Thus, there is a trade-off between the distribution of joints within a load path and the expression of the nonlinearity of the system as a whole. This is likely related to the same mechanisms observed in the formulation of the three joint rule for shock attenuation [52], and leads to the distribution versus nonlinearity hypothesis that increasing the number of joints in a load path will distribute the energy dissipated by each joint such that the expression of nonlinearity of the system will be reduced [176].

Strategies to approach modeling many-jointed systems will not be able to rely purely upon physics-based models, which are too computationally intensive for use in simulating many-jointed structures. As an example, the physics-based models developed in [56] can take several hours to simulate the response of a single joint using a high performance computing cluster. Instead, a physics-based model could provide the truth data for a single joint against which a calibrated model is fit (e.g., [42, 39]). Using the surrogate system hypothesis, these calibrated models could then be replicated many times throughout a large, many-jointed structure. Another possible path forward is a multi-scale approach in which machine learning models are employed instead of a calibrated modeling framework, trained with either high fidelity models or experimentally obtained measurements [177, 178]. The goal of such an approach would be to create a very computationally efficient and accurate representation of a joint as the nonlinear degrees of freedom associated with joints often account for the majority of computational expense in a model.

5.2.3 Sources of Damping in Built-Up Structures

One final barrier to predictability is a lack of understanding of how energy is dissipated in a built-up structure [179]. Existing models of damping assume convenient mathematical forms that are heuristic models and do not address what the actual source of energy dissipation is. Beyond frictional dissipation in interfaces, the mechanisms for damping are presently unknown. For example, the concept of 'linear material damping' is nebulous; any parameters describing low amplitude damping levels (which are associated with linear material damping) must be experimentally measured and provide no basis for predicting low amplitude damping levels in other structures even when made from the same lot of material. Dissipation in a structure can be conceptually divided into several sources:

- Material, in which energy is dissipated via granular or atomistic interactions;
- Fixturing, in which energy is dissipated by the interaction between the test specimen and the mounting structure (e.g., added friction due to non-ideal boundary conditions or even from accelerometer cables);
- Fluid, in which fluid/gas-structure effects dissipate energy due to the displacement of a volume of fluid or gas by the vibrating structure;
- Interfacial, especially germane to joints in which energy is dissipated due to interfacial interactions between subcomponents;
- Engineered vibration absorption, in which intentional features are introduced to dissipate energy, including additively manufactured structures [180]; and
- Damping treatments such as viscoelastic polymers coupled to the structure via a constraining layer.

In built-up structures, the low level damping measured for a subcomponent, which is often attributed to material damping even though it convolutes other effects such as fixturing and fluid/gas interactions, does not always correspond to the low level damping observed for an assembly of the same subcomponents [56]. Dissipation due to fixturing cannot be eliminated, but can be controlled for [43], with best practices reducing the dissipation due to fixturing to be below the material damping level. Damping due to fluid/gas-structure interaction is typically insignificant in air [181], though recent experiments comparing low amplitude damping in and out of a vacuum showed that the vacuum *increased* damping by 5.7-11.8% [182]. At high response amplitudes, damping due to interfacial effects account for most of the dissipation (up to 90% for high amplitudes and interfaces that exhibit more nonlinearity [38]). However, in large structures with many joints, it is likely that most joints do not experience large amplitudes of vibration, which makes understanding low amplitude damping even more important.

5.2.4 Observations on Designing Benchmark Structures

It is highly likely that new benchmark structures will be necessary to make further progress on the challenges articulated throughout this paper. As such, care must be taken in designing a benchmark structure that will be both useful and impactful. When the Brake-Reuß beam [13, 38] and the S4 beam [124] were introduced as benchmark structures, there were approximately two dozen benchmark structures in use at the time [13], several of which are shown in Fig. 15. Each of the systems of Fig. 15 can provide valuable lessons for guiding the next generation of benchmarks. For instance, while the Ampair 600 Wind Turbine presents an industrial example that is well documented by the substructuring community (see [183, 184, 185]), the composite material makes modeling the individual subcomponents much more difficult and thus restricts modeling approaches to model updating/calibration schemes. The Sumali beam [36], in its original configuration, exhibited very little damping and was thus non-ideal as a benchmark to study jointed structures. Systems such as the Gaul resonator [186, 111], dumbbell oscillator [12], and cut beam frictional benchmark system [187] are able to provide tremendous insights into the dynamics of a single bolt lap joint; however, the expense associated with fabricating the systems (or, in the case of the cut beam frictional benchmark system, instrumenting and exciting it), much less replacing the joint once it is worn, prohibits their widespread use.

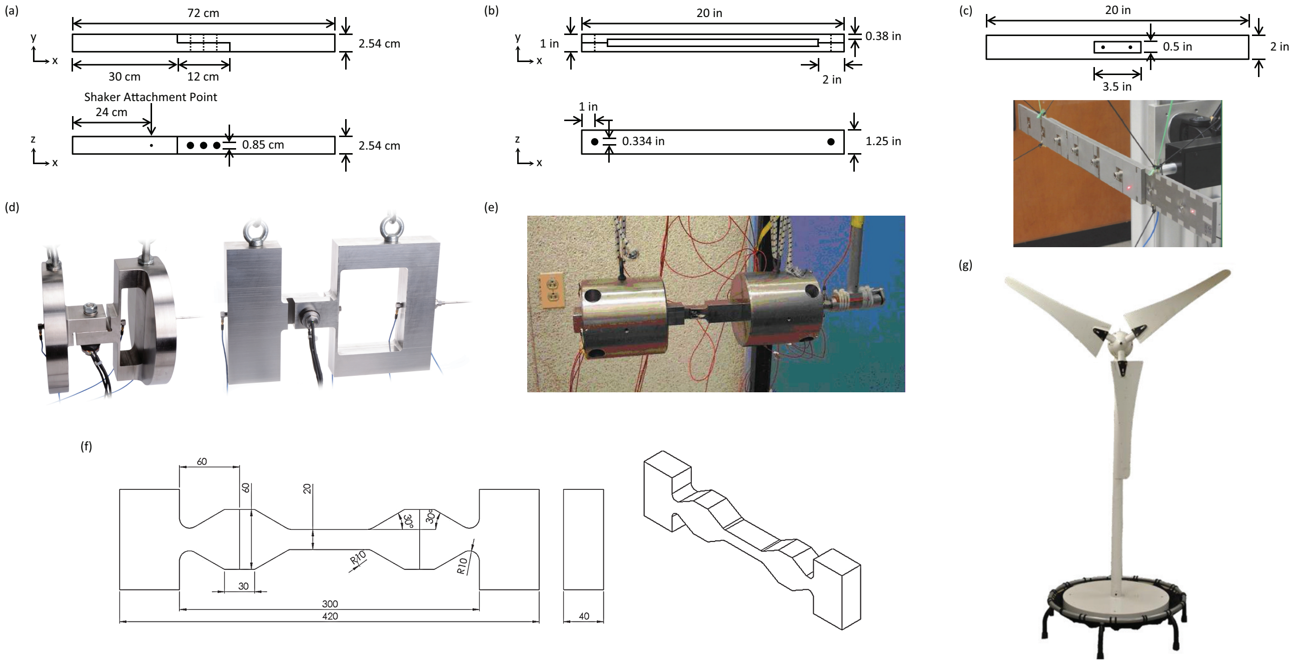


Fig. 15: A sample of some of the benchmarks used throughout the joints community over the last twenty years, including: (a) the Brake-Reuß Beam [13, 38]; (b) the S4 beam [124]; (c) the Sumali beam (and a chain of Sumali beams in the lower figure) [36]; (d) the Gaul resonator [186, 111]; (e) the dumbbell oscillator [12]; (f) the cut beam frictional benchmark system [188, 187]; and (g) the Ampair 600 Wind Turbine [183, 184, 185].

Lessons learned from these benchmark structures are briefly summarized here:

- New benchmark structures should be designed with a specific, and far-reaching, research question in mind.
- In order for benchmark structures to be impactful, community engagement is needed to ensure that the resulting structure will be used beyond a single research group.
- The adage of 'Keep It Simple...' is paramount.
- If possible, the knowledge gained from previous benchmarks should be built upon.

Community engagement is paramount for the impactfulness of a benchmark structure. Through having data collected on multiple copies of the structure by multiple research groups, experimental error and manufacturing tolerances are better explored and accounted for. Once a critical amount of data is gathered, perturbations to the system's design can lead to the causal inferences that are the foundation of postulating new modeling frameworks and hypotheses. With regards to keeping the structure simple, previous experiences have shown that if the structure itself is complicated away from the joint, most of the effort expended by a new researcher is focused on modeling the (uninteresting) parts of the structure away from the joint [189]. By keeping the structure itself simple, the majority of the effort expanded on modeling can focus on the portions of the structure that contain unknown physics (i.e., the joint itself). Additionally, a benchmark structure needs to be simple and cheap to fabricate if it is to be adopted widely.

Perhaps the most important question to be asked in designing a new benchmark structure is: 'what is the question to be addressed by experiments on this structure?' As the community looks forward to the challenges over the next decade, several important questions exist: are the mechanisms present in lap joints (such as the Brake-Reuß Beam) the same as in other types of joints (such as flange, spline, riveted)? If existing tribomechadynamic modeling frameworks are unable to make accurate predictions on other types of joints, then there are likely important physics absent in the modeling frameworks. As the lap joint has been thoroughly studied [13, 38, 124], future benchmarks should strongly consider other joint types. A second set of questions relates to scalability and the interaction of multiple joints within a structure. Lastly, the recent tribomechadynamics challenge structure [107] is already proving to be a good candidate for understanding the interplay between interfacial nonlinearities, geometric nonlinearities, and the influence of temperature. In designing a new benchmark structure and formulating the question to be addressed by it, it is also beneficial to consider how the knowledge gained from previous benchmarks can be leveraged. As an example, a multi-joint benchmark structure might build upon the existing single joint benchmark structures by replicating the already well-characterized joints multiple times throughout the new structure [190].

5.3 Computational Barriers to Simulation of Large Scale Structures

Advancing an understanding of both physics and structural dynamics techniques for modeling jointed structures is insufficient by itself. In order to have an effective impact on societal challenges, the computational challenges associated with modeling real structures must be simultaneously advanced. These challenges could be broadly described as multi-scale modeling frameworks, underlying numerical methods (such as time integration, continuation, linear algebra operations, etc.), and uncertainty propagation. In particular, major research questions include:

- As models are starting to span nano-meters to deca-meters and nano-seconds to years, what can we learn from the multi-scale modeling community?
- Can machine learning and data-driven approaches be effectively used to address the scalability challenge?
- As continuation techniques are requisite throughout most nonlinear dynamics analyses, how can we make the algorithms more robust?
- Most of the efficiency observed in recent techniques impose significant constraints on the system. Is it possible to develop a new solution technique that relaxes the constraints imposed by quasi-static or other simplifying solution methods?
- Continued refinement of transient techniques is needed to make them more widely accessible and more efficient for general physics-based models as opposed to specially formulated models.
- Ultimately, simulation frameworks will need to propagate uncertainty through joint models; how can this be done efficiently?

5.3.1 Multi-Scale Modeling Frameworks

Recent experiments on the long-term evolution of friction and wear properties within a jointed structure have shown that the time-scales associated with the evolution of tribological properties are well-separated from the time-scales of the structural response (on the order of hours compared to milliseconds respectively) [5]. To date, though, there have been no systematic studies of how to approach the multi-scale in time challenge. This challenge is particularly pronounced in aeroturbines, where the joints are designed to dissipate as little energy as possible and thus last for years without maintenance. By contrast, the multi-scale in space challenge has been approached by many recent studies. Tribological modeling efforts have shown amenability to representing multi-physics, high-dimensional, small-scale simulations with parameterized models [20, 21, 154]. This result implies that a hierarchical multi-scale modeling approach (similar to [55, 121, 56]) may be an appropriate method to include the tribological interactions within an interface; however, as highlighted in [56], features at the scale of 10s of μm and larger must be included in models to capture the interfacial contact pressures[7].

For concurrent multi-scale modeling frameworks, in which models at multiple scales are solved concurrently, there are several approaches to address the challenge of incorporating meso-scale and macro-scale (and even nano/micro-scale) simultaneously. As an example, [191] presents a concurrent coupling approach in which the micro-structure of various engineering applications were stochastically varied (through the use of stochastic ROMs) in order to assess their fatigue life and failure probability. One of the more promising methods for concurrent multi-scale analysis is the Schwarz alternating method [192], which provides a rigorous, non-intrusive, scalable and efficient approach to introduce two-way coupling across multiple scales. One of the key insights in the Schwarz alternating method is the introduction of a global time step across which simulations at multiple scales are synchronized (e.g., so simulations at one scale can be conducted at much smaller time steps than simulations at a different scale).

5.3.2 Underlying Numerical Methods

The foundational numerical methods used in simulating the response of jointed structures include:

- Continuation;
- Frequency domain simulation methods;
- Machine learning;
- Model reduction methods;
- Optimization;
- Quasi-static simulation methods;
- Transient simulation methods;

While machine learning is not currently a foundational method, there are promising results indicating that it could become a foundational technique [193]. Early work with machine learning in the context of nonlinear and jointed structures focused on structural health monitoring [194]. More recently, machine learning has been successfully extended to nonlinear modal analysis [195] as well as predicting structural dynamics [196, 197], including predicting the backbone curves of a novel jointed structure using training data from both high-fidelity and low-fidelity models in order to provide both accuracy and breadth in the training data [198]. Preliminary results also indicate that machine learning could provide a viable path for developing nonlinear ROMs of jointed structures [178].

Model reduction methods for linear structures are now well established. Fixed-interface methods have been used routinely since the seminal works by Hurty, Craig, and Bampton [199, 200, 201][8]. Likewise, free-interface methods are also well established and used in some industries, see for example the dual form as introduced in [202, 203]. Despite the maturity of model reduction techniques, interfaces still pose a significant challenge. Specifically, for the Hurty/Craig-Bampton method all interface degrees of freedom must be retained, so the method becomes ineffective if the substructures are joined across a continuous interface (see, e.g. [121, 56]). Reduction techniques for structures with continuous interfaces remain an active challenge [143]. As these methods continue to evolve, the most common approach in industry is currently to spider the interface, or regions of the interface, down to a few discrete nodes, and then to retain only those nodes in the reduction. However, structures with frictional interfaces can exhibit distributed, time-varying normal and tangential forces across the entire interface, and the spatial variation of these forces must be accurately represented for predictability [56, 73]. For example, even hyper-reduction techniques (in which the nonlinear forces are able to be evaluated directly in the reduced coordinate system, see §4.2 for more details) have so far only been able to reduce the interface degrees of freedom from thousands to many hundreds. Each of these methods is further illustrated in Fig. 16.

Transient time integration is also pervasive in many joints applications - both in terms of providing solutions as part of the alternating frequency-time algorithm for harmonic balance methods [132] or simulating the response to shock, random vibration, or other transient loading. The industry standard for time integration is the HHT implicit integration scheme for second order systems [204, 205]. For large systems, the memory requirements associated with recasting

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[7] The number 10 μm is an observation based on benchmark structures such as the Brake-Reuß beam; it is likely that more observations of larger scale structures is needed before this can be taken as a set limit.

[8] Though Hurty's work predated Craig and Bampton's publication of a similar concept, Hurty received very little recognition until only recently. Consequently, the preferred terminology for their method is the Hurty/Craig-Bampton method.

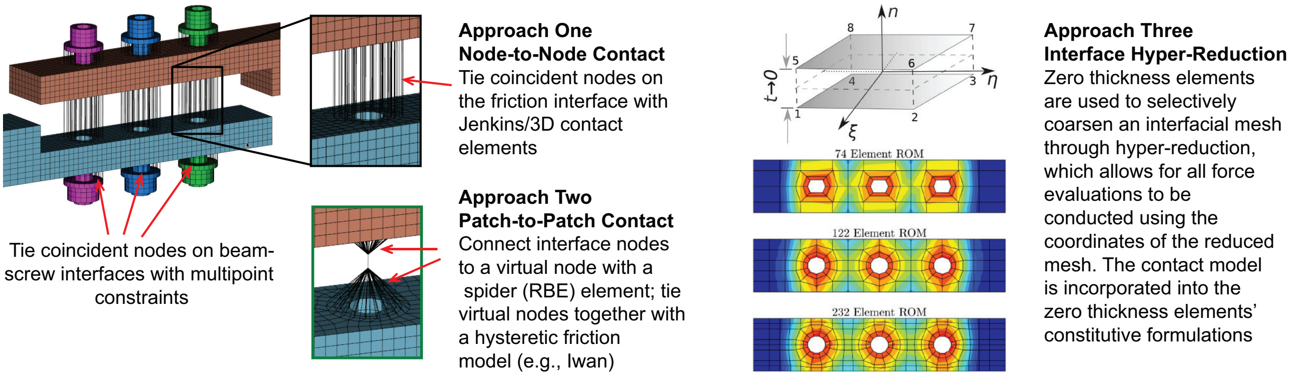


Fig. 16: Illustration of three different techniques for joining two components across an interface: node-to-node contact, patch-to-patch contact (i.e., spiders), and zero thickness elements. Adapted from [42, 121].

the system as a first order set of differential equations via a state space representation is often too computationally expensive. There are other second order solution methods, though, such as the Runge-Kutta-Nyström family of methods [206, 207], which include both implicit and explicit, adaptive time stepping schemes. Little progress has been made, however, on advancing the fundamental time stepping algorithms for use with jointed structures. One notable exception is the recent work of [141, 142], in which a ROM, such as a Hurty/Craig-Bamptom reduction for each substructure, is augmented with contact modes to add connectivity and flexibility to the joints. The contact modes impose a type of hyper-reduction on the interface degrees of freedom that allows for a much smaller set of equations to be integrated using transient integration methods such as the HHT method.

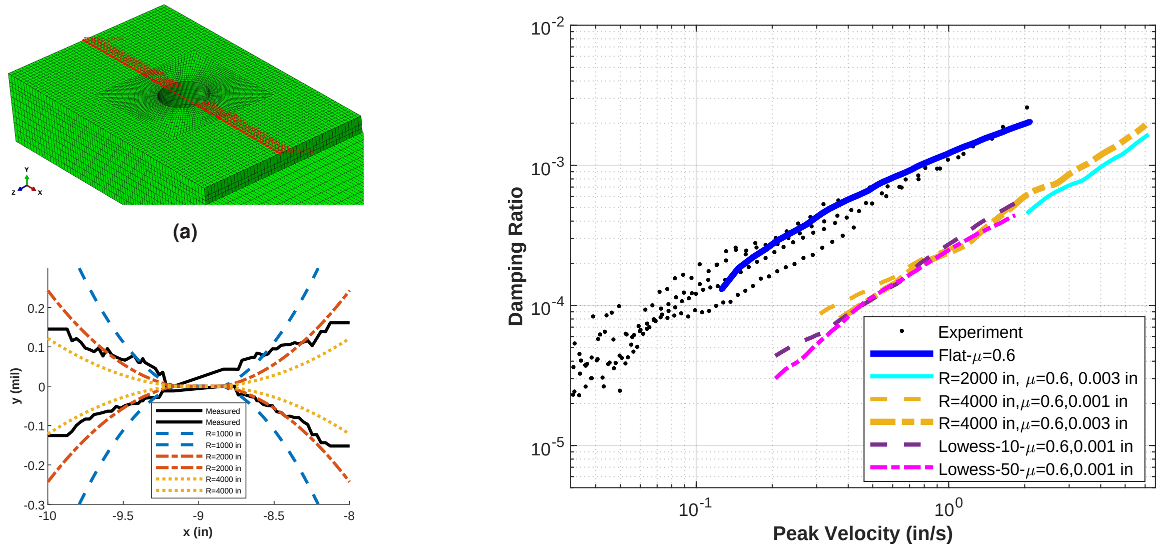
Mature algorithms exist for frequency domain solvers, quasi-static solvers, and model reduction; however, there are still opportunities for improvement here. The harmonic balance method [132] is considered a very accurate frequency domain solution method, but it is computationally expensive. The primary method used to provide more efficient frequency domain simulations is the extended periodic motion concept (EPMC) [133, 134], but this method is unable to predict the response in the presence of internal harmonics and modal interactions. As an alternative, the recently proposed variable phase resonance nonlinear modes (VPRNM) [136, 73] is able to capture the response of a structure exhibiting superharmonic resonances and modal interactions in an approximate manner. Further work is still needed to mature this methodology and improve the accuracy.

With regards to the more foundational numerical methods, continuation especially stands out as it is found throughout many aspects of joints research (from error exploration [51], to frequency response curves [136, 73], to even continuation of experimental controls schemes [92, 93, 91]). The continuation algorithms themselves, though, are still a maturing technology that often requires significant expertise to ensure that the quantities of interest are correctly tracked; further investment to make them more robust or accessible would have far-reaching effects on joints research. Likewise, one of the greatest bottlenecks associated with all numerical methods are the linear algebra matrix solves for very large systems of equations. There have been many recent advances to improve the efficiency of matrix manipulation within the mathematics community [208, 209, 210, 211], but these have not yet been implemented in engineering contexts.

5.3.3 Uncertainty Analysis and Design Conservancy

Jointed interfaces have demonstrated an extreme sensitivity to meso-scale features that are often within manufacturing tolerances [41, 56]. With regards to the goal of joints research, i.e., enabling more efficient, lighter weight, and more wear-resistant designs of structures and engines, this must be achieved in the presence of manufacturing variability. It is infeasible to require manufacturing tolerances to be small enough to remove uncertainty due to meso-scale features. Instead, the last challenge outlined in this paper is propagating the uncertainty of meso-scale features through models in order to discover an interface topology that is robust to manufacturing tolerances.

To motivate this topic, Fig. 11 shows a blind prediction from [73] for the nonlinear dynamics of a system with measured asperity distributions and meso-scale topography. The system exhibits the highest sensitivity to the meso-scale topography (defined as features on the same scale as machining features, approximately 10-40 µm). Presently, though, it is too computationally intensive to propagate uncertainties in either the asperity distribution or the meso-scale topography through this model, and instead only a limited number of simulations are able to be conducted to analyze specific design points. A less expensive computational framework, described in [104] and Fig. 17, shows a



(a)

(b)

(c)

Fig. 17: Effect of Meso-scale Topography on the nonlinear damping predicted for the S4 Beam [124, 104]. (a) shows the FE model of one contact patch while (b) shows measurements of the topography along the slice shown in (a), as well as spherical approximations to it. (c) shows the damping predicted for models including a smoothed representation of the measured topography (i.e. "Lowess-10" in the figure) as well as when it is approximated as spherical (i.e. "R=2000").

significant dependency on the *presence* of meso-scale topography, but relatively little dependency on the amplitude of the meso-scale topography.

Traditional approaches to mitigating the effects of uncertainty in jointed structures have been to over-design each subcomponent so as to either effectively rigidize the interface or to provide so much excess strength that the structure survives even if its damping is minimal. This design conservancy approach has the unintended consequence of stacking of uncertainties from each subcomponent within an assembly, resulting in an over-designed structure that weighs significantly more than it needs to. From an unpublished case study conducted with Rolls Royce by Ed Green and David Ewins, if the mass of the joints alone within an aeroturbine could be reduced by 10%, this could lead to fuel savings that would recover the research cost associated with designing the joints by more than a factor of ten [13]. Each added kg of mass in the commercial aerospace industry incurs $7,000 of additional fuel costs (using prices from 2023) over the lifetime of a plane; in the United States, this is approximately 3,000 gallons of fuel consumed per day for each extra kg of mass for the entire commercial air fleet [13].

There are many tools for understanding the propagation of uncertainties in jointed structures. The first step to applying these tools is to identify the source of uncertainty to be dealt with. Model form error, i.e., epistemic uncertainty, is introduced by either simplifying a high fidelity model to create a tractable ROM or by unknowingly (or intentionally) neglecting some of the physics of the problem being modeled. There are two recent approaches for addressing epistemic uncertainty - maximum entropy, and hybrid machine learning frameworks. Maximum entropy [212, 114, 213] and similar methods [214] use random matrices and polynomial chaos to distinguish between aleatoric (i.e., parametric) uncertainty and model form error. More recently, machine learning approaches have been used to assess and propagate uncertainty through models. In [215], a physics-based model is updated by means of a deep Markov model and experimental data in order to reduce the epistemic uncertainty of the initial physics-based model. In the absence of experimental data, [198, 216] creates a hybrid machine learning model using data from both a low fidelity ROM and a high fidelity physics-based model. By using two different fidelity models, ‘truth’ data is able to be generated by the computationally expensive physics-based models, and additional data to enable the training of the machine learning model is cheaply calculated using the low-fidelity ROM. Both approaches ([215, 198, 216]) seek to reduce the epistemic uncertainty of a low order model via supplementing a machine learning model with more accurate data (from either experiments or a high-fidelity model).

Uncertainty in model parameters, aleatoric uncertainty, can cover a range of variables, from material properties to accounting for machining variations in interface topographies. Two approaches for assessing parameter-based uncertainty are polynomial chaos expansions [126, 217] and parameterized ROMs [218, 219, 220]. In both approaches, uncertainties are propagated through a ROM (formulated via polynomial expansion or Taylor series expansion as is the case for parameterized ROMs) using a Monte Carlo analysis to assess the sensitivity of the response to uncertainties for a given parameter, most commonly through the calculation of the Sobol indices [126, 217], which decompose the global sensitivity of the analysis including all uncertain parameters to deduce the relative contribution of a specific uncertain variable.

6 The Path Forward and Open Challenges for the Community

With the goal of creating a modeling framework that can enable more efficient, lighter weight, and more wear-resistant designs of structures and engines, there are still many open challenges for the community to address. Building on the discussion of §5, six grand challenges have been identified for the community:

1. **Tribology Grand Challenges** The impediments to predictive modeling approaches for the joints community are closely aligned with the grand challenges within the tribology community - namely predictive models of wear and friction, and improved models of the combined normal and tangential contact mechanics specifically for the case of rough surfaces with direction-dependent features that give rise to anisotropic friction (e.g., machining grooves, ellipsoidal asperities) and elastic-plastic material properties that include work hardening. These grand challenges are further elaborated upon in [1].

   If the joints community is to be successful in developing predictive models for the dynamics of jointed structures, it must work more closely with the tribology community to both better understand friction and wear and to postulate improved models to predict friction and wear behavior. It is worth noting that the aims of the two communities are distinct. For prediction of the response of jointed structures, first order assessments of friction and wear will likely suffice. At some level, uncertainty in the topography of the interface will outweigh the uncertainty due to model form error in predictions of wear. Thus, it will be important to stay focused on the discovery of a minimally sufficient predictive model of friction and wear.

2. **Multi-Physics and Complex Loading** Understanding the tribological properties of a joint are insufficient for predictive modeling; detailed knowledge of how joints interact as part of a larger structure is necessary too. This interaction can be described by tribomechadynamics to include the local kinematics of the joint, the reflection, transmission, dissipation, and shunting of pressure waves, and the relative distribution of energy across multiple joints within a structure [55, 166].

   One particularly salient challenge for understanding the multi-physics environment of joints and their responses to complex loading is that *superposition* does not hold for nonlinear structures. Due to the nonlinearities inherent in jointed structures (i.e., from the interfaces, materials, and geometric sources), the response of a simple jointed specimen undergoing both axial and transverse bending excitations cannot be calculated from the response to axial excitations in isolation and transverse bending excitations in isolation. However, much of the work on characterizing jointed structures or probing the physics of an interface focus only on one modality of excitation. Significant work is needed to understand the response of joints under combined loading environments.

3. **Building Block Approaches** A second challenge stemming from the lack of superposition in jointed structures is understanding how to incorporate multiple joints into a model of an assembly. Academic research has primarily focused on single joint structures, while industrial problems, which contain hundreds if not thousands of joints, have demonstrated that repeating the model of a single joint multiple times for each nominally identical joint in a structure does not accurately predict the response of an assembly. This disconnect must be addressed.

   As research codes are refined, it is paramount that scalability of models be addressed. Predictive models of jointed structures will be useless if they are unable to inform the predictions for the dynamics of a large, multi-jointed assembly. One element of this challenge is the continued refinement of substructuring and reduced order modeling approaches to handle nonlinear elements (as most substructuring approaches still require linearized interfaces [143]).

4. **Designing Repeatable Joints** A recurring goal for the joints community is to redesign joints to be both predictable and consistent/robust to manufacturing tolerances [4, 14, 15, 16, 17]. Since the previous workshop on joint mechanics [17], a significant advance within the joints community has been in understanding the role that the meso-scale topography has on the dynamics of a jointed structure. At the same time, the community has also come to understand that the sub-micro-scale description of an interface (i.e., the asperities) does not need to be modeled deterministically (e.g., [56]).

Predictive models of joint dynamics are now sufficiently accurate that it is possible to perform design optimization across the meso-scale features of an interface. It is unclear, though, if these models are able to provide 1) a solution in a tractable amount of time due to their high computational cost and 2) a feasible to manufacture solution that will not incur higher costs than those associated with tightening the manufacturing tolerances on an interface's flatness. Consequently, the challenge of designing repeatable joints is now dependent upon advancing the numerical methods necessary for undertaking this challenge such that solutions can be readily found for each joint of interest.

5. **Large Data Challenges and Pedigree of Experimental Data** The application of data science is still a burgeoning research area, and it is unclear how data science will affect the prediction of jointed structures. Investment in data science approaches should not replace investigations into the fundamental physics of how joints interact within a larger structure; however, data science does offer an opportunity to learn how the above-mentioned challenges can be addressed. Recent work has shown that it can offer the potential for developing accurate reduced order models from a small amount of truth data combined with a large amount of data from inexpensive, lower accuracy simulations [198].

   One challenge facing the collection and curation of experimental data for data science applications has been the high experimental error associated with traditional studies of jointed structures. For instance, the tightness of bolts is typically controlled via a torque wrench, which can result in bolt strains between 10% and 102% of the expected preload value [103, 55]. Additionally, much of the characterization data for jointed structures come from hammer impacts, which impart broadband, multi-frequency responses, and can contain significant measurement error from the way in which the experimentalist strikes the structure. Thus, efforts must be made to ensure higher pedigree data that is well controlled for and that minimizes experimental error.

6. **Application-Centric Solutions** Much of the recent progress within the joints community has focused on understanding the fundamental physics of jointed structures and how to incorporate these physics into models. However, there are a number of pressing challenges within industry that are being neglected - understanding the response of jointed structures to shock and random vibration, to combined loads and environments, to bolt loosening (and the mechanisms that govern bolt loosening), and to regimes of large relative motion (such as articulated joints). The community cannot neglect these challenges as the other challenges on this list are addressed as there are many tasks that can be conducted in parallel, such as improved time integration techniques for studying shock and random vibration.

   A second consideration is that industry will not adopt new techniques if the techniques are disruptive to the way that industry currently works or it is not clear that the new techniques will bring a significant benefit. It is possible that lower fidelity but less intrusive/easier to implement techniques will be better received by industry. Research on joint mechanics is grounded in practical applications. As such, it is paramount that the results of this community's research be applicable to solving industrial problems and that the community of academics and industrial practitioners work together to make the state-of-the-art techniques appropriate for use in industrial best practices.

There are, of course, many other challenges within the joints community (e.g., understanding what "material" damping is and how to predict it); however, the above six challenges are both needed and timely such that the conditions are ripe for significant progress to be made towards them this decade. Unlike the outlook from previous workshops ([4, 14, 15, 16, 17]), the current state of modeling has demonstrated that predictive models of joint dynamics are possible. Progress, though, is both due to and dependent upon large-scale collaborations. Since the previous workshop on joint mechanics [17], significant effort has been invested in establishing large-scale collaborations and community-wide challenges (e.g., [107]). This trend must continue if the aforementioned challenges are to be overcome. Doing so will allow the joints community to advance the performance of complex engineering structures, optimize their design for cost, weight, and wear resistance, and ultimately reduce the resources needed to operate the next generation of engineering systems, contributing to the goal of creating environmentally sustainable systems. This is of particular importance given the existential crises related to climate change that our society is facing.

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Authors' Contributions

M. R. W. Brake was responsible for conceptualization, investigation, resources, data curation, writing - original draft, writing - review and editing, visualization, supervision, and project administration — and running with D. Quinn.
M. S. Allen was responsible for conceptualization, investigation, and writing - review and editing.
D. D. Quinn was responsible for conceptualization, investigation, writing - review and editing — and running with M. Brake.
D. R. Roettgen was responsible for conceptualization, visualization, and funding acquisition.
D. Nowell was responsible for conceptualization, investigation, and visualization.

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