



Frequency Based Nonlinear Decoupling Using Experimentally Measured Quasi-Linear FRFs

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Abstract

Decoupling methods based on measured Frequency Response Functions (FRFs) are well established for linear systems, yet their extension to nonlinear systems remains in its infancy. Existing frequency-based decoupling techniques for nonlinear systems are limited to localized nonlinearities and specific configurations that require prior knowledge of the nonlinearity location; a general decoupling method for nonlinear systems does not yet exist. This paper introduces a general nonlinear decoupling framework. The proposed Nonlinear Frequency-Based Decoupling (NL-FBD) framework uses quasi-linear FRFs of the coupled system via response-controlled testing (RCT), and measured or calculated FRFs of the known substructure; and then applies a linear decoupling formulation, to calculate the quasi-linear FRFs of the unknown nonlinear structure at several response levels. Subsequent linear modal identification yields response-dependent nonlinear modal parameters, which are then used to synthesize constant-force FRFs of the unknown nonlinear substructure. The proposed approach is capable of; i) handling multiple nonlinear elements located arbitrarily within the structure, ii) addressing systems even with distributed nonlinearities, iii) accommodating nonlinearities in both substructures, and iv) achieving decoupling at multiple degrees of freedom. The applicability limits of the proposed approach are examined in detail, together with an assessment of its accuracy through numerical case studies. In particular, nonlinearities in both substructures are demonstrated for the single degree-of-freedom connection case, whereas multiple-degree-of-freedom connections are investigated for systems with a known linear substructure. Experimental validation on a nonlinear beam–linear beam assembly confirms that NL-FBD accurately decouples the unknown nonlinear beam, yields a reliable nonlinear modal model, and predicts the forced response of a new assembly incorporating the decoupled nonlinear beam, with excellent agreement to measurements even at high forcing levels.

Keywords: Nonlinear structural dynamics; nonlinear decoupling; nonlinear substructuring; quasi-linear FRF; response-controlled testing

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1 Introduction

Several mechanical systems, such as aircraft structures, often consist of assemblies composed of interconnected components. Each component demonstrates unique dynamic behavior, which affects the system's overall performance. Performing a full reanalysis of the coupled system whenever a component is modified incurs significant computational cost and time. Substructuring techniques are available to avoid analyzing the whole system for each iteration. These

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techniques aim to divide and conquer the system [1]. Besides that, substructuring techniques help calculate the dynamics of substructures, whose characteristics are more important to obtain when connected to the main system, e.g., isolators, bolts, and bearings [2].

Traditional linear substructuring methods rely on the principle of superposition, where the overall system's response is calculated by adding up the responses of its substructures. Some of the first and most well-known techniques include the Craig-Bampton method [3] and Component Mode Synthesis (CMS) [4, 5], developed in the 1960s. These approaches use a set of fixed-interface normal modes and constraint modes to reduce each substructure's degree of freedom (DoF). Later, the advancement of experimental tools like sensors and signal processing techniques opened the way for physically testing individual subcomponents and incorporating their measured responses into the overall system model. With the progress in experimental techniques, frequency-based substructuring, which uses frequency response functions (FRFs) to characterize and connect substructures in the frequency domain, became more important, and new methods emerged. Some of these methods are listed in [6] as impedance coupling [7], admittance coupling [8], Structural Modification Using Response Functions (SMURF) [9], and Lagrange-Multiplier Frequency Based Substructuring (LM-FBS) [10]. The impedance coupling method [7] uses the dynamic stiffness (impedance) of substructures for coupling, which requires multiple matrix inversions and can be prone to numerical issues. The admittance coupling method [8] uses the receptance matrices of substructures to couple them, like assembling global matrices in the finite element method, but this is in the frequency domain. The SMURF method [9] computes the receptance matrices of the coupled system from the receptance matrices measured at the connection interfaces. LM-FBS [10] uses Lagrange multipliers to enforce interface constraints in the frequency domain, allowing for the coupling of multiple substructures. Later, D'Ambrogio & Fregolent [11] applied the LM-FBS formulations with "negative" substructures to decouple known components and thus calculate the response of unknown substructures. The method proposed by Özgüven [12] couples the FRFs of one substructure with the system matrices of the other, offering an efficient approach for coupling a large structure with a significantly smaller substructure that represents a modification to the main system. As decoupling methods evolved, new methods were suggested to include the effects of rotational DoF during decoupling. The Central Difference Method [13] and Virtual Point Transformation [14] are some of these methods. Besides the methods used to calculate the rotational DoFs in structures, methods like the Mixed Interface Method [15] evolved to compensate for the rotational effects.

Although several substructuring methods have been developed and successfully used by many researchers, all these methods rely on the assumption of linearity in structures. However, many structures exhibit nonlinear behaviors—like varying stiffness or coulomb friction—which linear methods cannot accurately capture. Nonlinearities in structural systems can come from various sources, such as material properties, geometric deformations, boundary conditions, and contact interactions [16, 17]. Unlike linear systems, where responses scale directly with input forces, nonlinear systems show behaviors that depend on the input force amplitude [18], which results in significantly more complex dynamic behavior hence obtaining accurate numerical models of the system or subsystems is even more important in nonlinear systems compared to linear systems.

Several methods have been developed to apply substructuring techniques to nonlinear systems. The fixed-interface Craig–Bampton method was extended by incorporating invariant-manifold nonlinear normal modes (NNMs) [19] for each substructure, which are coupled through linear interfaces [20]. This nonlinear CMS approach [20] produced reduced-order models that more accurately represented the dynamics of nonlinear assemblies compared to traditional linear CMS. Building on this, [21] advanced modal substructuring by proposing two approaches: (1) assembling substructures using their NNMs within a quasi-linear framework and (2) constructing nonlinear reduced-order models for each substructure based on a limited number of linear modes before coupling. Collectively, these contributions effectively extended classical CMS to the nonlinear domain. Another work [22] by Latini et al. employed a modal-domain substructuring technique to two linear substructures connected by a nonlinear element exhibiting cubic-hardening behavior. Their iterative, energy-based approach accurately predicted resonance shifts and mode shapes, closely matching results from harmonic balance simulations and physical testbed measurements.

The first study on nonlinear coupling through analytical quasi-linearization of nonlinear systems dates back to 1995 [23]. In this work, the modal coupling method was combined with the concept of Nonlinearity Matrix based on describing functions [24] to compute the harmonic responses of linear systems coupled with nonlinear elements. In 2014, the hybrid substructuring technique proposed in 1990 [12] was extended to nonlinear substructuring, coupling FRFs of a linear system with the dynamic stiffness matrix of the nonlinear modifying system calculated using the Nonlinearity Matrix concept [25]. Wang et al. [26] developed an inverse substructuring method for nonlinear systems, assuming that the type and location of the nonlinearity are known. In this approach, the nonlinearity is represented as a single nonlinear element connecting two coordinates. The proposed method relies solely on system-level FRFs to extract the subsystem-level FRFs.

The concept of measuring quasi-linear FRFs using response-controlled tests and employing them in harmonic

response analysis, structural coupling, and structural modification was first introduced in 2008 by Arslan and Özgüven [27]. Although the proposed method was limited to systems with a grounded local nonlinear element, this pioneering study led to the idea of combining substructuring techniques with measured quasi-linear FRFs, offering a promising approach for addressing the challenges of nonlinear substructuring. The FRF Decoupling Method for Nonlinear Systems (FDM-NS) [28, 29] stands as the first notable example of integrating quasi-linear FRFs with advanced linear substructuring techniques for nonlinear decoupling. However, FDM-NS is limited to systems with a local nonlinearity represented by nonlinear element(s) between two DoFs. This limitation arises because the method relies on keeping the relative displacement of specific nonlinear elements constant during testing, a condition that is practically not possible to enforce in systems with distributed geometric nonlinearities or elements at inaccessible locations. Di Manno et al. [30] proposed an extension of FDM-NS to systems where multiple nonlinearities may exist in different DoFs, but it is still restricted to nonlinearities between two points of a structure whose relative displacement can be directly controlled. In contrast, the present framework does not require prior knowledge of the structure, or location of the nonlinearities in the unknown substructure; their combined effect is represented through the measured quasi-linear FRFs and the response-dependent modal model. Moreover, the proposed approach overcomes the need for direct measurement at the joint DoFs that may be inaccessible. After the generalization of the response-controlled testing method presented in [27, 31] to systems with multiple distributed nonlinear elements through response-controlled stepped-sine testing RCT [32], it was possible to extend the experimental nonlinear structural decoupling method FDM-NS to systems with many nonlinear elements. A preliminary and limited version of this approach was presented in [33] as a conference paper, building upon the concept originally proposed in [34].

The present work substantially extends and further develops that initial study into a more complete experimental Nonlinear Frequency-Based Decoupling (NL-FBD) framework, by generalizing FDM-NS toward a broader approach for nonlinear decoupling, and by systematically investigating its practical limitations, sensitivity to measurement noise, and applicability under different connection configurations. In contrast to FDM-NS [28], which is methodologically restricted to localized nonlinearities where relative displacement can be controlled, the proposed framework is capable of:

- i) handling multiple nonlinear elements connected at single or multiple points within the structure,
- ii) addressing systems even with distributed nonlinearities,
- iii) allowing nonlinearities in both substructures, and
- iv) achieving decoupling even when the substructures are coupled at multiple degrees of freedom.

To the best of our knowledge, this study is the first nonlinear decoupling work to investigate this generalization systematically in an experimental frequency-based framework. It should also be noted that this generalization is not so straightforward: in FDM-NS, the relative displacement of the nonlinear element is kept constant in the response-controlled test, yielding a quasi-linear FRF in any frequency range, whereas in NL-FBD, RCT is employed, which gives quasi-linear FRFs only at near-resonant frequencies, introducing certain limitations and accuracy issues. These are the costs of this generalization, and are investigated in detail in this paper for substructures with connections at single or multiple DoFs. Various discrete and continuous system example cases involving nonlinearities in one or both substructures are used in these studies. The key assumptions underlying the measurement and use of measured quasi-linear FRFs of nonlinear systems – namely, well-separated modes, negligible contribution from higher harmonics, and the absence of internal resonances – are also adopted in the proposed decoupling framework.

The nonlinear decoupling approach presented in this work, NL-FBD, together with the Nonlinear Frequency-Based Coupling approach (NL-FBC) – first applied in [27] for a special case involving a single grounded nonlinear element and later conceptually generalized in [34] – constitute the experimental Nonlinear Frequency-Based Substructuring (NL-FBS) framework. While the decoupling component of NL-FBS is examined in detail in this paper, the coupling component is also briefly explained and then employed in the experimental validation of the proposed decoupling framework.

Section 2 presents both decoupling and coupling components of the Nonlinear Frequency-Based Substructuring framework, emphasis being on decoupling. The limitations and accuracy of the decoupling framework are investigated in Section 3 using numerical applications for decoupling of nonlinear systems coupled at a single DoF. Both of the cases where there is nonlinearity in one substructure and in both substructures are considered. Section 4 studies the same features of NL-FBD applied on systems with multiple connection-DoFs, and having a known linear and an unknown nonlinear substructure: (1) lumped systems connected at multiple points and (2) a numerical beam example with continuous interface. Experimental validation of NL-FBD is provided in Section 5 by also using NL-FBC. Conclusions are given in Section 6.

2 Nonlinear Frequency-Based Substructuring Framework

Nonlinear Frequency-Based Substructuring framework, NL-FBS, consists of decoupling (NL-FBD) and coupling (NL-FBC) components. The theoretical foundations of both experimental nonlinear decoupling and coupling approaches presented in this study build upon the measurement of quasi-linear FRFs of nonlinear systems using response-controlled testing (RCT), frequency-based substructuring techniques developed for linear systems, and modal model of nonlinear systems.

2.1 Quasi-Linearization in Nonlinear Systems

Earlier studies on experimental quasi-linearization of nonlinear systems were limited to configurations with either grounded nonlinear element(s) connected to a single DoF or nonlinear element(s) located between two DoFs. Accordingly, quasi-linear FRFs were obtained through response-controlled tests by keeping either the displacement of the DoF to which the nonlinear element was attached or the relative displacements between these two DoFs constant. Therefore, the frequency based nonlinear decoupling method FDM-NS had the same limitations.

However, according to Single Nonlinear Mode (SNM) theory [35], when the modes are well-separated, higher harmonics are negligible and no internal resonance occurs, the near-resonant response of a nonlinear structure is dominated by a single nonlinear normal mode (NNM), and the corresponding mode shape does not change significantly at near-resonant frequencies. This provided the theoretical basis for obtaining quasi-linear FRF by keeping the response amplitude of a selected point (usually, the excitation point) constant in RCT [32].

This can also be shown theoretically, considering the equation of motion of a MDoF nonlinear system:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + i\mathbf{H}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \mathbf{f}(t) \quad (1)$$

where $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the mass matrix, $\mathbf{H} \in \mathbb{R}^{n \times n}$ is the hysteretic damping matrix, and $\mathbf{K} \in \mathbb{R}^{n \times n}$ is the stiffness matrix of an n -DoF system. $\mathbf{f}_{nl} \in \mathbb{C}^{n \times 1}$ is the nonlinear restoring force vector depending on displacement ($\mathbf{x}(t)$), velocity ($\dot{\mathbf{x}}(t)$), and time (t). $\mathbf{f}(t)$ is the external forcing vector, assumed to be harmonic at a constant excitation frequency, expressed as $\mathbf{F}e^{i\omega t}$. Transforming Eq. (1) into the frequency domain and expressing the nonlinear restoring force vector in terms of Nonlinearity Matrix [24] that can be calculated using describing functions, the equation takes the form,

$$-\omega^2 \mathbf{M}\mathbf{X} + i\mathbf{H}\mathbf{X} + \mathbf{K}\mathbf{X} + \Delta(\mathbf{X})\mathbf{X} = \mathbf{F} \quad (2)$$

where $\Delta(\mathbf{X}) \in \mathbb{C}^{n \times n}$ is the response-level dependent Nonlinearity Matrix, and $\mathbf{X} \in \mathbb{C}^{n \times 1}$ is the harmonic response amplitude vector of the system. The elements of $\Delta(\mathbf{X})$ can be written in terms of the describing functions, ν , of the individual nonlinear components [24]:

$$\Delta_{jk} = \begin{cases} \sum_{m=1, m \neq j}^n \nu_{jm}(|X_j - X_m|) + \nu_{j0}(|X_j|) & \text{if } j = k \\ -\nu_{jk}(|X_j - X_k|) & \text{if } j \neq k \end{cases} \quad (3)$$

Here, n represents the total number of DoFs of the system, and m serves as a summation index accounting for all other DoFs connected to the j^{th} DoF. The term ν_{jk} represents the describing function of the nonlinear element connecting the j^{th} and k^{th} DoFs, which depends on their relative displacement amplitude $|X_j - X_k|$. Similarly, ν_{j0} represents the describing function of a grounded nonlinear element at the j^{th} DoF, depending on its absolute displacement amplitude $|X_j|$. This expression is used in the numerical examples to construct and interpret the quasi-linear representation of the simulated nonlinear systems. However, in the experimental application of the proposed NL-FBD framework, the nonlinearities of the unknown substructure are not explicitly modelled or individually identified through this matrix, and their number, types and locations do not need to be known a priori. Instead, their combined effect is represented by the experimentally measured quasi-linear FRFs and the corresponding response-dependent modal model.

In a response-controlled stepped-sine test, the forcing amplitude is automatically adjusted so that the response amplitude follows a target level or remains within the prescribed tolerance limits. Since at near-resonant frequencies, the mode shape does not change notably and the structure vibrates at a single nonlinear normal mode (NNM) [35], the response can be expressed as,

$$\mathbf{X} = \tilde{q}_r \bar{\boldsymbol{\phi}}_r(q_r) \quad (4)$$

where $\bar{\phi}_r(q_r)$ is the r^{th} mass normalized nonlinear modal vector, $\tilde{q}_r$ is the complex modal displacement response, and $q_r = |\tilde{q}_r|$ is the amplitude of the modal displacement response at the r^{th} nonlinear mode. Eq. (4) shows that the overall response amplitude X remains constant if the response amplitude of any selected DoF is kept constant for frequencies near resonance. Consequently, the Nonlinearity Matrix $\Delta(X)$ remains almost constant, effectively quasi-linearizing Eq. (2). Then, measuring FRF of the system for a given response level at an r^{th} mode by exciting it from k^{th} DoF and making measurement at j^{th} DoF, mathematically corresponds to the FRF obtained using the solution of Eq. (2) at that mode,

$$\bar{\alpha}_{jk}(q_r, \omega) = \frac{\bar{\phi}_{jr}(q_r)\bar{\phi}_{kr}(q_r)}{\bar{\omega}_r^2(q_r) - \omega^2 + i\bar{\eta}_r(q_r)\bar{\omega}_r^2(q_r)} \quad (5)$$

Here, $\bar{\alpha}_{jk}$ represents the receptance between DoFs j and k for the r^{th} mode; $\bar{\phi}_{jr}$ and $\bar{\phi}_{kr}$ are the corresponding mass normalized nonlinear modal vector components for j^{th} and k^{th} DoFs at the r^{th} mode; $\bar{\omega}_r(q_r)$, $\bar{\eta}_r(q_r)$, ω represent the r^{th} nonlinear natural frequency, r^{th} nonlinear modal damping, and excitation frequency, respectively.

The force required to maintain a constant response amplitude at the selected DoF in RCT can be related to the complex modal displacement response $\tilde{q}_r$, calculated from Eq. (4), by the following equation:

$$\tilde{q}_r = \frac{\bar{\phi}_{kr}(q_r)F_k}{\bar{\omega}_r^2(q_r) - \omega^2 + i\bar{\eta}_r(q_r)\bar{\omega}_r^2(q_r)} \quad (6)$$

2.2 Frequency-Based Substructuring

The decoupling and coupling formulations used in this study are based on the dual domain decomposition framework introduced by De Klerk and Rixen [10], and specifically follows the development presented in [11] by D'Ambrogio and Fregolent. This dual domain formulation expresses the coupling and decoupling problems within the same mathematical structure, allowing the dynamic contribution of a substructure either to be added (coupling) or removed (decoupling). Consider a coupled linear assembly C formed by two substructures A (known) and B (unknown). A block system is formed by the measured coupled assembly C and the negative of the known substructure A , which is used for decoupling. The equation of motion of this block system can be written in the frequency domain as,

$$\begin{bmatrix} \mathbf{u}^C \\ \mathbf{u}^A \end{bmatrix} = \alpha^{CIA} \left(\begin{bmatrix} \mathbf{f}^C \\ \mathbf{f}^A \end{bmatrix} + \begin{bmatrix} \mathbf{g}^C \\ \mathbf{g}^A \end{bmatrix} \right) \quad (7)$$

Herein, $\mathbf{u}^C$, $\mathbf{f}^C$, and $\mathbf{g}^C \in \mathbb{C}^{n^C \times 1}$ denote the displacement, external force, and internal force vectors of the coupled system C , respectively, while $\mathbf{u}^A$, $\mathbf{f}^A$, and $\mathbf{g}^A \in \mathbb{C}^{n^A \times 1}$ denote the corresponding vectors of the known substructure A . Therefore, the stacked displacement, external force, and internal force vectors in Eq. (7) have sizes of $(n^C + n^A) \times 1$. The FRF can be expanded as, $\alpha^{CIA} = \text{diag}(\alpha^C, -\alpha^A)$, where $\text{diag}(\cdot)$ denotes the block-diagonal concatenation of matrices. Specifically, $\alpha^{CIA} = \text{diag}(\alpha^C, -\alpha^A) = \begin{bmatrix} \alpha^C & \mathbf{0} \\ \mathbf{0} & -\alpha^A \end{bmatrix} \in \mathbb{C}^{(n^C+n^A) \times (n^C+n^A)}$, where the off-diagonal blocks are zeros.

The interface equilibrium and compatibility conditions at the interface can be written as,

$$\mathbf{g}_c^C + \mathbf{g}_c^A = \mathbf{0}, \quad \mathbf{u}_c^A = \mathbf{u}_c^B = \mathbf{u}_c^C \quad (8)$$

where $\mathbf{u}_c$ and $\mathbf{g}_c$ vectors of size of the connection DoFs denote the interface displacements and forces, respectively. The objective of decoupling is to calculate the FRF of the unknown substructure, $\alpha^B(\omega)$, using the measured FRF of the coupled system, $\alpha^C(\omega)$, together with the FRF of the known substructure, $\alpha^A(\omega)$. To express Eq. (8) in a general form, the equilibrium and compatibility conditions are put into matrix form by introducing signed Boolean matrices $\mathbf{B}^A$ and $\mathbf{B}^C$ that select the corresponding interface DoFs from the full displacement vectors of substructure A and the coupled system C , respectively, and map them onto common compatibility coordinates. The compatibility condition and equilibrium condition with Lagrange multipliers, $\lambda \in \mathbb{R}^{n_c \times 1}$ (n_c being the number of connection DoFs) can be shown with Boolean matrix $\mathbf{B}$ as,

$$\begin{bmatrix} \mathbf{B}^C & \mathbf{B}^A \end{bmatrix} \begin{bmatrix} \mathbf{u}^C \\ \mathbf{u}^A \end{bmatrix} = \mathbf{0}, \quad -\begin{bmatrix} \mathbf{B}^C & \mathbf{B}^A \end{bmatrix}^T \lambda = \begin{bmatrix} \mathbf{g}^C \\ \mathbf{g}^A \end{bmatrix} \quad (9)$$

Using the dual (Lagrange multiplier) representation, the equation of motion in Eq. (7) with Lagrange multipliers as internal forces can be rewritten as,

$$\alpha^{CIA-1} \begin{bmatrix} \mathbf{u}^C \\ \mathbf{u}^A \end{bmatrix} + \mathbf{B}^T \lambda = \mathbf{f} \quad (10)$$

After isolating the displacement vector from Eq. (10), it is substituted into the compatibility condition, and λ is calculated as,

$$\lambda = \left(B \alpha^{C|A} B^T \right)^{-1} B \alpha^{C|A} f \quad (11)$$

Inserting this equation into Eq. (10) and solving for the FRF of unknown substructure B yields Eq. (12), which enforces both compatibility and equilibrium:

$$\widehat{\alpha}^B = \left(I - \alpha^{C|A} B^T \left(B \alpha^{C|A} B^T \right)^{-1} B \right) \alpha^{C|A} \quad (12)$$

Herein, $\widehat{\alpha}^B \in \mathbb{C}^{(n^C + n^A) \times (n^C + n^A)}$ is an intermediate matrix resulting from the decoupling calculation in Eq. (12). It is important to note that this matrix is not the direct, physical FRF matrix of substructure B. From this matrix, the FRFs of substructure B, $\alpha^B \in \mathbb{C}^{n^B \times n^B}$, where n^B represents the number of DoFs of substructure B, can be obtained by extracting only the entries that correspond to the degrees of freedom belonging to substructure B within the upper-left portion of $\widehat{\alpha}^B$. The detailed derivation of Eq. (12) is given in [11]. In the case of systems with just translational DoFs, the standard interface decoupling method [15] can be used to decouple the substructures from their connection points. However, when rotational DoFs are involved, the standard interface decoupling approach becomes inadequate, as it is often experimentally impractical to excite the structure in rotational DoFs. Rotational DoFs are to be estimated using the Central Difference method [13] or Virtual Point Transformation [14] to decouple, for instance, a beam-like structure where rotational effects cannot be disregarded. An alternative approach to compensating for rotational effects is the implementation of the mixed interface decoupling method [15]. This method introduces selected internal DoFs into Eq. (12) by measuring them and changing their Boolean components from 0 to 1. It enables the decoupling of beam-like structures with minimal error without the need to measure rotational DoFs at the interface [15]. Although the focus of this study is on decoupling, the LM-FBS framework can naturally also be used for the coupling of substructures. When the individual FRFs of substructures A and B are known, their coupled FRF can be obtained by applying the same projection operator, but without the negative sign used in the block-diagonal receptance matrix for decoupling:

$$\widehat{\alpha}^C = \left(I - \alpha^{A|B} B^T \left(B \alpha^{A|B} B^T \right)^{-1} B \right) \alpha^{A|B} \quad (13)$$

where $\alpha^{A|B} = \text{diag}(\alpha^A, \alpha^B)$. In this context, α^C represents the coupled FRF. However, since $\alpha^{A|B}$ includes the interface DoFs from both substructures, $\widehat{\alpha}^C$ contains the interface DoFs twice. Hence, the redundant interface rows and columns must be removed to yield α^C . Eq. (13) describes the dual coupling relation, which assembles two known substructures into a single coupled system by enforcing displacement compatibility and force equilibrium at the interface. Thus, coupling and decoupling appear as perfectly symmetric operations within the same dual domain framework: decoupling isolates a substructure by adding the negative of the known substructure to the coupled structure, while coupling unites the substructures through the same projection mechanism. Although this formulation is originally derived for linear systems, it can be employed for nonlinear systems when quasi-linear FRFs are used. Measuring quasi-linear FRF of a nonlinear system via RCT by keeping the response amplitude of a selected point (usually, the excitation point) fixed as frequency changes, and repeating the test for several values of response amplitudes, will give a series of quasi-linear FRFs. In the Nonlinear Frequency-Based Substructuring framework, these measured quasi-linear FRFs are employed in decoupling (or coupling) as explained above to calculate the quasi-linear FRFs of the decoupled (or coupled) structure, each corresponding to a different response amplitude.

2.3 Obtaining Modal Model of Nonlinear Systems and Calculating Frequency Response

The decoupling (or coupling) results are obtained in the form of a new set of quasi-linear FRFs as shown in Fig. 1a, which represent the direct output of the nonlinear decoupling (or coupling) operation. Then a set of modal parameters can be identified from each quasi-linear FRF by using any modal identification method for linear systems, yielding response-dependent modal parameters, $\bar{\omega}_r$, $\bar{\eta}_r$, and A^r , as shown in Fig. 1b, which define the modal model of the nonlinear system. Here, the modal constant A_{jk}^r is defined as

$$A_{jk}^r = \bar{\phi}_{jr}(q_r) \bar{\phi}_{kr}(q_r) \quad (14)$$

The modal model obtained can be employed to synthesize the constant-force FRFs of the decoupled (or coupled) nonlinear system for a specified harmonic forcing by using Eq. (5), as shown in Fig. 1c. Using suitable numerical

techniques, frequency responses can be synthesized even when there are unstable branches due to strong nonlinearity and/or high forcing levels causing jumps in the FRFs. As the modal parameters are response-dependent, an iterative solution is required using a numerical solver. However, instead of an iterative numerical solution, it is more straightforward to solve for the excitation frequency, ω , that corresponds to a prescribed modal response amplitude, q_r following the amplitude-parametrization approach used in [36, 37]. This approach provides a closed-form expression for the amplitude-frequency curve. This expression can be derived by rearranging the single-mode response equation

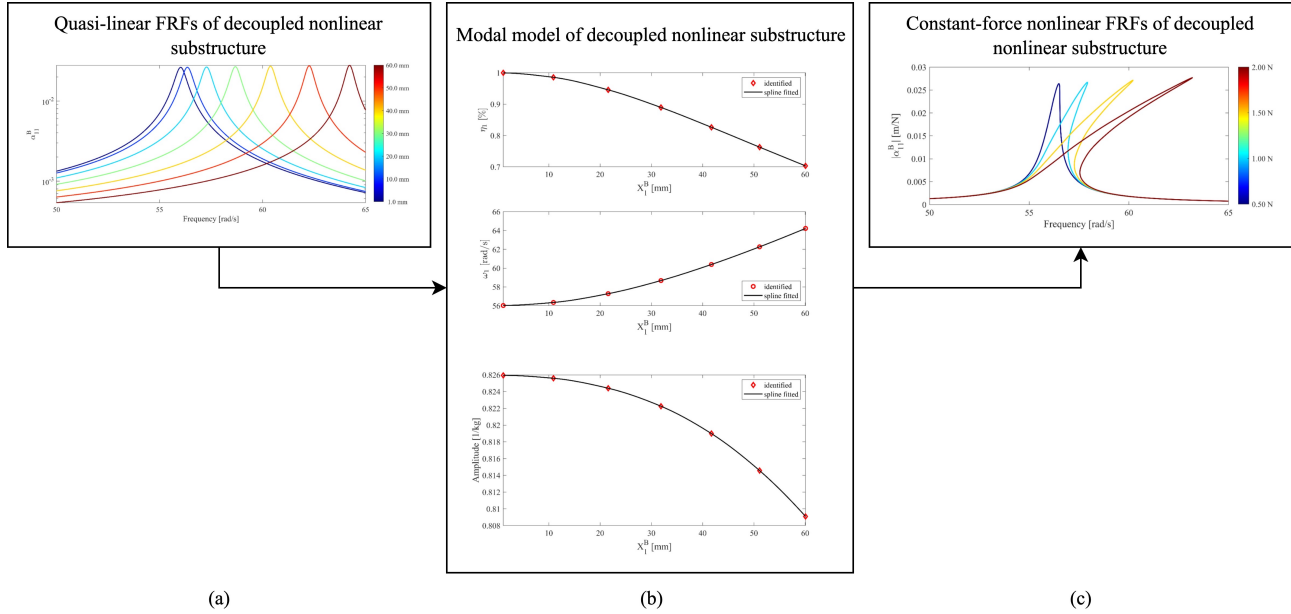


Fig. 1: (a) Quasi-linear FRFs, (b) response-dependent modal model, and (c) synthesized constant-force FRFs of the decoupled nonlinear substructure

(Eq. (6)). Since hysteretic damping is used in this paper, the equations from [36, 37], which were derived for viscous damping, are modified accordingly. By taking the magnitude of Eq. (6), using $q_r = |\tilde{q}_r|$, and solving for the excitation frequency ω , two solutions can be obtained. These solutions define the left (ω_L) and right (ω_R) branches of the resonance peak for a given modal amplitude, q_r :

$$\omega_L = \sqrt{\tilde{\omega}_r^2(q_r) - \sqrt{\frac{|\tilde{\phi}_{kr}(q_r)F_k|^2}{q_r^2} - \tilde{\eta}_r^2(q_r)\tilde{\omega}_r^4(q_r)}} \quad (15)$$

$$\omega_R = \sqrt{\tilde{\omega}_r^2(q_r) + \sqrt{\frac{|\tilde{\phi}_{kr}(q_r)F_k|^2}{q_r^2} - \tilde{\eta}_r^2(q_r)\tilde{\omega}_r^4(q_r)}} \quad (16)$$

By calculating ω_L and ω_R for a range of prescribed amplitudes q_r , the complete constant-force FRF can be constructed directly without iteration. This method is computationally efficient and naturally captures unstable response branches, if any. It is important to note that these ω_L and ω_R solutions are physically valid only when the radicands (the terms inside the square roots) are non-negative, which defines the existence limits of the response at a given amplitude.

2.4 Nonlinear Frequency-Based Decoupling

In this section, the steps of NL-FBD framework are summarized and its theoretical limitations are discussed. Throughout the study, the dynamics of substructure A are assumed to be known, while substructure B represents the unknown nonlinear substructure. Substructure A may be either linear or nonlinear, and both scenarios are investigated in this paper. The proposed decoupling framework involves the following steps:

- (1) **FRF Measurement:** Measure the FRFs of substructure A (quasi-linear FRFs if it is nonlinear) and the quasi-linear FRFs of the coupled system for various displacement amplitudes using RCT.

- (2) **FRF Decoupling:** Apply Eq. (12), originally developed for linear systems [11], to compute the quasi-linear FRFs of the unknown substructure B, corresponding to the displacement levels used in the first step.
- (3) **Modal Identification:** Identify a set of modal parameters from each quasi-linear FRF of substructure B and obtain its modal model.
- (4) **Response Prediction:** Use Eqs. (15) and (16) to calculate the frequency response of substructure B under any given harmonic forcing.

To identify the modal parameters accurately, the FRFs of substructure B must be obtained around its resonance frequencies, where the response can be treated as quasi-linear. FRFs computed at off-resonance frequencies may not exhibit quasi-linear behavior and thus are not suitable for step (3). Therefore, in general, the mode of substructure B should be close to a mode of the coupled system, and also to a mode of substructure A if it is nonlinear as well. As these modal regions deviate from each other, the decoupling accuracy decreases. In practice, the relevant frequency range can be selected from preliminary low-amplitude broadband measurements. At sufficiently small response levels, the coupled system can be treated as approximately linear, and a preliminary linear decoupling step can be used to estimate the resonance region of the unknown substructure B.

The main requirement for the proposed decoupling framework in nonlinear systems to be successful is to ensure almost constancy of the Nonlinearity Matrix throughout the analyses and/or measurements for decoupling (which ensures the nonlinear system acts as a quasi-linear system). The RCT method, coupled with "Nonlinear Normal Mode Theory" facilitates small variations of the Nonlinearity Matrix around the resonance frequencies of subsystems and the coupled system. Since RCT ensures the constant displacement of the response-controlled DoF and the Nonlinear Normal Mode Theory ensures ratios of the amplitudes of the other DoFs with respect to the RCT-controlled DoF are almost constant, the constancy of the displacement amplitudes, hence the constancy of the Nonlinearity Matrix, is accomplished around the resonance frequency of a certain mode.

There is also another requirement for the proposed decoupling approach to be successful: Nonlinearity Matrix should not change significantly during the decoupling process. In other words, the value of the elements of the Nonlinearity Matrices of the subsystems should not change significantly when the subsystems are coupled. This condition requires the displacement amplitudes of DoFs where the nonlinear elements are present to be similar when the subsystem is analyzed/tested by itself and when it is coupled with the other subsystem. This condition is automatically satisfied when the subsystems are connected from a single connection DoF and RCT is applied at the connection DoF.

As a result, the displacement at any point in the assembly under RCT is identical to the displacement at that point when RCT is applied to the isolated substructure. Therefore, even if the FRFs measured this way at off-resonance frequencies of A and the coupled system are not quasi-linear, and FRFs measured at each frequency correspond to a different equivalent linear system (having a different Nonlinearity Matrix), these equivalent systems will be the same in the coupled system and the measured substructure. Consequently, as long as the selected frequency range corresponds to a mode of substructure B, the FRFs calculated for substructure B will still be quasi-linear, so that we can proceed with step (3). It is important to emphasize that this is true only for a single-DoF connection, which is a special case.

However, when the substructures are connected at multiple DoFs, this condition can no longer be fully satisfied because the response is controlled at only one of the connection DoFs during RCT. As a result, the internal displacement amplitudes governing the nonlinear stiffness terms differ between the coupled and decoupled cases, leading to variations in the nonlinearity matrices. This discrepancy introduces errors in the estimated FRFs of the decoupled system, even under noise-free conditions. The degree of error will be determined by how much change occurs in the elements of the Nonlinearity Matrix of the subsystems when they are coupled.

Nonetheless, if the chosen frequency range is close to an anti-resonance of either substructure A or the coupled system, practical issues may arise even for systems with single-DoF connection. These include the experimental difficulty of providing a very large force required for a constant displacement level, and increased sensitivity to measurement errors. If substructure A is also nonlinear, then in theory, unstable branches may arise when RCT is applied at anti-resonance frequencies. However, since it is practically impossible to perform RCT at anti-resonances, this theoretical limitation has no practical impact. Finally, the need for drive-point measurements at the connection DoFs is inherited from frequency-based substructuring (which is also valid for linear systems). When rotational interface DoFs cannot be measured or excited directly, mixed-interface decoupling can be used, as discussed in the following sections.

3 Numerical Applications on systems with single connection DoFs

This section evaluates the effectiveness of the proposed nonlinear decoupling approach for systems with a single DoF connection through case studies involving both linear–nonlinear and nonlinear–nonlinear coupled systems, based on simulated FRF measurements. The impact of potential measurement errors on the accuracy of the decoupling results is also demonstrated in these case studies.

3.1 Decoupling systems consisting of two nonlinear substructures

Fig. 2 illustrates the system analyzed in this study. This example has been previously used in our earlier publication [33], where the corresponding constant-force nonlinear FRFs were also presented. The mass, m is 0.1 kg, and the stiffness, k is 1×10^4 N/m. Also, cubic stiffnesses with a value of $k_c = 1 \times 10^7$ N/m³ are assumed between several DoFs of the system. In this example, the coupled system and substructure B have close resonances, and the decoupling occurs at the 4th DoF.

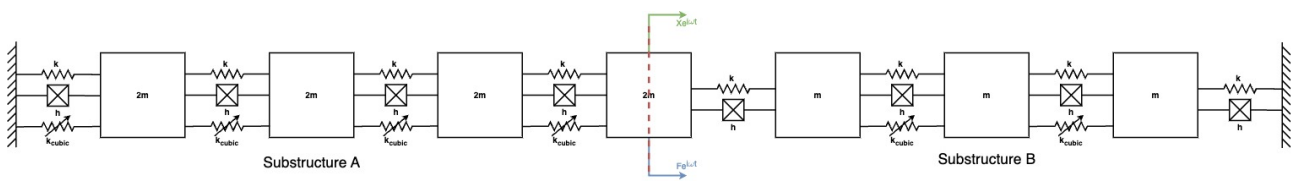


Fig. 2: Lumped system connected from single connection DoF with close modes

While the response of the 4th DoF is kept at a constant displacement amplitude in applying RCT, the excitation level is measured at the same DoF. The measurements are repeated for various, constant-response levels, and the direct point FRFs of the coupled system at the connection DoF are obtained as given in Fig. 3. In this and all subsequent FRF plots, each curve corresponds to a specific constant response amplitude, as indicated by the values in the color bar.

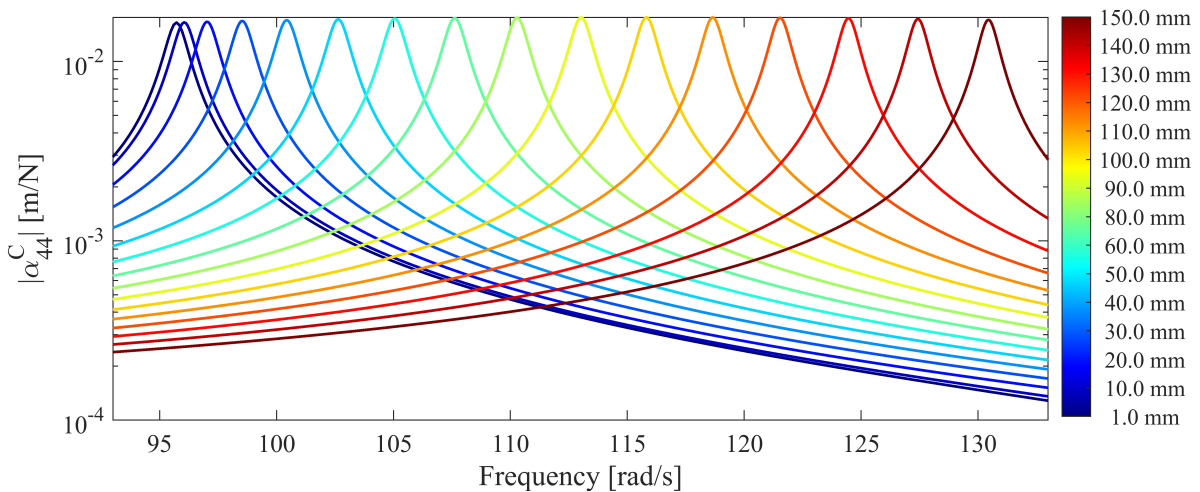


Fig. 3: Quasi-linear FRFs of the nonlinear coupled system at connection DoF

To decouple two substructures utilizing the dual decoupling formulation (Eq. (12)), constant-response FRFs of substructure A are also needed. Substructure A is isolated at the connection DoF, and RCT is applied at that DoF under several constant-response levels. The resulting quasi-linear FRFs of the nonlinear substructure A are presented in Fig. 4.

Using these quasi-linear FRFs (Fig. 3 and Fig. 4), Eq. (12) is employed to decouple the two nonlinear substructures.

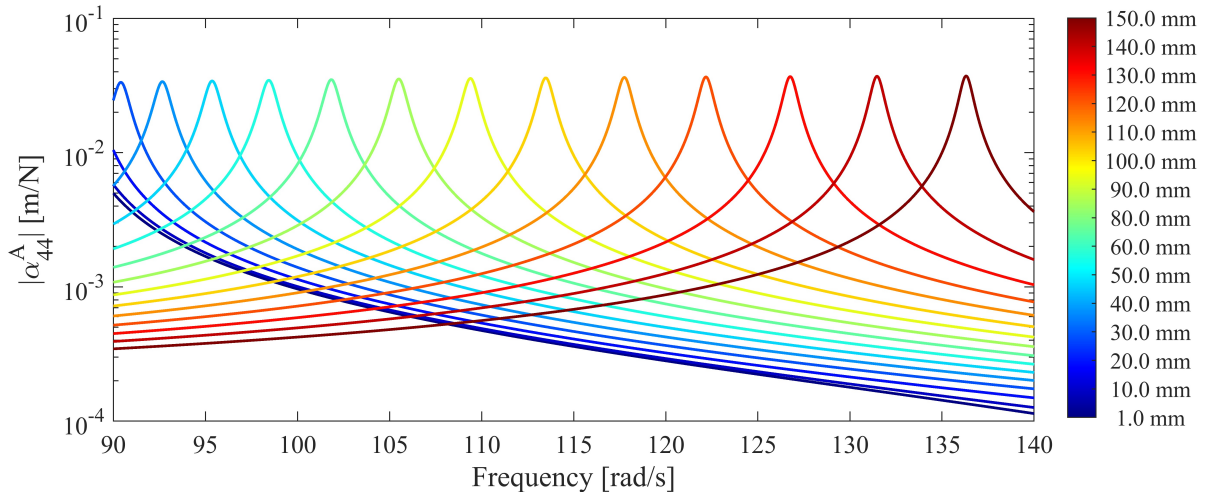


Fig. 4: Quasi-linear FRFs of nonlinear substructure A at connection DoF

The following Boolean matrix is used in Eq. (12) to isolate the unknown nonlinear substructure,

$$\mathbf{B} = \begin{bmatrix} \overbrace{0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0}^{\mathbf{B}^C} & \overbrace{0 \ 0 \ 0 \ -1}^{\mathbf{B}^A} \end{bmatrix} \quad (17)$$

The ordering of the inputs in $\mathbf{B}$ follows the numbering of the response DoFs in the global response vector. In this example, the response DoFs of the 7-DoF coupled system and the 4-DoF substructure A are numbered sequentially in increasing order, and the Boolean matrix is constructed accordingly. The quasi-linear FRFs of the decoupled substructure B are given in Fig. 5.

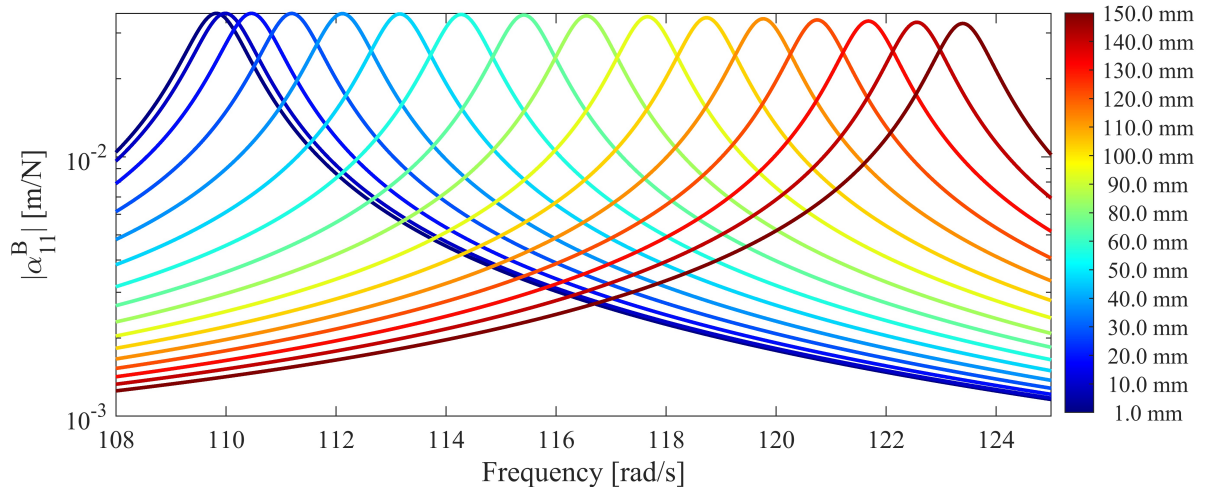


Fig. 5: Estimated quasi-linear FRFs of the unknown substructure

In the next step, the nonlinear modal parameters of the unknown substructure are identified from the estimated quasi-linear FRFs. The well-known peak-picking method is employed to identify the modal parameters of the unknown nonlinear substructure B from the calculated quasi-linear FRFs at each constant-response level. The identified nonlinear modal parameters and their corresponding fitted curves are given in Fig. 6 and Fig. 7. Fig. 6 shows the variation of the modal constant with the response level, whereas Fig. 7 shows the variations of the damping ratio and the natural frequency. The structure exhibits a stiffening behavior, and the natural frequency increases with an increase in response level, as expected. However, the damping ratio decreases with increasing response level;

because the damping in the system is linear, but stiffness increases as response level increases, yielding a decreasing modal damping ratio.

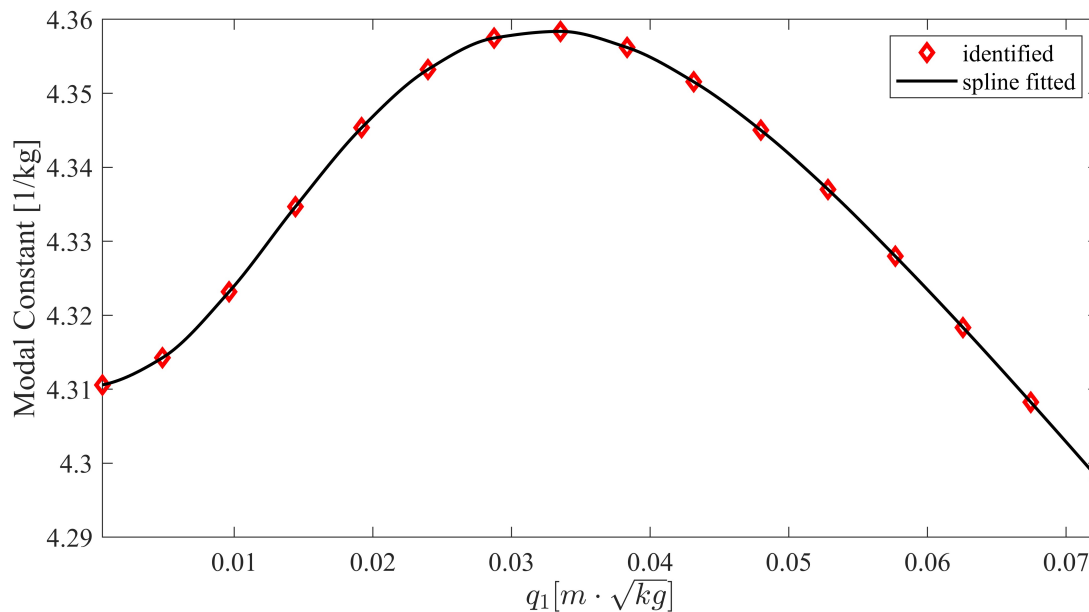


Fig. 6: Variation of modal constant with response level of connection DoF of unknown substructure B

Finally, by employing the nonlinear modal model in Eqs. (15) and (16), the constant-force FRFs of the unknown substructure B are synthesized for four different forcing levels without any iterations. These constant-force nonlinear FRFs are also obtained directly using system matrices of substructure B. The comparison of these constant-force FRFs is given in Fig. 8.

From these results it can be concluded that, except for the off-resonance regions—where single nonlinear mode theory weakens due to contributions from other modes—the decoupling performance of the method remains robust for nonlinear systems with single DoF connections.

The method's performance is also assessed with simulated measurement errors, by adding artificial noise with standard deviations of 0.5%, 1%, and 2%, referenced to the mean real and imaginary quasi-linear FRF components over the considered frequency range, to the quasi-linear FRFs given in Fig. 3. The noisy FRFs of the coupled system are given in Fig. 9 for the lowest and highest noise levels of 0.5% and 2%, respectively. The decoupled quasi-linear FRFs of the unknown substructure B are shown in Fig. 10, where pronounced noise effects can be observed in the resonance regions.

Although the FRFs in Fig. 9 exhibit relatively low noise level, the decoupling process amplifies this noise. This occurs because, when calculating the quasi-linear FRFs of substructure B in its resonance region, the corresponding FRFs of the coupled system at off-resonance frequencies—where the signal-to-noise ratio is lower—are used in the calculation.

The FRFs with high noise levels make the peak-picking method for modal identification not easy to apply. Therefore, the nonlinear modal parameters of the unknown substructure B are estimated by employing the least-squares fitting method (LSQF). Using the nonlinear modal parameters of the estimated unknown substructure B, the constant-force nonlinear FRFs are synthesized, and compared to the constant-force nonlinear FRFs obtained directly using system matrices of substructure B in Fig. 11, for all three noise levels.

To study the performance of the decoupling approach at resonance, the errors in the frequencies of peak amplitudes and peak amplitudes themselves for constant-force nonlinear FRFs are calculated and plotted in Fig. 12 for several forcing levels. Interestingly, as the forcing level increases, the errors decrease, since the modal parameters can be identified more accurately in higher constant-response levels where the resonance frequencies of the coupled system and unknown substructure B are closer, as can be seen from Fig. 10 and Fig. 11.

To see the performance of NL-FBD when the measurements in the anti-resonance region of the coupled system are used in decoupling, the locations of the nonlinear components of the system are modified, and the frequency range of interest is shifted. The procedure and results for that system are given in the following subsection. However,

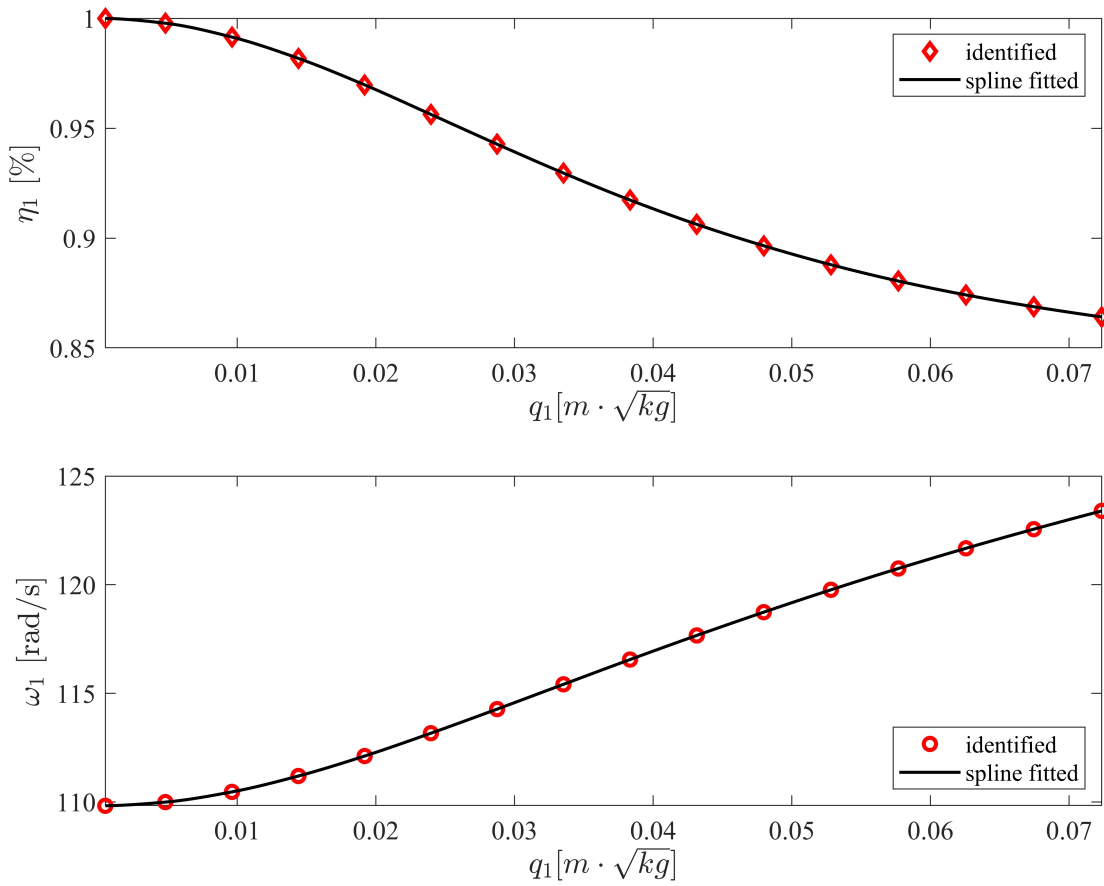


Fig. 7: Variation of damping ratio and natural frequency with response level of connection DoF of unknown substructure B

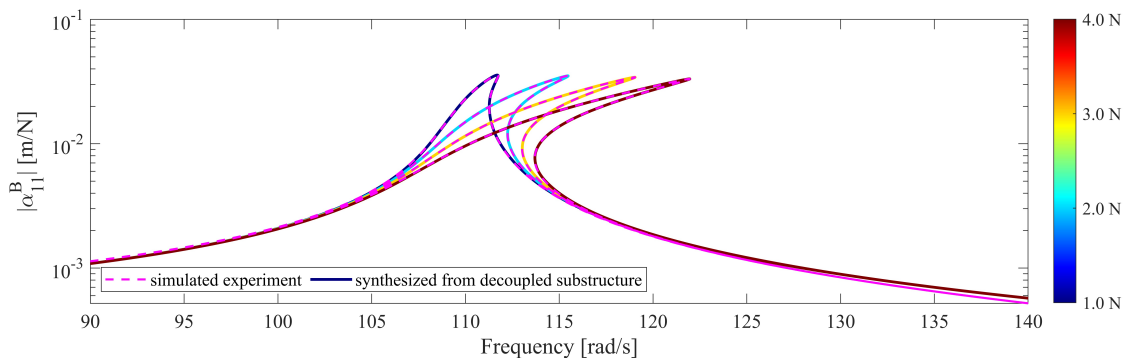


Fig. 8: Comparison of the synthesized constant-force nonlinear FRFs of the unknown substructure B (solid line) with the constant-force FRFs obtained directly from the system parameters of substructure B (dashed line)

as mentioned before, when substructure A is nonlinear, then in theory, unstable branches may arise when RCT is applied at anti-resonance frequencies. Therefore, to assess the performance of NL-FBD when measurements in the anti-resonance region of the coupled system are used, the known substructure A is considered linear. However, it should be remembered that it is practically impossible to perform RCT at anti-resonances due to extremely high force requirements to keep the displacement level constant, and therefore the following section has only theoretical but not practical importance.

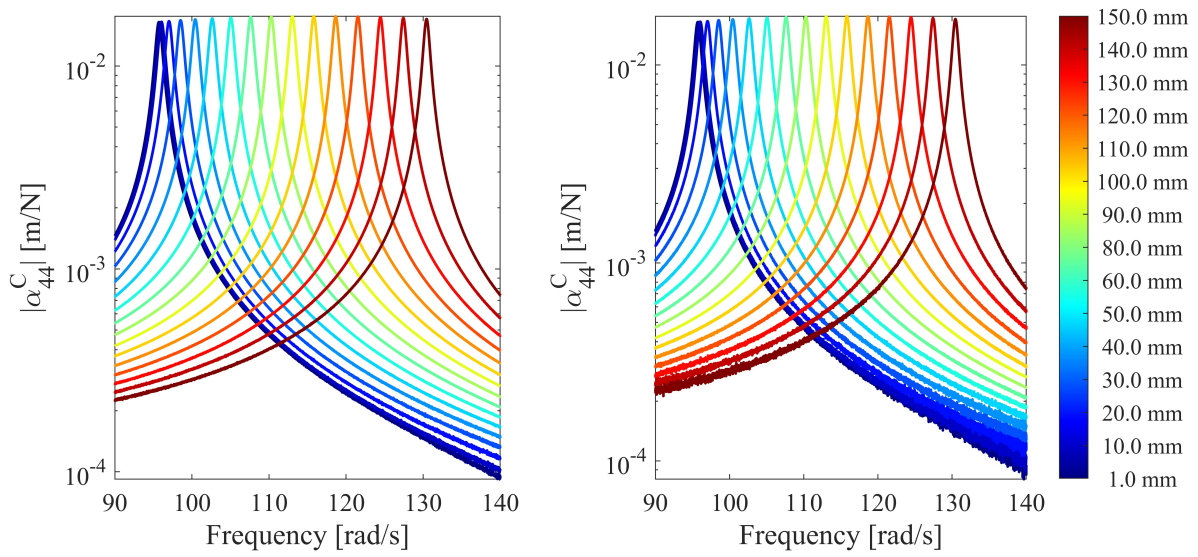


Fig. 9: Quasi-linear FRFs of the coupled system with simulated measurement errors of 0.5% (left) and 2% (right)

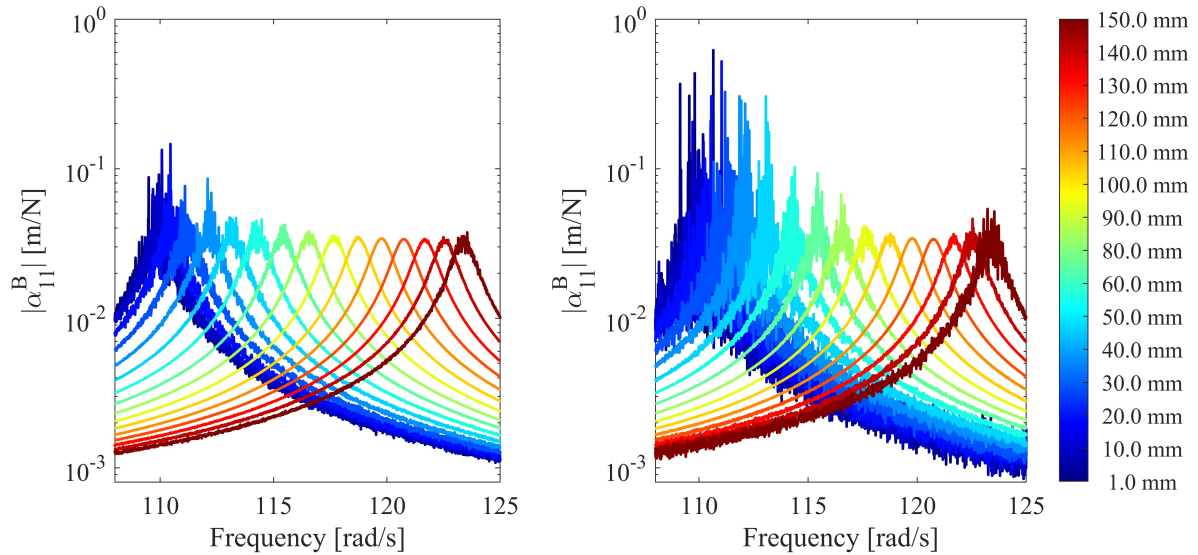


Fig. 10: Estimated quasi-linear FRFs of the unknown substructure B from the coupled system's FRFs with simulated measurement errors of 0.5% (left) and 2% (right)

3.2 Investigation of Decoupling Performance near Anti-Resonance

In this subsection, the system in Fig. 2 is modified to have a linear substructure A, resulting in the coupled system shown in Fig. 13. A frequency band is selected so that the frequency range corresponding to the resonance region of substructure B coincides with the anti-resonance region of the coupled system. The system is again decoupled at the 4th coordinate.

The constant-response FRFs of the coupled structure are presented in Fig. 14. The constant-force nonlinear FRFs synthesized by using the modal parameters identified from the decoupled quasi-linear FRFs of subsystem B, and their comparison with those calculated directly from system parameters are given in Fig. 15. Although theoretically a very good agreement is observed around the resonance frequency, as in section 3.1, when measurement errors are included, the performance obtained in the previous case cannot be attained, as will be shown below.

When noise representing measurement error is introduced to the constant-response FRFs of the coupled system (Fig. 16), the error levels in the decoupled quasi-linear FRFs of the unknown substructure B increase even further (Fig. 17), since the anti-resonance regions are more prone to measurement errors. Consequently, the effect of

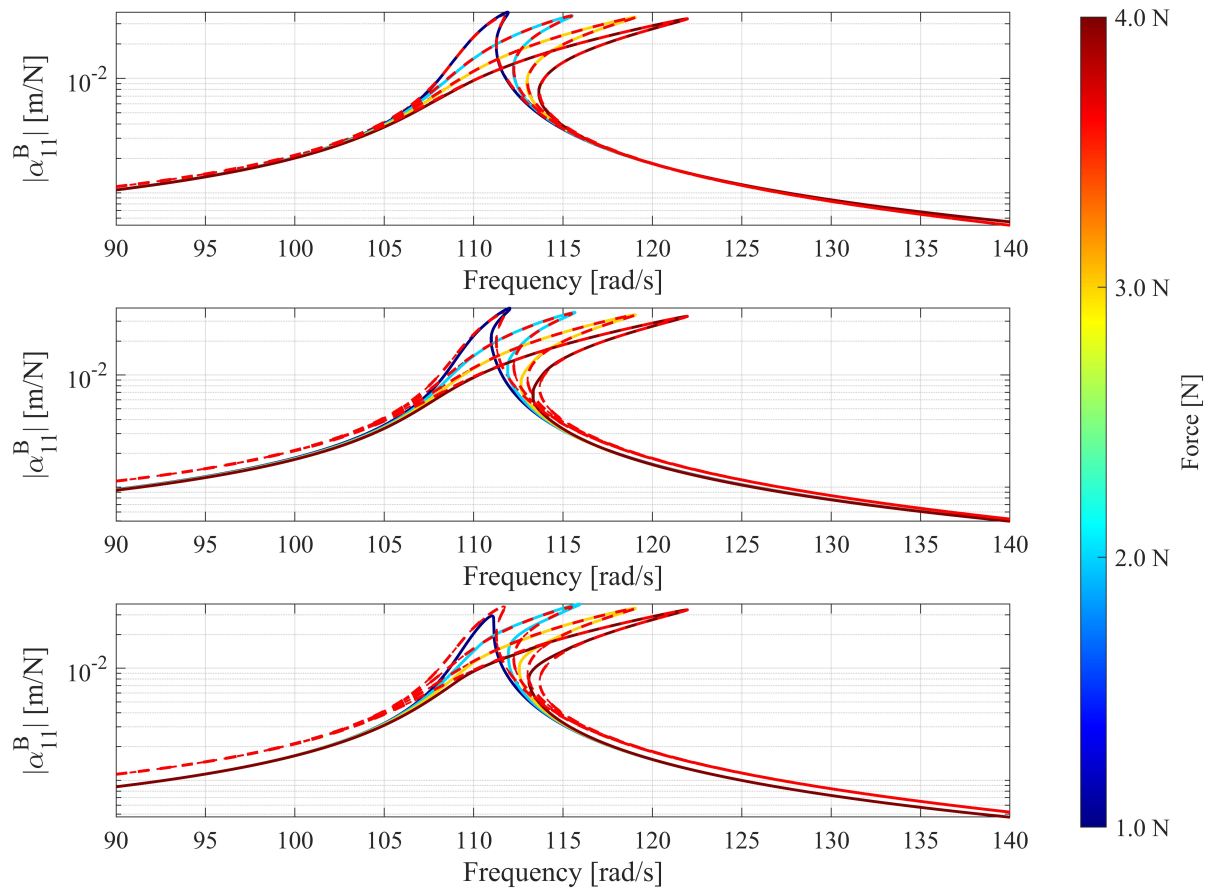


Fig. 11: Synthesized (solid lines) and directly calculated (dashed lines) constant-force nonlinear FRFs of the unknown substructure B for 0.5% noise (top), 1% noise (middle), and 2% noise (bottom)

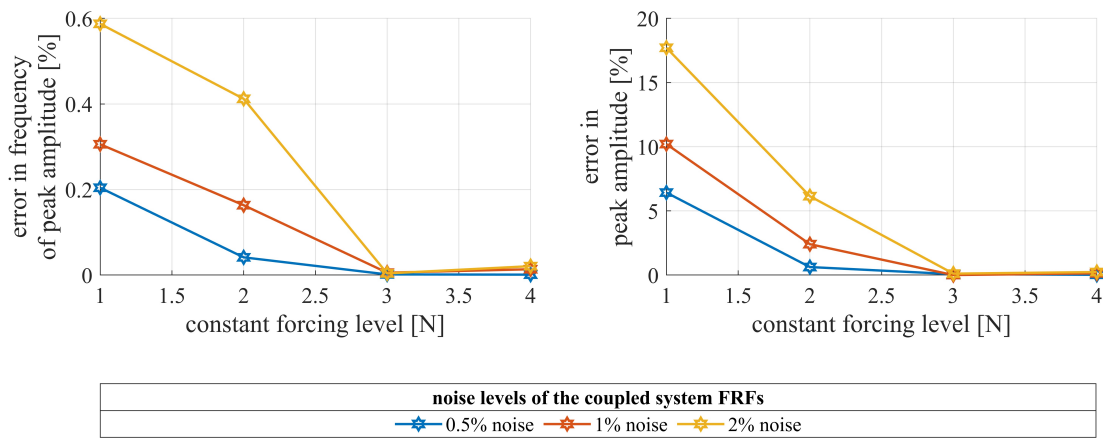


Fig. 12: Errors in the frequencies of peak amplitudes and the peak amplitudes themselves for the constant-force nonlinear FRFs synthesized using the modal parameters identified from noisy quasi-linear FRFs of the coupled system

noise representing measurement error increases dramatically when the interested frequency region of the unknown

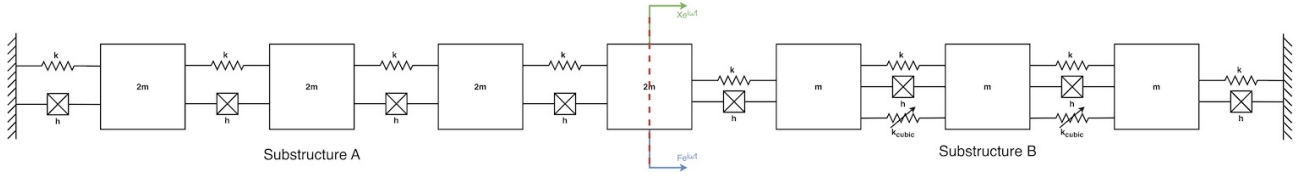


Fig. 13: Lumped system connected from single connection DoF with separated modes

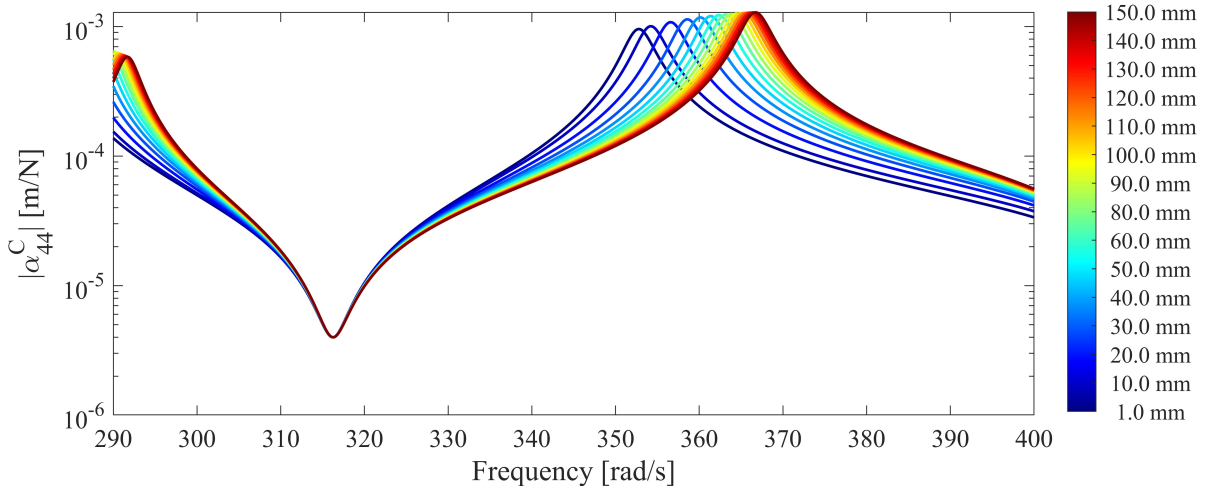


Fig. 14: Constant-response FRFs of the nonlinear coupled system at connection DoF

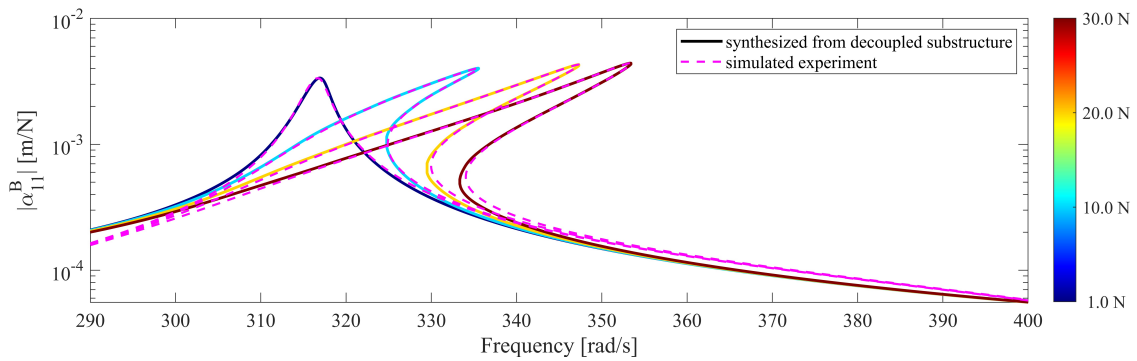


Fig. 15: Comparison of the synthesized constant-force nonlinear FRFs of the unknown substructure B (solid line) with the constant-force FRFs obtained directly from the system parameters of substructure B (dashed line)

substructure is located around an anti-resonance region of the coupled system.

The nonlinear modal parameters corresponding to the noisy mode are identified again using LSQF, with data points showing excessively high damping values omitted during the modal fitting process to ensure reliable parameter estimation. Fig. 18 presents the comparison of the synthesized constant-force nonlinear FRFs obtained using the nonlinear modal model with those calculated directly using system matrices of substructure B, for all three noise levels. The errors in the frequencies of peak amplitudes and the peak amplitudes themselves for different forcing levels are given in Fig. 19. Except for the lowest forcing level (1 N) under the highest noise condition (2%, yellow line), the errors in both the frequency of the peak amplitude and the peak amplitude itself are several times higher compared to the previous case.

These results demonstrate that the method is sensitive to measurement errors, a common issue in all frequency-based decoupling techniques for linear systems. The error becomes more pronounced when the resonance region of the unknown substructure coincides with the anti-resonance region of the coupled system. However, it is important to emphasize once again that this has no practical significance, as applying the RCT near anti-resonance frequencies is

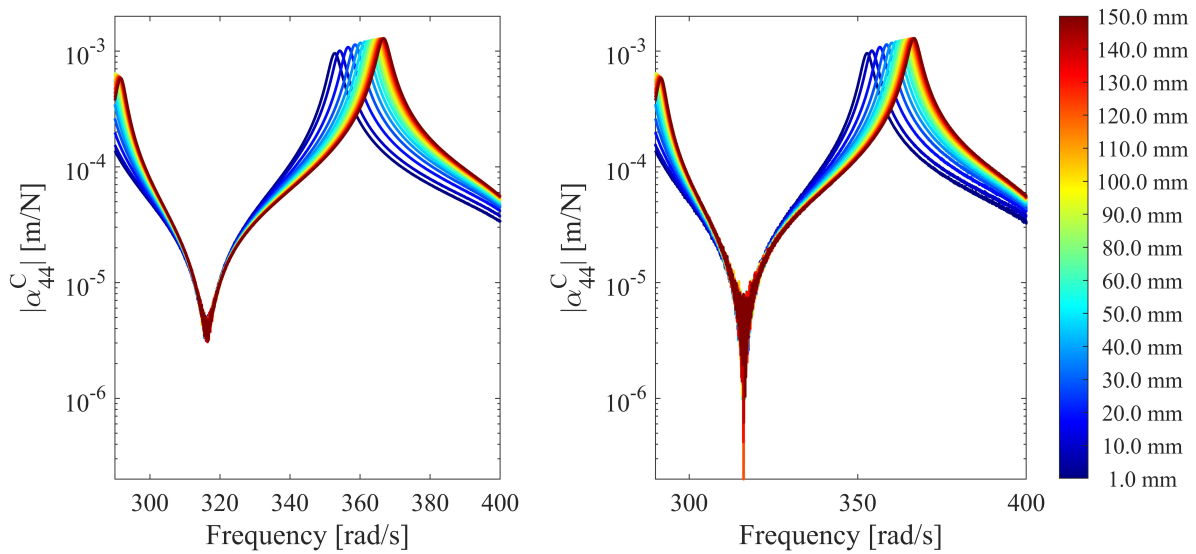


Fig. 16: Constant-response FRFs of the coupled nonlinear system with simulated measurement errors of 0.5% (left) and 2% (right)

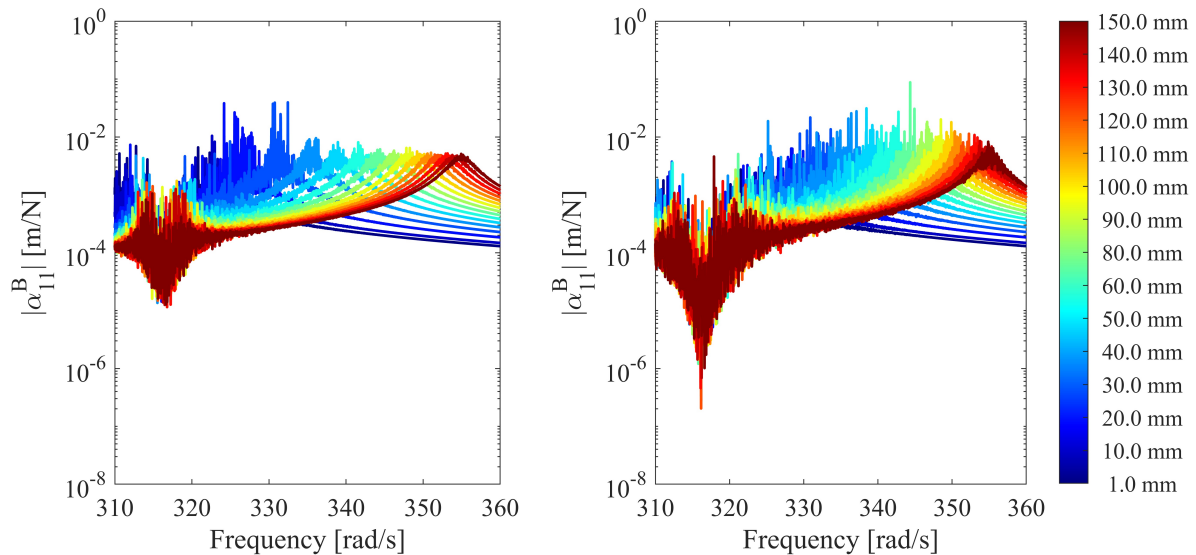


Fig. 17: Estimated quasi-linear FRFs of the unknown substructure B from the coupled system FRF with simulated measurement errors of 0.5% (left) and 2% (right)

impractical. This impracticality arises from the extremely high force required to maintain a constant displacement level at such frequencies, leading to very high-power requirements and high risk of structural damaging. Therefore, even though a system with a single DoF connection is a special case, and the proposed decoupling framework is, in theory, applicable to a broad range of scenarios, the presence of an anti-resonance region in the coupled system within the frequency range of the unknown substructure's resonance poses a limitation for the practical application of the approach, even in this case.

An interesting conclusion that can be derived from this case study is that, when the interface is defined by a single DoF and the known substructure is linear, applying RCT to the coupled system at the connection point yields quasi-linear FRFs, even if the coupled system has no resonance close to the testing frequency interval, provided that the resonance of the nonlinear substructure is within this frequency interval. This conclusion appears to contradict general RCT assumptions, but can be explained by the limited variation of the mode shape of the nonlinear substructure in this frequency region. This result suggests that RCT may be valid under broader conditions than previously recognized: the FRFs measured on a nonlinear coupled system remain quasi-linear at even off-resonance frequencies of the

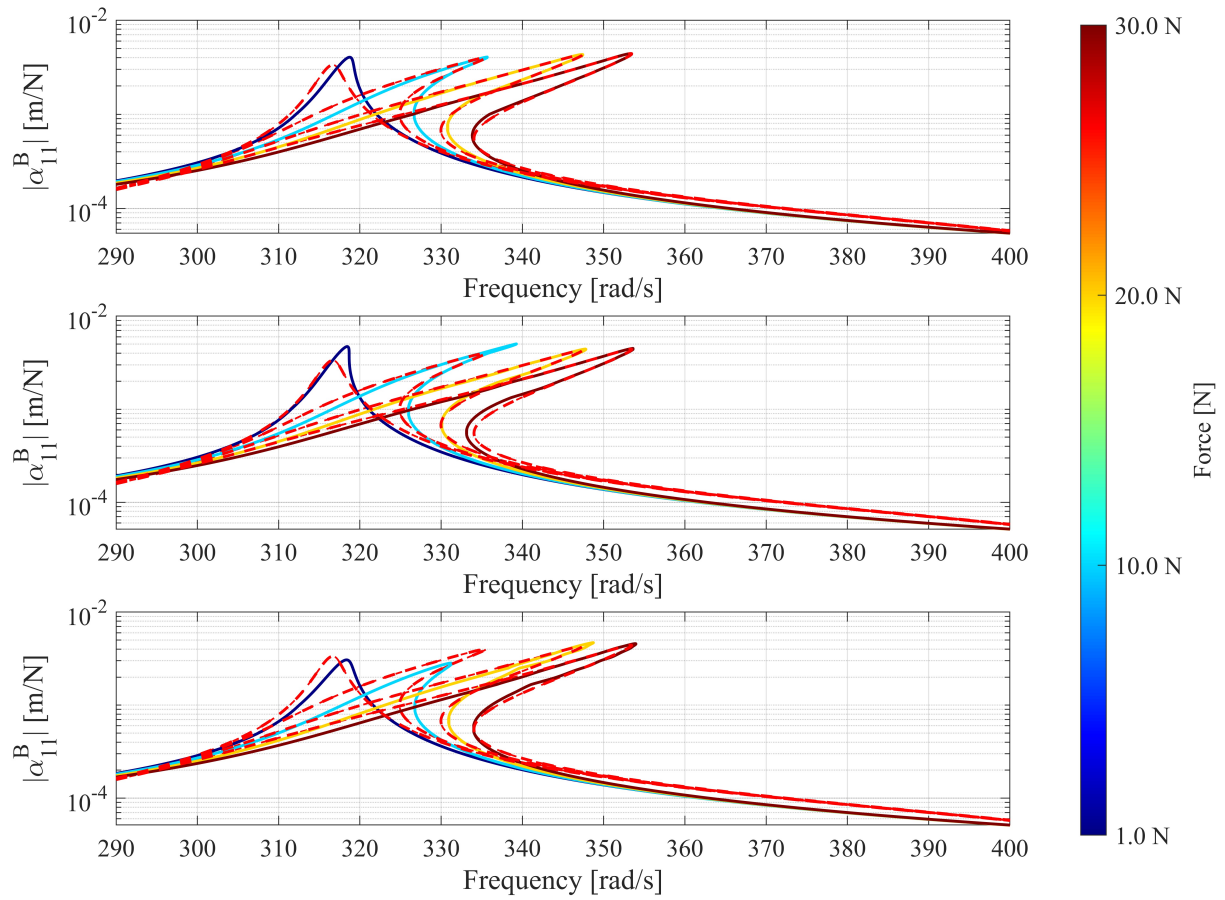


Fig. 18: Synthesized (solid lines) and directly calculated (dashed lines) constant-force nonlinear FRFs of the unknown substructure for 0.5% noise (top), 1% noise (middle), and 2% noise (bottom)

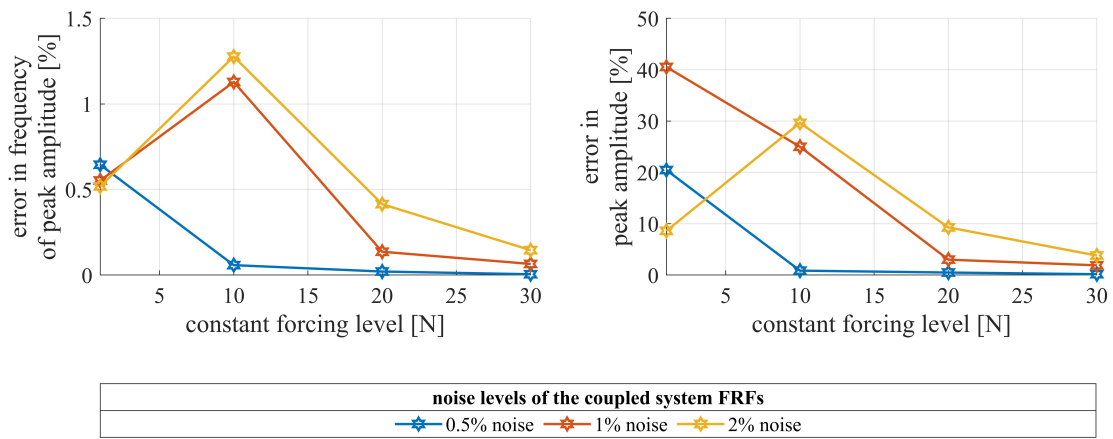


Fig. 19: Errors in the frequencies of peak amplitudes and the peak amplitudes themselves for the constant-force nonlinear FRFs synthesized using modal parameters identified from noisy quasi-linear FRFs of the coupled system

coupled system, provided that the frequency range coincides with a mode of the nonlinear substructure. Leveraging

this property of a nonlinear system coupled to a linear system at a single DoF, the decoupling approach provides highly accurate quasi-linear FRFs for the decoupled nonlinear subsystem within a frequency range associated with one of its modes—regardless of whether this range coincides with a mode of the coupled system. However, if the selected frequency band is not around a resonance of the nonlinear substructure, quasi-linearity cannot be ensured. Nevertheless, decoupling at off-resonance frequencies of the nonlinear system is of no use, since our purpose is to obtain a modal model of the decoupled nonlinear system.

4 Numerical Applications for systems with multiple connection-DoFs

In the previous section, it is shown that decoupling of nonlinear systems can be error-free for noise-free measurements when the subsystems are coupled from a single connection DoF, unless the known subsystem is also nonlinear and has an anti-resonance in the frequency range of interest. However, as discussed in Section 2.4, this is a special case because applying RCT at the single connection DoF imposes the same displacement boundary condition on the coupled system and on the isolated subsystems.

For systems with multiple connection-DoFs, the same condition cannot generally be satisfied exactly when only one connection DoF is response-controlled. As a result, the internal displacement amplitudes governing the nonlinear stiffness terms may differ between the coupled and decoupled cases, leading to variations in the Nonlinearity Matrices. This discrepancy introduces errors in the estimated FRFs of the decoupled system, even under noise-free conditions. The degree of error is determined by how much change occurs in the elements of the Nonlinearity Matrix of the subsystems when they are coupled.

These restrictions and limitations of the NL-FBD are discussed in this section with two case studies. In the first case study, the decoupling of two substructures from multiple connection points is considered. In the second case study, the practical problem of decoupling from connection points where some connection degrees of freedom cannot be actively controlled or excited (such as rotational degrees of freedom) is addressed.

4.1 Decoupling of a lumped system with multiple connection-DoFs

Although lumped systems can be assumed to be coupled from a single connection DoF, that might not be true for systems in real-life examples. In this section, the discrete system used in [33] is adopted, incorporating an additional nonlinearity configuration (shown in Fig. 20), to conduct a detailed assessment of the NL-FBD framework's accuracy for lumped systems with multiple connection points. The system is coupled rigidly from the 6th, 7th, and 8th DoFs. To decouple the system using NL-FBD, it is necessary to construct the receptance submatrices corresponding to these interface DoFs in Eq. (12). This requires the measurement of all direct and cross receptances among the selected interface points, resulting in a 3×3 FRF block. To obtain these values under quasi-linear assumptions, a modified RCT procedure is applied. The steps are as follows:

1. RCT is first performed at one of the interface DoFs (DoF 6 in this example) by applying harmonic excitation directly at this DoF.
2. The test is repeated for a set of discrete displacement amplitudes, keeping DoF 6 at the target vibration amplitude in each case.
3. The procedure is then repeated by exciting the rest of the interface DoFs (DoFs 7 and 8) individually at the same amplitude levels in a non-collocated setup, while maintaining DoF 6 at a constant displacement amplitude throughout.

The properties of the linear counterpart of the system illustrated in Fig. 20 are summarized in Table 1. To assess the influence of nonlinearities on the error levels, three different system configurations, as shown in Fig. 20, are constructed. Configuration 1 has only the yellow nonlinearity; configuration 2 has yellow and red nonlinearities; configuration 3 has all three nonlinearities. All three nonlinearities in the system are cubic stiffness nonlinearities with a stiffness value of $k_c = 1 \times 10^7 \text{ N/m}^3$.

The resulting interface FRF submatrices of both the coupled system and the known substructure are then assembled to be used in the LM-FBS formulation, as shown in Fig. 21. Three different colors in the figure indicate three different sets of RCT simulations. In addition, the choice of DoFs used for excitation and displacement control may be critical to avoid having an anti-resonance frequency too close to the resonance frequency of the mode of interest. This choice can be guided by preliminary low-amplitude linear FRF measurements, where a suitable control

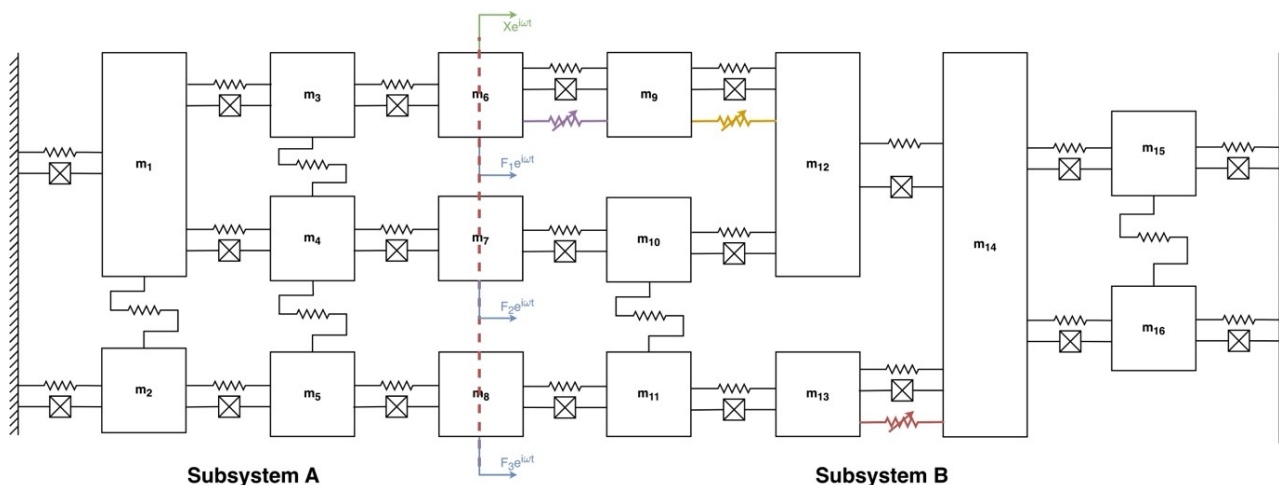


Fig. 20: Nonlinear lumped system with multiple connection points

Table 1: System properties of the lumped system with multiple connection points

Index	1-5 (1-5 for A)	6-8 (6-8 for A, 1-3 for B)	9-16 (4-11 for B)
Mass [kg]	1	1.5	1
Stiffness [N/m]	10000	10000	10000

DoF should show a clear resonance of the target mode and should not have a nearby anti-resonance of the coupled system. This can be clearly observed in the present case study. While successful results are obtained when the system is excited from and controlled at point 6, if the system is forced and controlled at point 7, the decoupling approach fails to perform with the required accuracy due to having an anti-resonance frequency of the coupled system (71.94 rad/s) very close to its resonance frequency (71.14 rad/s), resulting in FRFs measured with RCT not being quasi-linear within the frequency range required for successful decoupling application. Therefore, the controlled DoF is chosen to be the 6th DoF so that the anti-resonance of the FRF has minimal impact on the performance.

$$\left[\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ \cdots & \alpha_{66}^C & \alpha_{67}^C & \alpha_{68}^C & \cdots \\ & \alpha_{76}^C & \alpha_{77}^C & \alpha_{78}^C & \\ \cdots & \alpha_{86}^C & \alpha_{87}^C & \alpha_{88}^C & \cdots \\ & & & & \end{array} \right] \quad \left[\begin{array}{c} \\ \\ \\ \alpha^A \\ \\ \end{array} \right]$$

Fig. 21: Assembled matrices for nonlinear decoupling using LM-FBS

The quasi-linear FRFs of the coupled system shown in Fig. 22 are used in Eq. (12), together with the reference FRFs of substructure A, to estimate the quasi-linear FRFs of substructure B. The estimated quasi-linear FRFs of substructure B are presented in Fig. 23. Overall, the quasi-linear FRFs show very good agreement with reference data; however, noticeable discrepancies appear at higher displacement amplitudes.

RCT can only be applied to a single DoF of the nonlinear structure at a time, which means only one displacement can be kept at a constant amplitude. In this study, RCT is applied by exciting the system from the 6th, 7th, and 8th

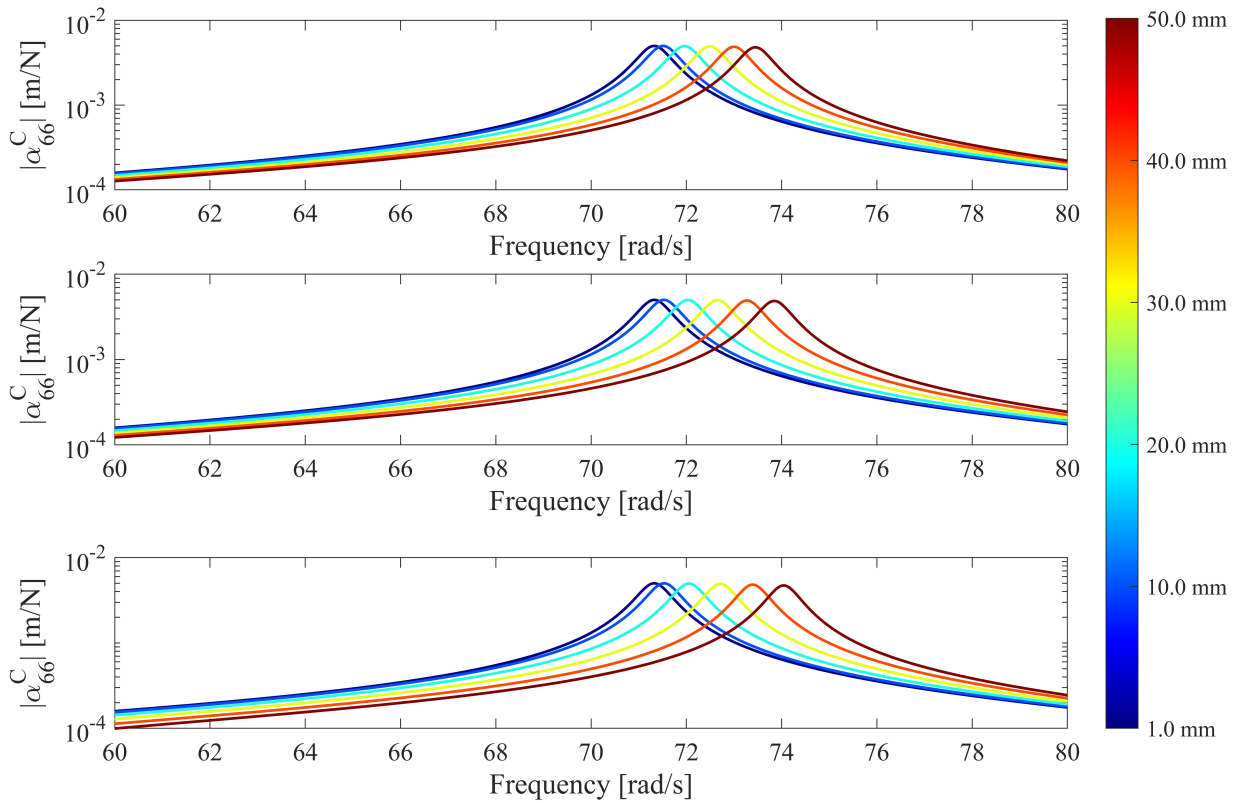


Fig. 22: Quasi-linear FRFs of the nonlinear coupled system in configuration 1 (upper), configuration 2 (middle), and configuration 3 (lower)

DoFs in turn, but keeping the displacement amplitude of the 6th DoF constant in all forcing cases throughout the test. This results in coupled response of the nonlinear subsystem B to be different than its response when it is individually excited with response-controlled forcing, as discussed above. Therefore, the Nonlinearity Matrix will be different in the coupled system compared to its counterparts in the decoupled subsystem B. This difference becomes more pronounced as the effect of nonlinearity increases with the increasing displacement amplitudes. Interestingly, in this case, a fictional second mode emerges in the quasi-linear FRF when the displacement amplitudes increase, as shown in Fig. 23. After applying modal identification to the quasi-linear FRFs given in Fig. 23, they are used in synthesizing the constant-force FRFs of the unknown nonlinear substructure. These estimated constant-force FRFs are given in Fig. 24 along with the reference FRFs obtained directly from the numerical model. Although the constant-force FRFs match with high precision for lower forcing levels, as expected the error increases as the forcing level increases, consistent with the errors in the quasi-linear FRFs given in Fig. 23.

The errors in the peak amplitudes and corresponding frequencies of the constant-force FRFs are presented in Fig. 25. The results show that the errors increase with forcing level in all cases. Moreover, the error is influenced by the number of nonlinear elements, becoming larger when the number of nonlinear elements increases from 2 to 3. It should be noted that the errors observed in this lumped-system example should not be interpreted as an inherent accuracy loss of the NL-FBD framework for all multiple-connection problems. This example represents a case in which only one connection DoF is response-controlled while the nonlinear restoring forces are affected by the relative motion of several DoFs. Therefore, the Nonlinearity Matrix of the unknown substructure changes more significantly between the coupled and decoupled configurations. Even in this case, the error in the peak frequency remains below 1%, while the peak-amplitude error remains below 15%. The accuracy of multi-DoF connection cases depends on the selected control DoF and the modal deformation pattern, since single-DoF RCT can keep only one displacement at a prescribed amplitude and may result in a response pattern in the nonlinear substructure in coupled condition that differs from its response when tested individually under response-controlled excitation.

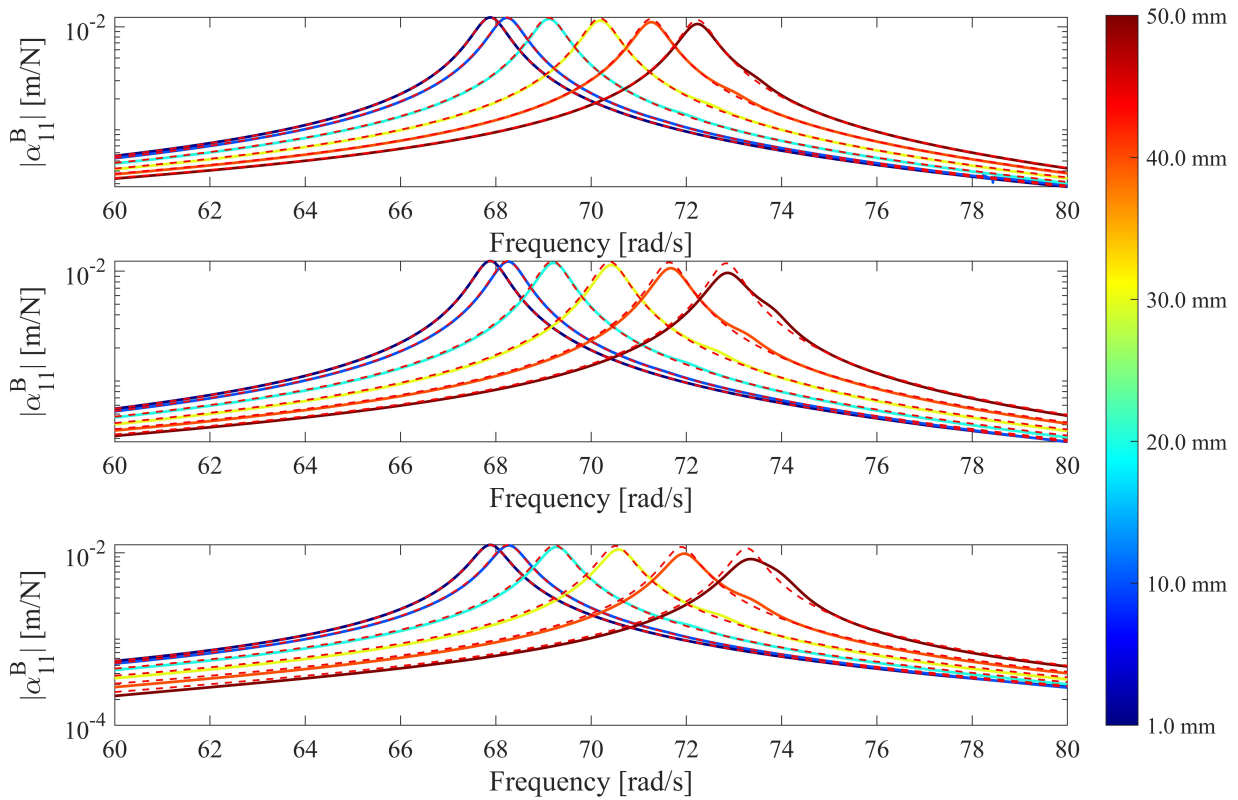


Fig. 23: Estimated quasi-linear FRFs (solid lines) and reference quasi-linear FRFs (dashed lines) of the nonlinear substructure B in configuration 1 (upper), configuration 2 (middle), and configuration 3 (lower)

4.2 Decoupling of continuous nonlinear systems

Another example used to investigate the accuracy of NL-FBD is a continuous numerical beam, discretized as shown in Fig. 26, for which the resulting constant-force FRFs of its mixed interface results are also given in [33]. The coupled beam consists of 28 Euler-Bernoulli beam elements. It has a linear substructure A (free-free beam) coupled rigidly to substructure B (nonlinear cantilevered beam) at the 9th node. The nonlinear cantilevered beam is connected to the ground using a cubic stiffness element from the translational DoF of the 10th node. The continuous beam is discretized and solved in MATLAB using a custom finite element implementation based on 2-node Euler-Bernoulli beam elements, each having one translational and one rotational DoF per node. The assembled stiffness and mass matrices calculated using Euler-Bernoulli Beam elements are also compared to the matrices generated in the NLVib© code [38] to validate the FE model of the beam. In addition to the mechanical properties of the beam given in Fig. 26, (classical) structural damping proportional to the stiffness matrix is assumed, with a damping ratio of 1%.

Each physical node has two degrees of freedom, representing the translation and the rotation of the physical node. When standard interface decoupling is used to decouple the beam from the 9th node, a 2×2 receptance matrix consisting of $\alpha_{17,17}$, $\alpha_{18,18}$ and their cross-coupling terms are needed. Although it is possible to apply forcing and make measurements in rotational direction in simulated experiments, this is not possible or practical in real applications. Therefore, obtaining the full 2×2 matrix FRF matrix required in the decoupling computations is impossible in real-life applications. In order to simulate real-life applications, only the translational DoF measurements are used in this case study. To decouple the system from its 9th node, RCT is applied to the 17th DoF (translational DoF of 9th node, indicated by the red dashed line in Fig. 26). The simulated measurement results for the quasi-linear FRFs of the coupled system at the 17th DoF, given in Fig. 27, are used in Eq. (12) along with the calculated FRF of substructure A to decouple the structure from a single translational DoF. The resulting quasi-linear FRFs of substructure B are given alongside the reference FRFs obtained directly from the numerical model in Fig. 28. Since the rotational effects are not negligible in beam elements, neglecting them resulted in unacceptable results.

To improve the results while decoupling continuous structures without directly measuring rotational DoFs,

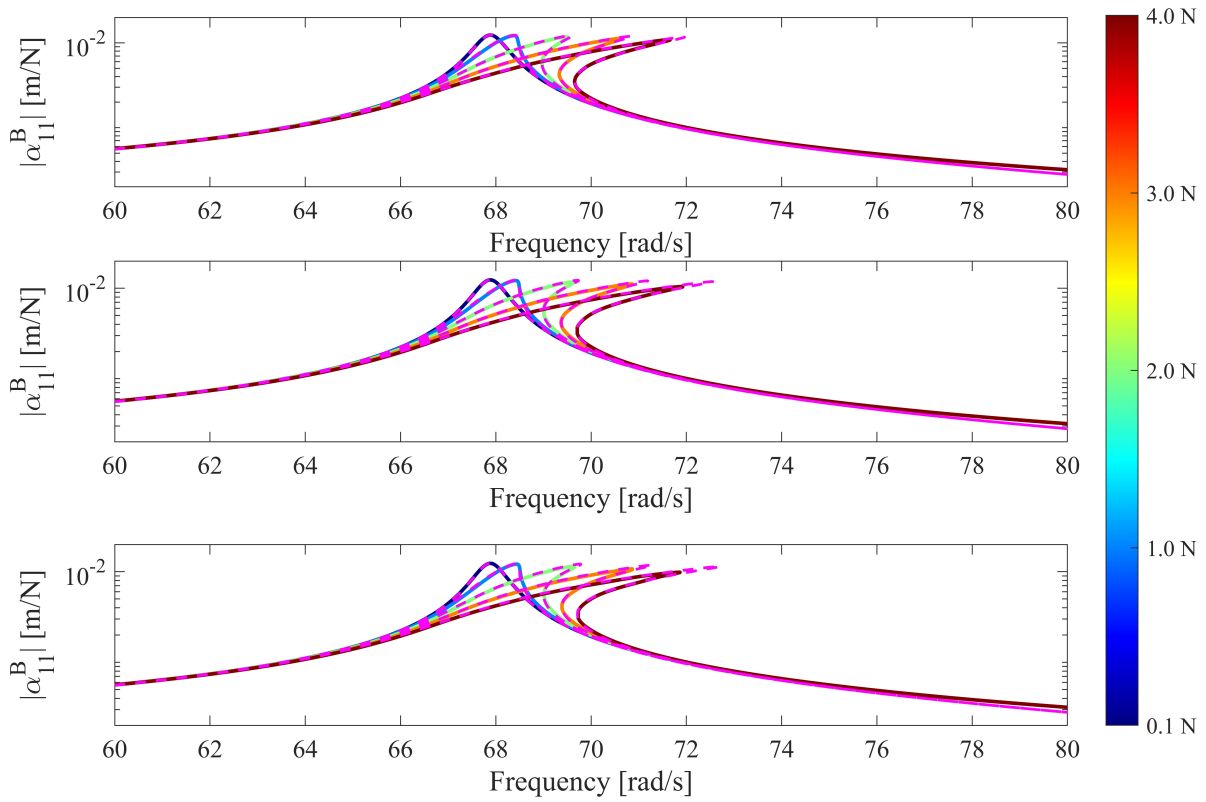


Fig. 24: Comparison of synthesized constant-force FRFs (solid lines) and reference constant-force FRFs (dashed lines) of the nonlinear substructure B in configuration 1 (upper), configuration 2 (middle), and configuration 3 (lower)

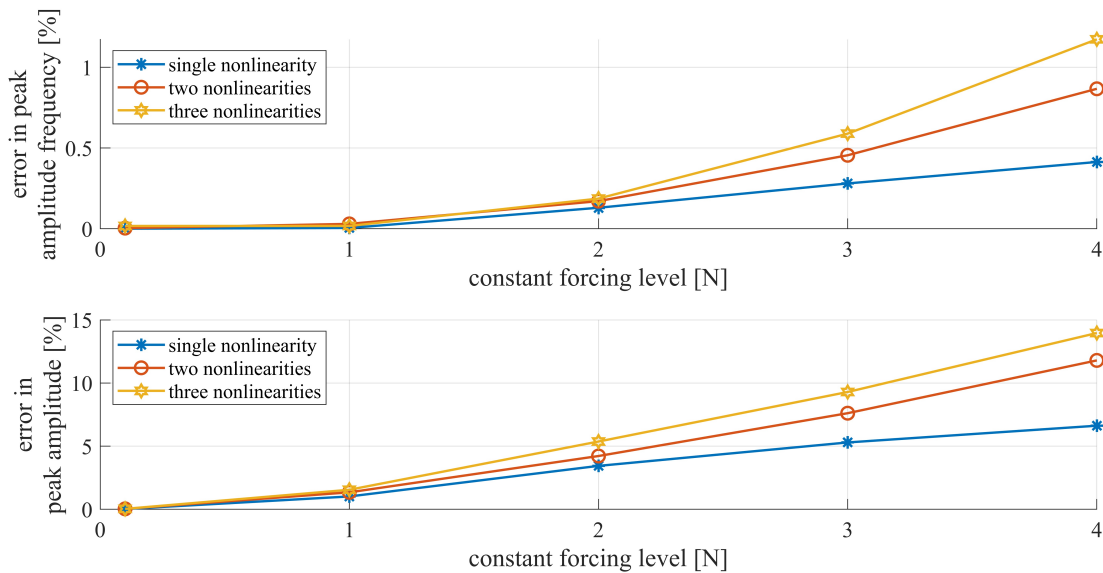


Fig. 25: Errors on peak amplitude frequency and peak amplitude for the nonlinear constant-force FRFs synthesized using a modal model identified using the estimated quasi-linear FRF

D'Ambrogio and Fregolent suggested the use of the mixed interface method for linear systems [15]. This method is successfully adapted in this paper for nonlinear systems. It uses a subset of internal DoFs along with a subset of interface DoFs to decouple two substructures. Since rotational DoFs cannot be measured at the interface of the

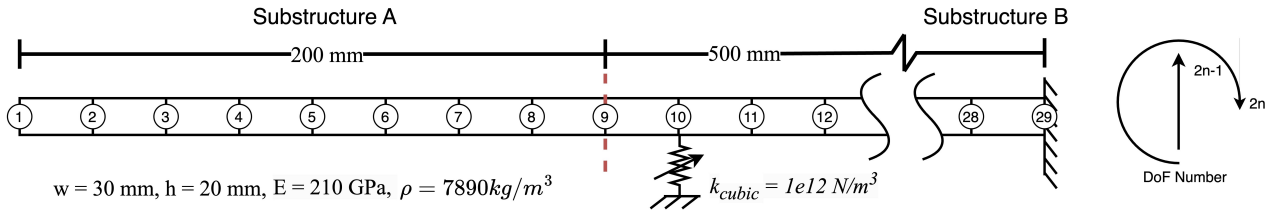


Fig. 26: Coupled continuous beam with cubic stiffness nonlinearity

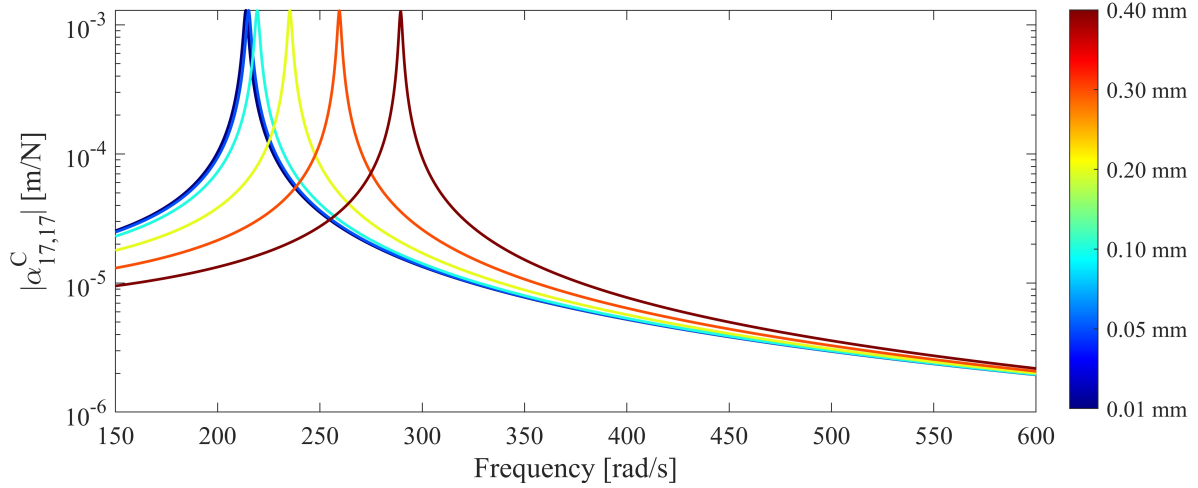


Fig. 27: Constant-response FRFs of coupled nonlinear beam

substructures, this method adds some of the translational DoFs in the internal nodes and thus compensates for the rotational effects at the interface.

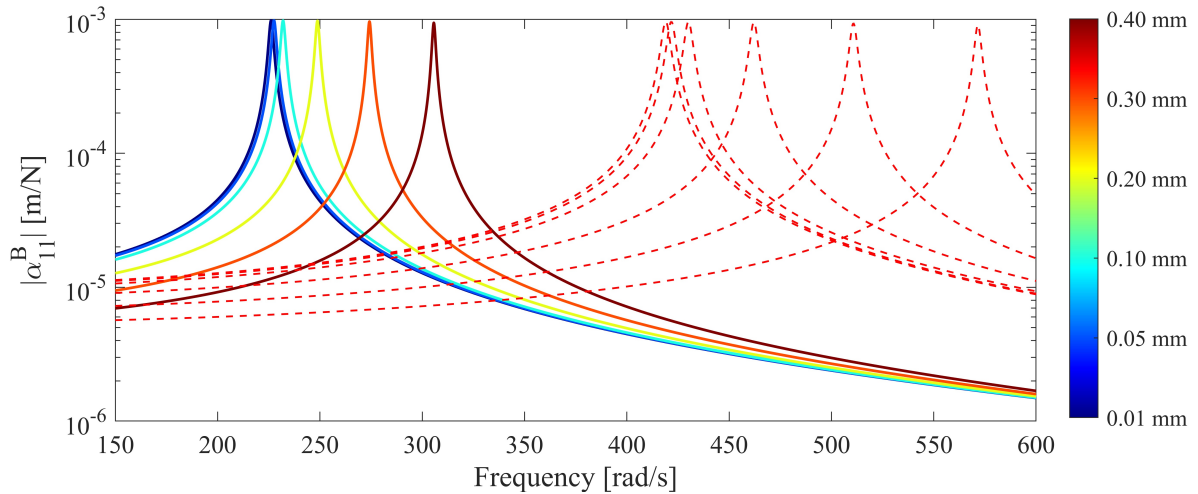


Fig. 28: Estimated (solid lines) and reference (dashed lines) constant-response FRFs of unknown substructure B

Using only a single translational DoF at the interface makes the problem mathematically under-determined, as the number of selected interface DoFs ($N_E = 1$) is smaller than the number of physical coupling DoFs ($n_c = 2$). Introducing an additional translational internal DoF (here, at node 5) increases N_E to 2, making it equal to n_c and thereby rendering the system mathematically determined.

This particular choice of internal DoF is physically meaningful: the unmeasurable rotation at the interface (node 9)

affects not only the translation at that node but also causes bending along the beam. The resulting bending motion is captured by the translational response at the internal node 5. Hence, by using both translational DoFs (at nodes 5 and 9), the system has sufficient independent information to resolve the rotational coupling effects without direct rotational measurements.

The Boolean matrix, $\mathbf{B} = [\mathbf{B}^C | \mathbf{B}^A]$ for decoupling the structure with mixed interface method is presented in Eq. (18). Only the first, last, and non-zero columns are provided for conciseness.

- Row 1 enforces compatibility at the internal DoF, linking the DoF in the coupled system (Col. 9, translational DoF of node 5) to its corresponding DoF in substructure A (Col. 65)
- Row 2 enforces compatibility at the interface DoF, linking the DoF in the coupled system (Col. 17, translational DoF of node 9) to its corresponding DoF in substructure A (Col. 73)

$$\mathbf{B} = \begin{bmatrix} \overbrace{\begin{matrix} \text{Col.1} & \text{Col.9} & \text{Col.17} & \text{Col.56} \end{matrix}}^{\mathbf{B}^C} & \overbrace{\begin{matrix} \text{Col.57} & \text{Col.65} & \text{Col.73} & \text{Col.74} \end{matrix}}^{\mathbf{B}^A} \\ 0 & \dots & 1 & \dots & 0 & \dots & 0 & 0 & \dots & -1 & \dots & 0 & 0 \\ 0 & \dots & 0 & \dots & 1 & \dots & 0 & 0 & \dots & 0 & \dots & -1 & 0 \end{bmatrix} \quad (18)$$

From the simulated RCT on the coupled system by keeping a constant displacement at the 17th DoF and exciting the system from the 9th DoF, the corresponding FRFs of the coupled system are obtained. These quasi-linear FRFs (from 9th DoF excitation) are then combined with the 17th-DoF excitation FRFs, for which the displacement at DoF 17 is also kept constant (which were obtained and discussed at the beginning of this section, see Fig. 27). The quasi-linear FRFs of the unknown substructure B are then calculated using the decoupling equation (Eq. (12)). This calculation requires the Boolean matrix (Eq. (18)) and the matrix α^{CIA} (introduced in Eq. (7)). α^{CIA} is formed by combining two input sets: (1) the linear FRFs of the known substructure A (α^A), and (2) the quasi-linear FRFs of the coupled system (α^C), where both input sets are computed from the 9th and 17th DoF excitations. The quasi-linear FRFs of the unknown substructure B, shown in Fig. 29, exhibit excellent agreement between the estimated and reference FRFs obtained directly from the numerical model results.

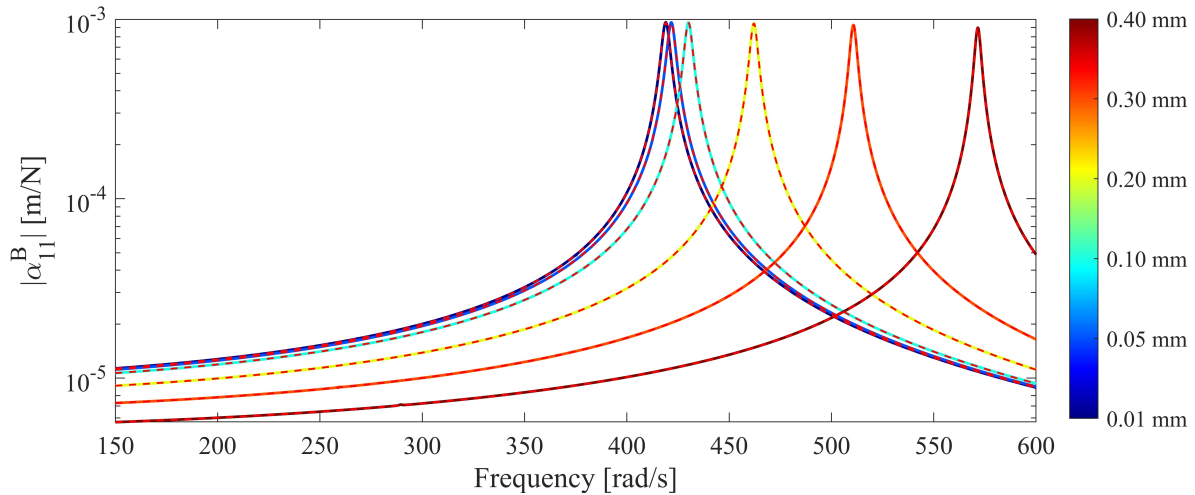


Fig. 29: Estimated (solid lines) and reference (dashed lines) quasi-linear FRFs of the unknown substructure B with mixed interface decoupling

These estimated FRFs are then used to identify the modal parameters of the substructure B. The resulting modal model is employed to synthesize constant-force nonlinear FRFs of the decoupled nonlinear substructure for four different force levels, (Fig. 30). The synthesized constant-force nonlinear FRFs also demonstrate high accuracy, as can be observed from Fig. 31, which compares the errors in peak amplitudes and corresponding frequencies obtained from two different decoupling approaches: one using only interface DoF 17, and the other employing a mixed-interface comprising DoFs 9 and 17. This comparison shows that the errors in frequencies are unacceptable when only interface DoF is used, while the errors in peak amplitudes are less than 3% for the forcing levels used. However, including internal DoFs in the decoupling process improves the accuracy considerably, dropping the errors in both peak amplitude frequency and peak amplitude below 1%, which can be regarded excellent. This high level of

accuracy is achieved because, as demonstrated in [15], the mixed-interface approach implicitly captures the rotational effects at the interface through the inclusion of internal translational DoFs, even though rotational DoFs are not explicitly included in the decoupling formulation. These findings demonstrate that the mixed-interface decoupling method, when used with the quasi-linear FRFs obtained through RCT, offers a more accurate and reliable framework for decoupling in nonlinear continuous systems.

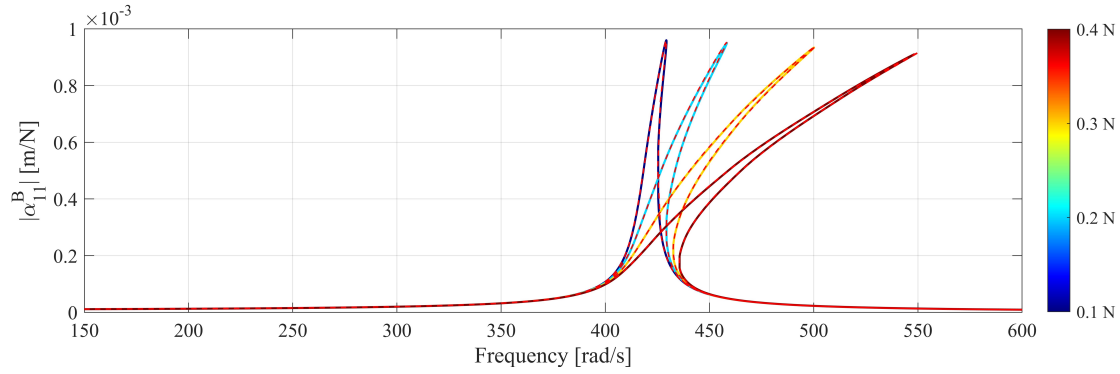


Fig. 30: Comparison of synthesized (solid lines) and reference (dashed lines) constant-force nonlinear FRFs of the unknown nonlinear substructure B

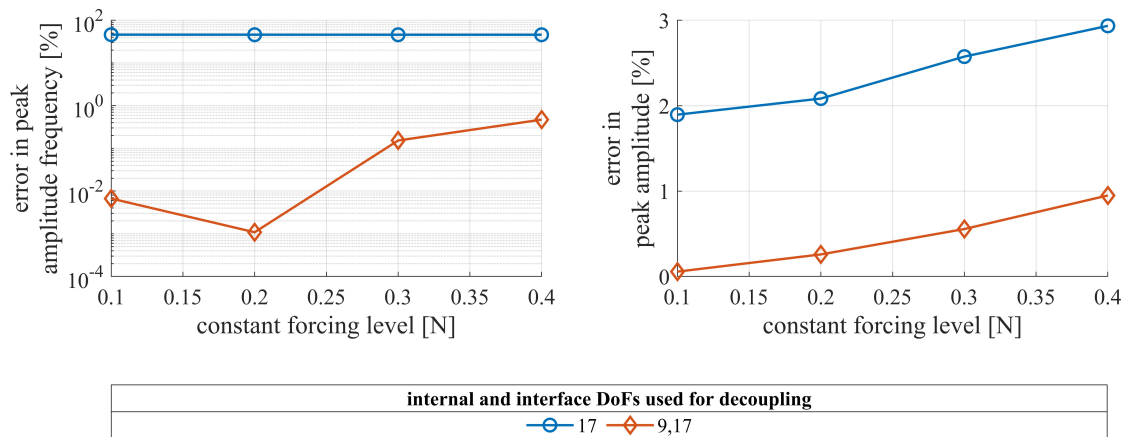


Fig. 31: Errors in peak amplitude frequency and peak amplitude for the nonlinear constant-force FRFs synthesized using standard and mixed interface decoupling

5 Experimental Validation

This section investigates the application of the nonlinear decoupling framework using RCT on an experimental benchmark structure. The unknown substructure consists of two cantilever beams of length 335 mm and 235 mm connected by two thin strips of length 115 mm, forming a composite system. The thin strips undergo transverse deflections that are large relative to their thickness, causing stretching effects and consequently introducing geometric nonlinearity. An additional linear cantilevered beam of length 300 mm is attached perpendicularly to this assembly, as shown in Fig. 32. The connection point between the cantilevered beam and the nonlinear beam is located 270 mm away from the left-hand side of the nonlinear beam. The cross-sections of the beam and thin strip regions are also given in Fig. 32.

A modal exciter (Brüel&Kjaer Modal Exciter Type 4824) is used to apply controlled dynamic excitation to the structure. The shaker is connected to the test specimen via a stinger to ensure efficient force transmission while minimizing both the forces in other directions and moments. A force transducer is attached between the stinger and

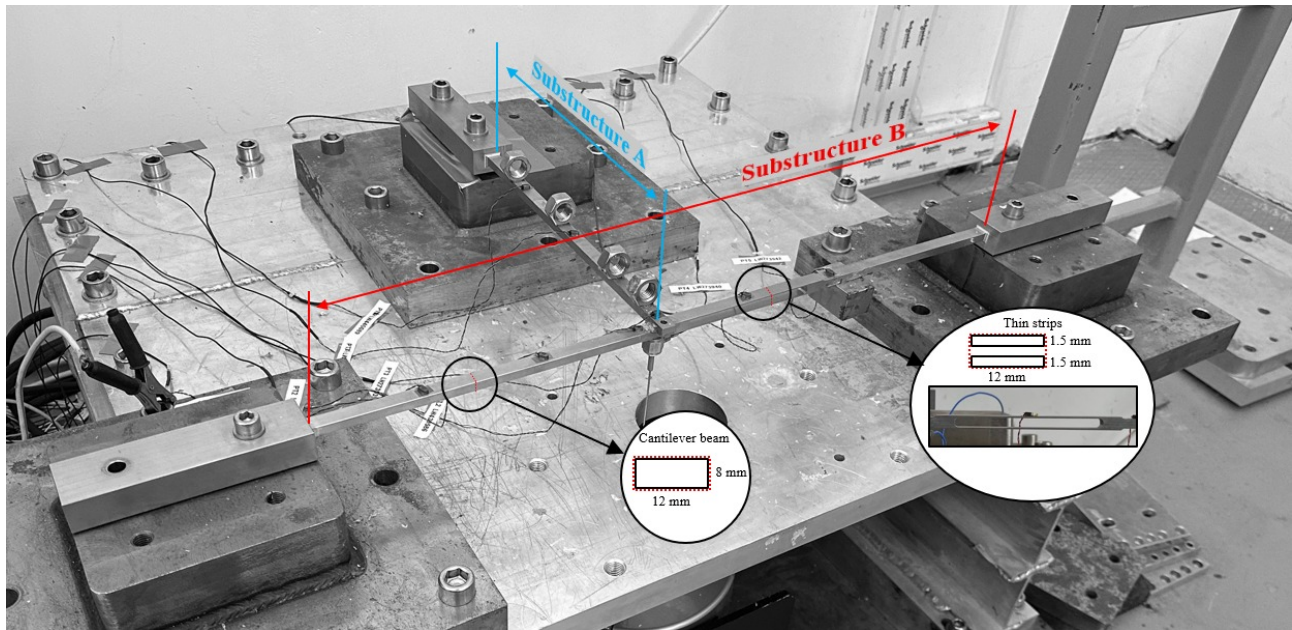


Fig. 32: Experimental setup used for verification of the method

the test structure for force measurement. The force transducer (PCB Piezotronics 208C01) has a force capacity of 44 N.

ICP accelerometers (PCB Piezotronics 352A73) are placed at different locations of the structure to measure the vibrational response of the beams. Low mass accelerometers are chosen to minimize the mass loading effects. These weigh only 0.3 grams and do not affect the dynamic behavior of the structure considerably. The locations of the accelerometers are given in Fig. 33. Among these accelerometers, the response at location 3 is used as the response-controlled coordinate in the RCT measurements and as the connection/interface coordinate in the decoupling and coupling calculations. Two accelerometers are placed close to the nominal location 3. The other accelerometer, labelled as 3_2, is used only as an auxiliary measurement to monitor the local response consistency around the same connection region; it is not introduced as an additional interface DoF in the substructuring calculations. The responses measured at locations 3 and 3_2 are observed to be very close in the frequency range of interest. The other accelerometers are used to monitor the motion of the structure during the tests and to check the consistency of the measured response levels, but they are not included as interface DoFs in Eq. (12).

The RCT measurements are performed as response-controlled stepped-sine tests using Siemens Testlab and the controller embedded in the Siemens SCADAS system. At each excitation frequency, the controller adjusts the shaker force amplitude until the displacement amplitude at the selected control DoF reaches the prescribed target level within the selected tolerance. After the steady-state response is reached, the force and acceleration signals are recorded, and the corresponding receptance value is calculated. This procedure is repeated for all frequency lines and for all prescribed response levels. The frequency resolution, response tolerance, and settling duration are selected so that unstable control near resonance is avoided and repeatable quasi-linear FRFs are obtained. The complete experimental campaign for the final data that is being used in the paper, including preliminary spectral testing, RCT measurements for configuration 1, and validation measurements for configuration 2, took approximately one working day. The two configurations are described in the following subsection.

To make the experimental decoupling problem more challenging, additional masses are attached to substructure A so that the resonance of configuration 1 is shifted away from the resonance region of substructure B. The additional masses attached to substructure A are labelled as masses 1–3, each weighing 31 grams. Configuration 2 is then used as the modified assembly for validation, where the prediction capability of the decoupled nonlinear model is assessed by coupling the estimated substructure B with the corresponding mass-modified substructure A. Mass 4, weighing 273 grams, is attached to the nonlinear substructure B for both configurations to enhance the effect of geometric nonlinearity by increasing the displacement at accelerometer location 5, as shown in Fig. 33. The positions of the additional masses are shown in Fig. 34.

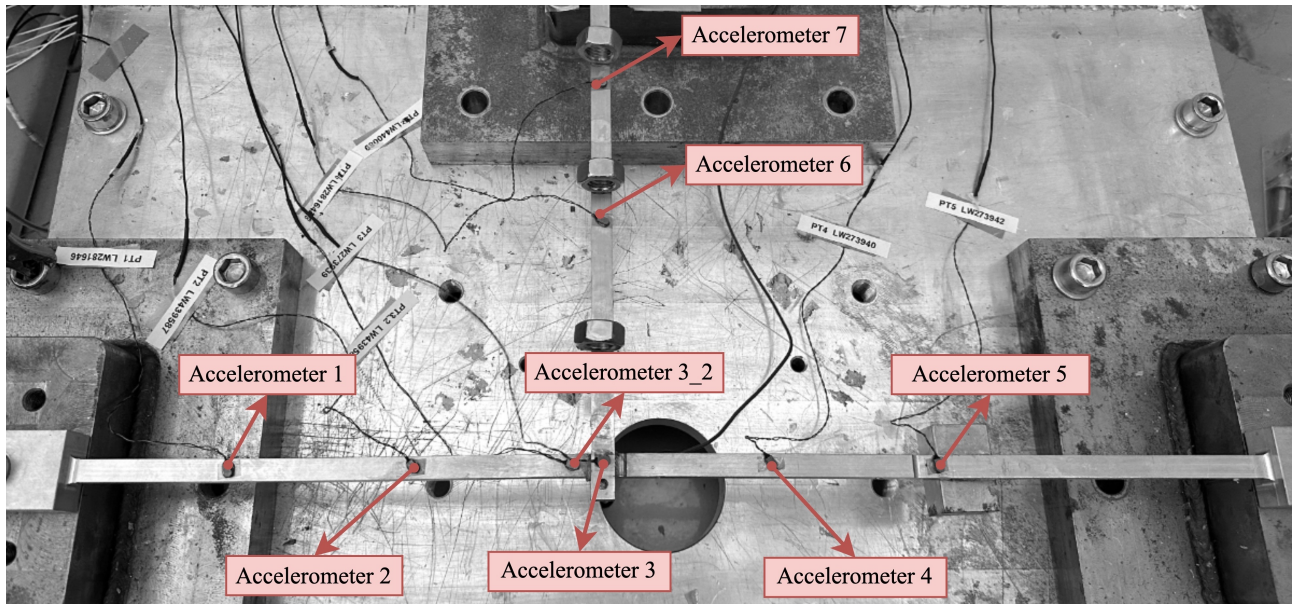


Fig. 33: Accelerometer locations on the test structure

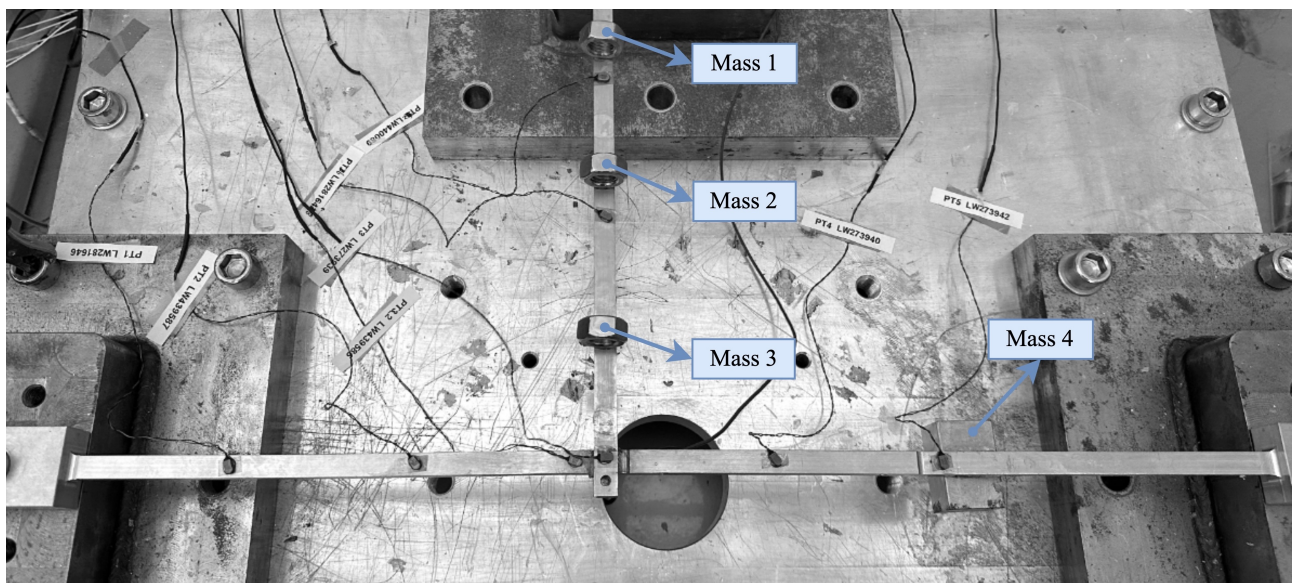


Fig. 34: Locations of the added masses on the coupled structure

5.1 Validation Approach for Decoupling Using Quasi-Linear FRFs

The two substructures A and B—a cantilevered linear beam and a clamped-clamped nonlinear beam—are joined using a metal-reinforced adhesive (LOCTITE 3463). Since this is not a standard mechanical joint, separating the beams requires significant force, which can alter their connection to the ground and cause the system to settle into a different "stick" state, i.e., into a state where there is no relative motion between the contact surfaces, and the interface behaves elastically. Even in bolted or otherwise detachable joints, physically decoupling the substructures often leads to changes in joint conditions, such as preload loss or altered contact behavior.

Validating a decoupling method typically requires measurements on both the coupled and decoupled configurations, and comparing the decoupling results against the measurements of the physically decoupled configuration. However, physically separating the substructures inevitably changes the original joint and boundary conditions and, consequently, dynamics of the substructures. Thus, it is not possible to keep the dynamics of the subsystems unchanged when they are physically detached.

On the other hand, performing measurements on the nonlinear substructure B before assembling it with substructure A does not provide a reliable alternative for validation. In RCT, the quasi-linear FRFs are obtained through stepped-sine tests performed at progressively increasing, controlled displacement amplitudes with the measurement repeated at each amplitude level. As the displacement amplitude is increased to cover the range required to characterize the nonlinearity, the boundary conditions of the structure can gradually change. In particular, at higher amplitudes, the bolted end connections do not remain perfectly invariant, and the contact/stick state may shift slightly between successive amplitude levels, especially at the highest levels. Consequently, a meaningful comparison would require the identical sequence of response-controlled measurements to be performed both on the isolated substructure B and on the coupled A–B assembly, while maintaining identical boundary states throughout the measurements. Such a condition cannot be achieved reliably in practice.

Therefore, the present study introduces an alternative validation approach that avoids physically detaching the substructures. Moreover, this approach keeps the joint conditions unchanged at a given response level, allowing for a more reliable evaluation of the decoupling method. The proposed procedure is illustrated in Fig. 35.

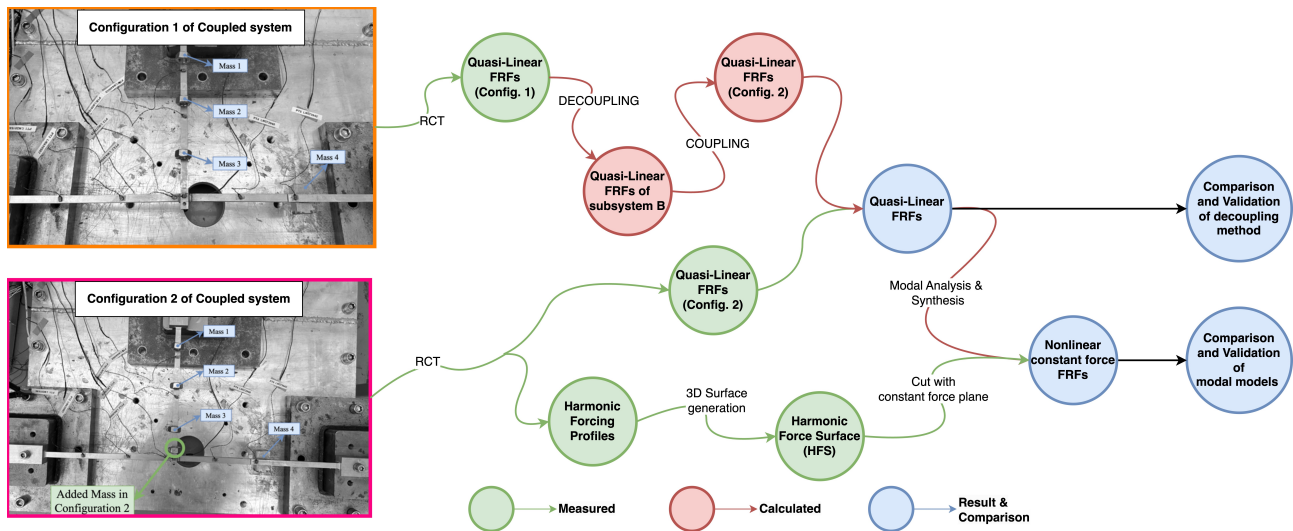


Fig. 35: Validation flowchart for the decoupling using RCT

The coupled system is composed of two substructures, which are shown and labeled in Fig. 32. In the experiments, two different configurations of substructure A are used. Configuration 1 is without the additional mass, and configuration 2 is with the additional mass on Substructure A, as shown in Fig. 35 (left part). The tests are conducted up to the force limit imposed by the power amplifier–shaker system, which corresponds to a 0.70 mm constant displacement amplitude. Therefore, the RCT measurements are conducted at constant harmonic displacement amplitudes of 0.10 mm, 0.20 mm, . . . , 0.70 mm at accelerometer location 3. The testing procedure is as follows,

1. Conduct RCT on configuration 1 (configuration without additional mass)
2. Add mass
3. Conduct RCT on configuration 2 (configuration with additional mass)
4. Remove mass
5. Repeat 1-4 for the next displacement level

As explained before, to eliminate the repeatability problems, the decoupling process needs to be validated without physically decoupling substructure A and B, which means the receptance of the linear cantilever beam (substructure A) cannot be obtained in isolation. This challenge is overcome by utilizing the finite element-based receptances of the linear cantilever beam for configuration 1 to avoid introducing additional repeatability uncertainties into the experimental validation. Subsequently, the quasi-linear FRFs of the coupled system in configuration 1 are measured. These, together with the linear FRFs of substructure A in configuration 1, are substituted into Eq. (12) to compute the quasi-linear FRFs of substructure B. In this experimental application, a standard translational interface is used and the decoupling/coupling operation is performed through the translational DoF at location 3, corresponding to the

physical connection point between the linear cantilever beam and the nonlinear beam. Therefore, the Boolean matrix in Eq. (12) contains only this measured translational interface DoF to calculate the quasi-linear FRFs of substructure B. Then, these calculated quasi-linear FRFs of substructure B are coupled with the linear FRFs of the mass-modified substructure A to obtain the estimated quasi-linear FRFs of configuration 2 of the coupled system. The quasi-linear FRFs of the mass-modified coupled system (configuration 2) are also obtained from direct RCT measurements of the modified coupled system. The measured and estimated quasi-linear FRFs of configuration 2 of the coupled system are used to both validate the proposed nonlinear decoupling approach and derive a modal model by identifying modal parameters from these FRFs. The modal model is then employed to synthesize the constant-force nonlinear FRFs of the coupled system in configuration 2. The synthesized responses are compared with the corresponding constant-force FRFs obtained using the Harmonic Forcing Surface (HFS) [32] concept to assess the validity of the approach further, focusing on the accuracy of the frequency responses predicted by using the decoupled system model. The measured V-shaped forcing profiles of configuration 2 of the coupled system are used to generate the HFS. This HFS is cut by constant forcing planes to obtain the measured constant-force FRFs of configuration 2 of the coupled system, which will be used in validating the NL-FBD framework.

5.2 Finite Element Model of Substructure A

The two configurations of substructure A are modeled in a Finite Element environment using solid elements and point mass elements connected with MPCs (multi-point constraint) to the beam. A finite element model of the beam is generated in ANSYS to calculate the harmonic response of the connection point of the cantilevered beam to a unit forcing input. The material's density and elastic modulus values are assigned to be 7916.5 kg/m^3 and 200 GPa , respectively. The mesh comprises 26,229 elements, of which nearly 95% are HEX20 elements. In the FEM, the beam is fixed at the exact locations where it is tightened with two bolts. This boundary condition is labeled with “E” in Fig. 36. Also, compression-only support is added to the sides of the beam since the lateral modes are not of interest and are neglected in the tests. Finally, point mass elements of 31 grams are attached to the points shown in Fig. 36 with letters “A”, “B”, and “C”.

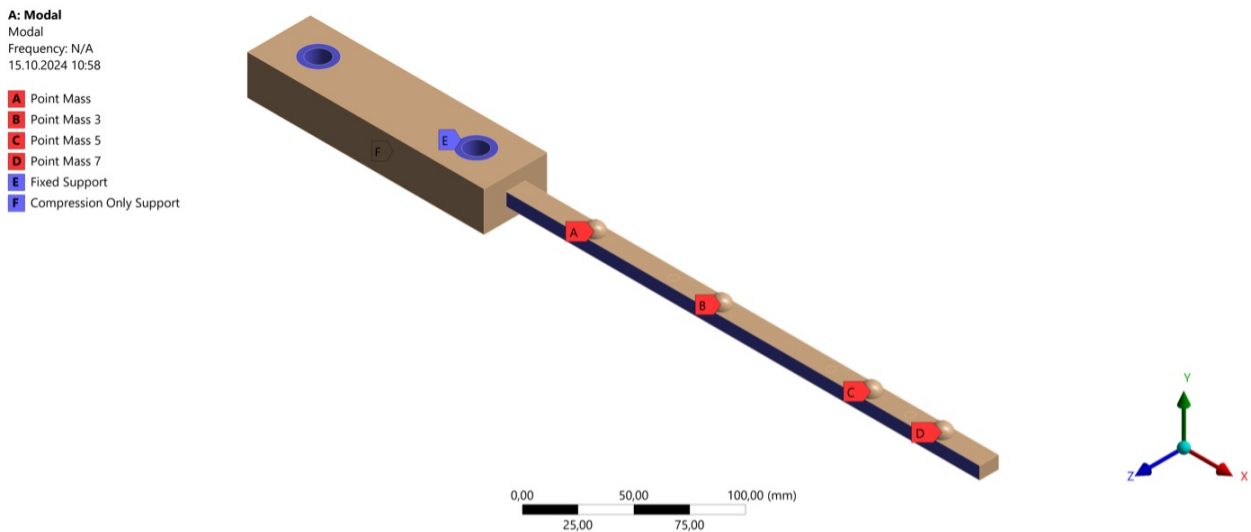


Fig. 36: Boundary conditions and added point masses of the linear cantilever beam

The first mode of substructure A is given for both configurations in Fig. 37. The mass addition shown with “D” in Fig. 36 shifts the beam’s natural frequency from around 65 Hz (Configuration 1) to 57 Hz (Configuration 2).

The FE model of substructure A is used just to obtain the receptance of the known linear substructure at the connection DoF required in the decoupling and coupling calculations. The frequency range of interest corresponds to an off-resonance region of substructure A; therefore, the calculated FRFs are not highly sensitive to small variations in damping. A damping ratio of 1% is assigned to the FE model, and the resulting FRFs are used in the decoupling and coupling calculations.

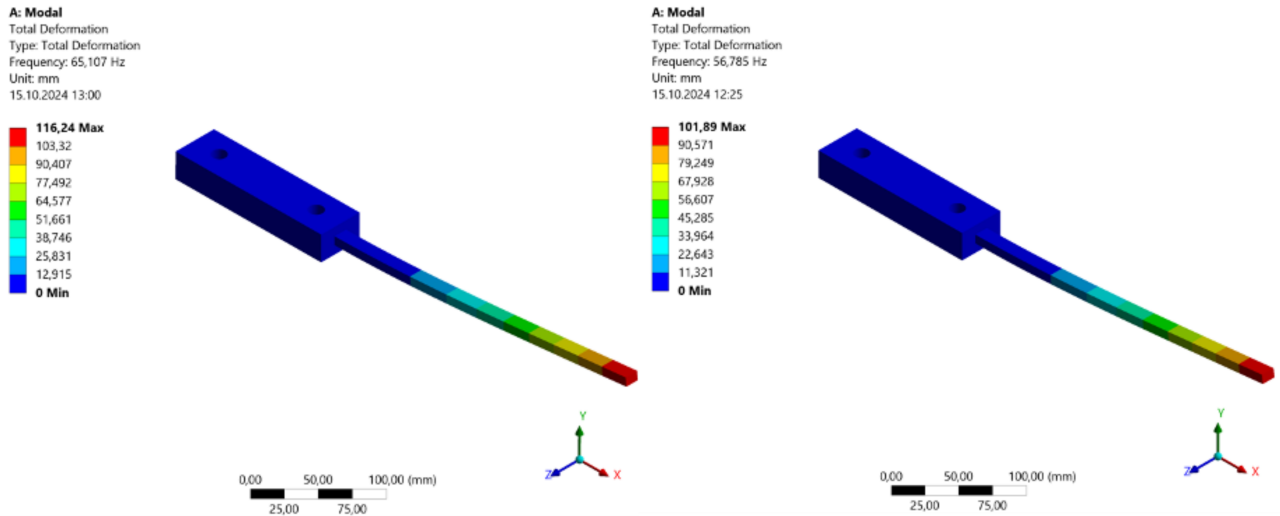


Fig. 37: The first mode of configuration 1 (left) and configuration 2 (right) of the cantilevered linear beam

5.3 Validation of the NL-FBD

The approach described in subsection 5.1 is used to validate the results. First, the quasi-linear FRFs of the coupled system's first configuration are measured (Fig. 38). These quasi-linear FRFs are then used with the FRF of substructure A in Eq. (12) to calculate the quasi-linear FRFs of substructure B (Fig. 39). It is observed that the resonance peaks become slightly wider at larger response levels in Fig. 39, indicating an increase in the identified equivalent damping. In the present framework, these effects are captured through the response-dependent modal damping identified from the quasi-linear FRFs.

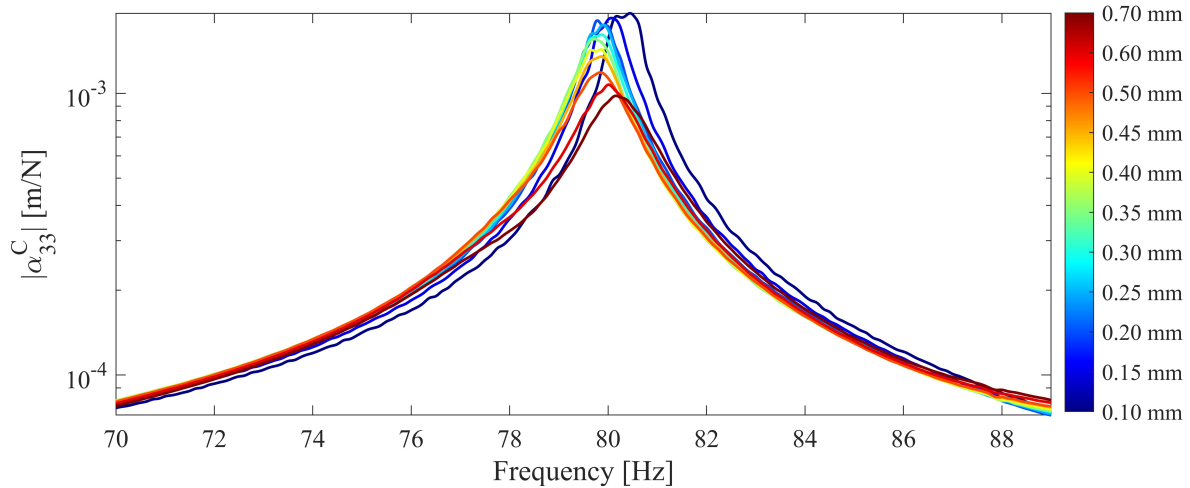


Fig. 38: Quasi-linear FRFs of configuration 1 of the coupled system measured with RCT

As described in the upper branch of Fig. 35, the calculated quasi-linear FRFs of substructure B are coupled with the modified substructure A, using the coupling component of NL-FBS framework. The resulting quasi-linear FRFs of configuration 2 of the coupled system are shown in Fig. 40 alongside the directly measured quasi-linear FRFs. As can be seen, the estimated and measured quasi-linear FRFs of configuration 2 match with high accuracy. For clarity only three displacement levels are shown in the figure. FRAC numbers are also calculated to compare the coupled system's estimated and measured quasi-linear FRFs in configuration 2. As presented in Fig. 41, the FRAC numbers are above 0.9 at every response level, and except for the lowest constant displacement amplitude, the FRAC numbers are very close to unity. This close match between the estimated and measured quasi-linear FRFs shows the accuracy of the nonlinear decoupling.

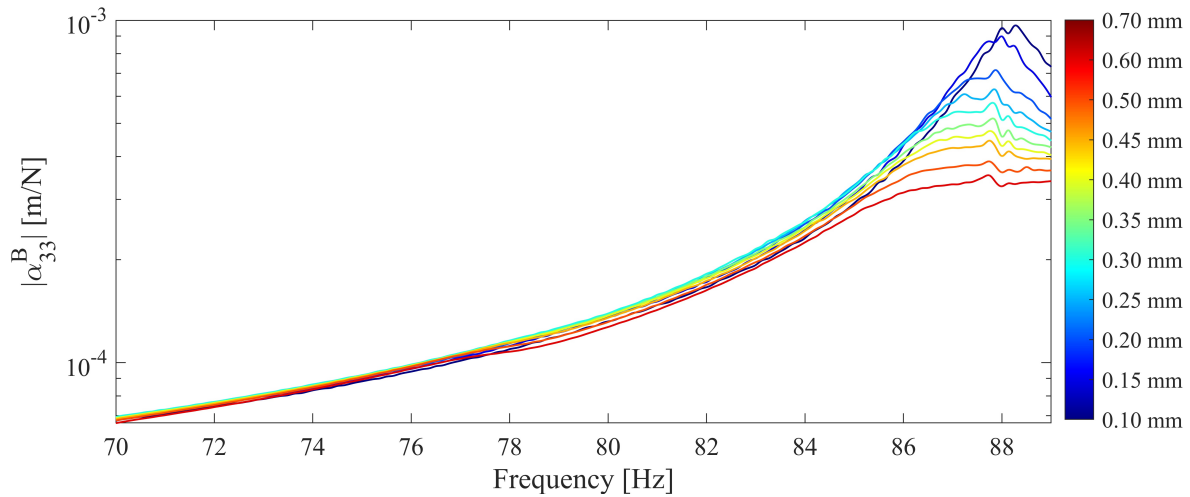


Fig. 39: Quasi-linear FRFs of substructure B calculated using the suggested decoupling method

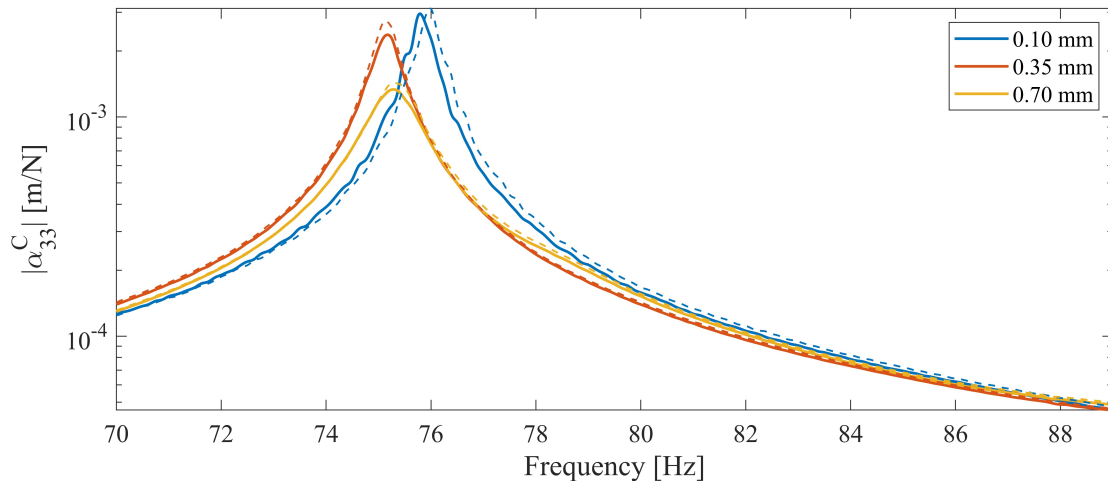


Fig. 40: Measured (solid line) and estimated (dashed line) quasi-linear FRFs of configuration 2 of the coupled system

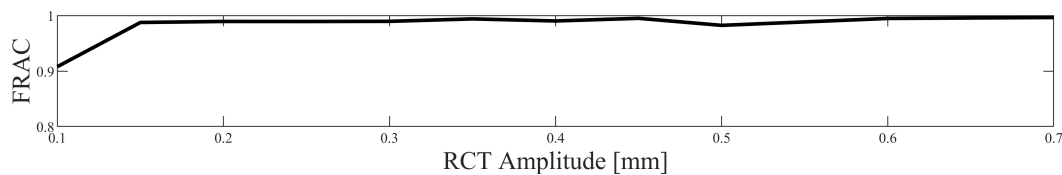


Fig. 41: FRAC numbers between the estimated and measured quasi-linear FRFs of configuration 2 of the coupled system

For further validation, first, the estimated quasi-linear FRFs given in Fig. 40 are used to identify the nonlinear modal parameters of the coupled system in configuration 2 employing peak-picking method. Then these nonlinear modal parameters are compared with those obtained from measured ones, in Fig. 42 and Fig. 43. The nonlinear modal parameters match better in terms of natural frequency and modal constant, whereas the damping ratio has a bias throughout the constant response levels. The spline curves (black line) fitted to the natural frequency and damping ratio capture the overall trends well. In contrast, the spline fitted to the modal constants does not follow the identified points closely, as the data points are scattered over a relatively narrow range with local irregularities rather than forming a smooth monotonic variation. This irregular distribution causes the fitted spline curve to deviate slightly from the individual identified points.

Furthermore, these nonlinear modal parameters are used to synthesize the constant-force FRFs of the coupled

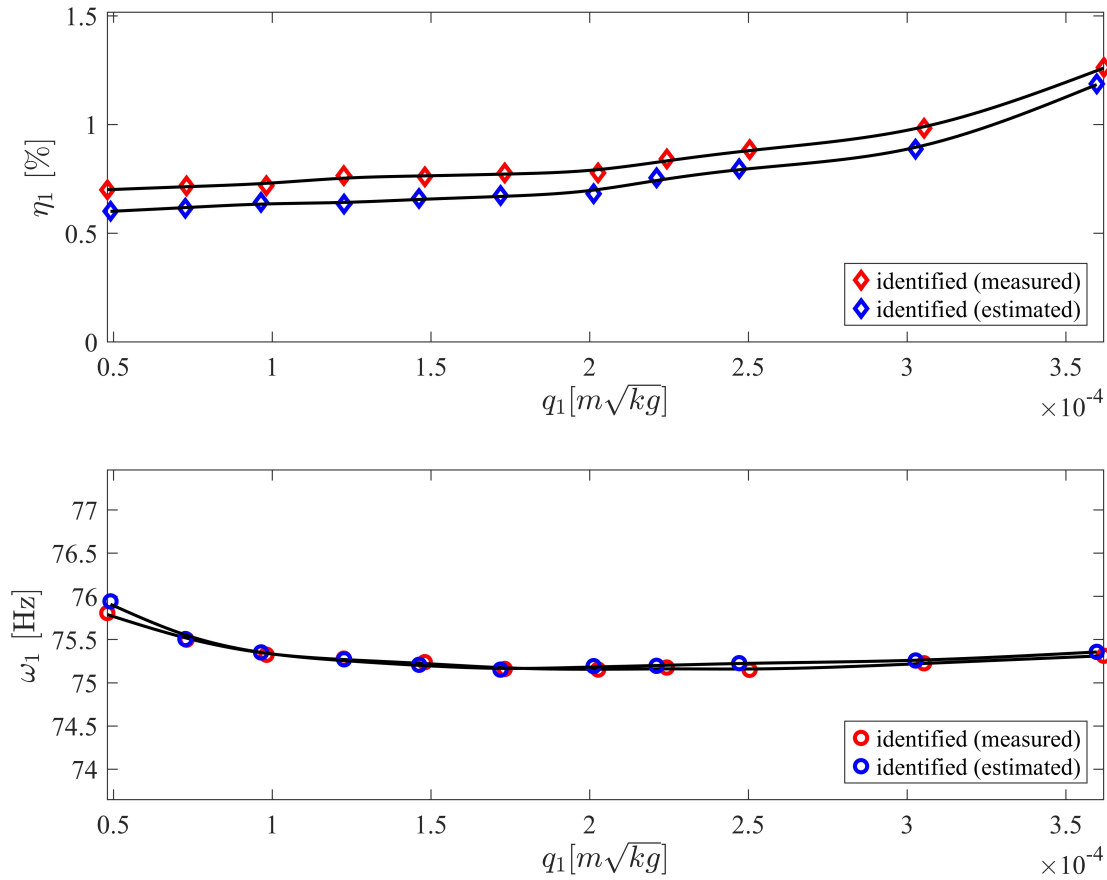


Fig. 42: Variation of natural frequency and damping ratio from the measured and estimated quasi-linear FRFs of the coupled system in configuration 2 with response level of connection DoF

system in configuration 2. The constant-force FRFs are also measured to assess the accuracy of the modal models derived from the quasi-linear FRFs, as well as their impact on the nonlinear response prediction, thereby validating the NL-FBD framework.

The Harmonic Forcing Surface (HFS) [32] is employed to obtain the measured constant-force FRFs. As each quasi-linear FRFs corresponds to a constant displacement amplitude, to keep the amplitude of the coupled system constant in RCT, the forcing amplitude is to be changed as the frequency changes. The forcing spectra obtained in the test are used to construct HFS shown in Fig. 44. Cutting the HFS with constant forcing planes, as shown in Fig. 44 as an example for 0.30 N, results in the measured nonlinear constant-force FRFs including unstable branches, if there are any.

The measured constant-force responses are compared with those obtained using the modal models from the measured and estimated quasi-linear FRFs, as shown in Fig. 45. An excellent agreement is obtained between the directly measured response of the nonlinear-linear assembly and those obtained by employing NL-FBD and NL-FBC together, for all three forcing levels. This demonstrates the accuracy of the proposed framework in the experimental application.

To assess the sensitivity of the experimental results to noise in the known substructure, noise at levels of 0.5% and 1% was added to the real and imaginary parts of the FRFs of substructure A, and the complete decoupling–coupling and synthesis procedure was repeated. The resulting constant-force responses are shown in Fig. 46 for the 1% noise level.

The error reported below is a sensitivity measure rather than an error relative to a physical measurement. Specifically, it is defined as the relative difference between the noise-free and noisy estimated constant-force syntheses of configuration 2. The sensitivity stays small: the response-peak frequency shifts by less than about

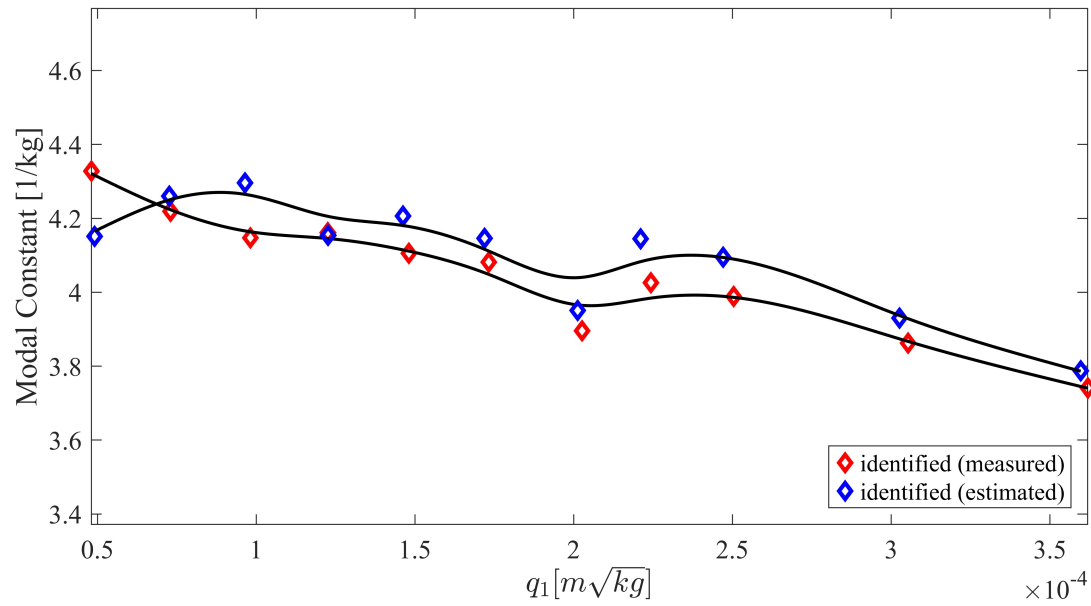


Fig. 43: Variation of modal constants from the measured and estimated quasi-linear FRFs of the coupled system in configuration 2 with response level of connection DoF

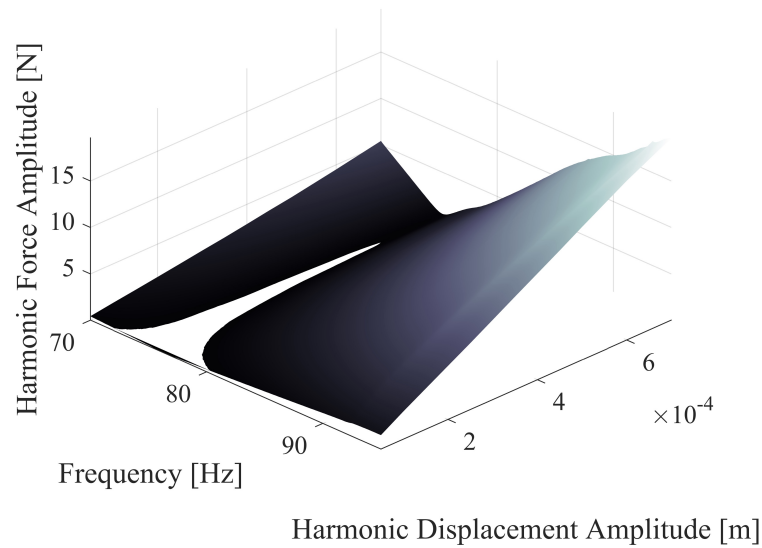


Fig. 44: Harmonic Forcing Surface of the coupled system in configuration 2

0.08% at every noise and forcing level, while the peak-amplitude error is largest at the lowest forcing level (about 5.6% at 0.05 N), falling to about 0.4% at the highest forcing level. For 0.5% noise, the peak-amplitude error remains below 1.1% across all forcing levels.

6 Conclusions

This paper introduces the Nonlinear Frequency-Based Decoupling (NL-FBD) framework as a component of a broader framework, the Nonlinear Frequency-Based Substructuring (NL-FBS). The NL-FBD extends the previously developed FRF Decoupling Method for Nonlinear Systems (FDM-NS) [28] for nonlinear decoupling, which was originally limited to systems with localized nonlinearities represented by nonlinear elements between only two DoFs, as it was based on response-controlled testing method suggested for systems with local nonlinearity [27]. The proposed framework

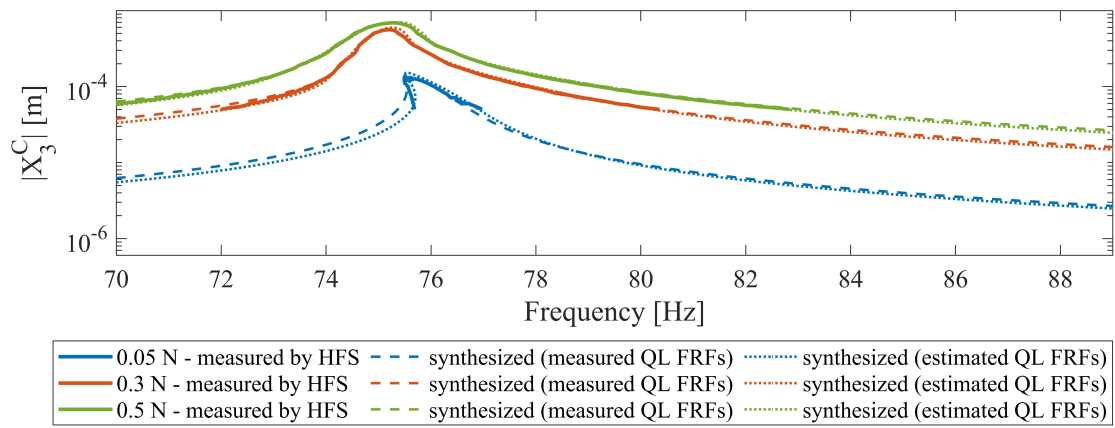


Fig. 45: Comparison of constant-force displacement of the coupled system in configuration 2: measured (solid), synthesized using the modal model of measured quasi-linear FRFs (dashed), and synthesized using the modal model of estimated quasi-linear FRFs (dotted)

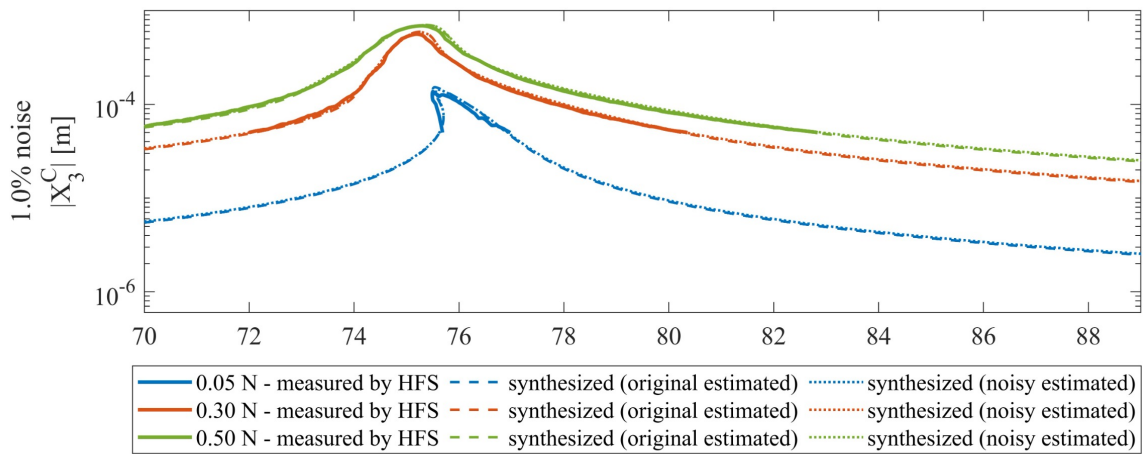


Fig. 46: Comparison of constant-force displacement of the coupled system in configuration 2: measured (solid), synthesized using the modal model of estimated quasi-linear FRFs (dashed), and synthesized using the modal model of estimated noisy quasi-linear FRFs (dotted)

generalizes the approach to systems with *multiple, arbitrarily located, or even distributed nonlinearities*; it enables *decoupling at multiple DoFs and, in separate investigated cases, allows nonlinearities to exist in both substructures*. NL-FBS employs measured quasi-linear FRFs of the assembled system together with either measured or analytically obtained FRFs of the known substructure. When the known substructure is also nonlinear, its quasi-linear FRFs must likewise be obtained experimentally. The key assumptions in the application of NL-FBD framework are well-separated modes, negligible contribution from higher harmonics, and the absence of internal resonances.

The coupling component of NL-FBS is also briefly outlined theoretically, and applied in experimental validation, demonstrating the integrated use of the complete framework. The theoretical foundations of both the nonlinear decoupling and coupling approaches presented in this study are built upon i) the measurement of quasi-linear FRFs of nonlinear systems using response-controlled testing RCT, ii) frequency-based substructuring techniques originally developed for linear systems, and iii) obtaining and using modal model of nonlinear systems.

The limitations and accuracy of NL-FBD are investigated using both discrete and continuous system examples, coupled at single and multiple DoFs, and is also validated through an experimental setup that consists of a nonlinear beam connected to a linear beam. The important conclusions obtained in this study are summarized below.

- NL-FBD is sensitive to measurement errors, which is a common problem in all frequency-based decoupling methods. The error becomes more pronounced when the resonance region of the unknown substructure coincides with the anti-resonance region of the coupled system. As it is practically impossible to perform RCT at anti-resonances, this limitation has no practical impact.
- A coupled nonlinear system with a single DoF connection is a special case and the proposed decoupling approach is applicable to a broad range of scenarios, unless there is an anti-resonance region of the coupled system within the frequency range of the unknown substructure's resonance.
- Having multiple connection points leads to increasing decoupling errors as the nonlinear mode shape of the unknown nonlinear subsystem may not be similar to that in the coupled configuration. It is also observed that the level of nonlinearity affects the accuracy of the decoupling process when there are multiple connection points.
- The key condition for the successful application of the NL-FBD framework is that the nonlinearity matrices in coupled and decoupled configurations remain similar. This, in turn, requires that the mode shapes of the nonlinear parts of the substructures remain approximately similar in both coupled and decoupled configurations around the mode of interest.
- In the application of the NL-FBD to a nonlinear continuous beam, it is observed that when only the translational DoF on the connection interface is used very high errors are obtained; however, the error is eliminated by employing the mixed-interface method in linear decoupling equations. This represents a novel application of the mixed-interface approach in nonlinear decoupling, reducing very high errors both in frequency of peak amplitude and peak amplitude value below 1%.
- Regarding computational efficiency, the burden of the proposed NL-FBD framework is primarily concentrated in the data acquisition phase (performing Response-Controlled Testing). Once the quasi-linear FRFs are obtained, the subsequent decoupling process relies solely on linear matrix algebra and standard linear modal identification. Furthermore, the use of the analytical synthesis formulations [36, 37] given in Eqs. (15) and (16) effectively eliminates the need for iterative numerical solutions typically required by nonlinear modal models. Consequently, the computational processing time of the NL-FBD framework is reduced to negligible levels compared to standard iterative identification or model updating techniques.
- Care should be taken in selecting the control DoF in applying RCT. As the anti-resonances of a vibratory system depend on the choice of the DoFs, when a structure is excited from a connection DoF in an RCT setup, the control DoF can be selected from among several convenient options. However, depending on this choice, it is possible for a resonance frequency to lie very close to an anti-resonance frequency — even when the modes are well separated. In such cases, RCT may fail to produce quasi-linear FRFs, as the usable frequency range around the resonance becomes too narrow due to the proximity of the anti-resonance. This conclusion has very important implications not only in the application of the NL-FBD, but also it brings an additional condition in the general use of RCT.
- The experimental study validated the proposed nonlinear substructuring framework, demonstrating excellent agreement between the directly measured frequency responses of a nonlinear-linear assembly and those obtained using both the NL-FBD and the NL-FBC. It is also observed that the accuracy increases at higher forcing levels.

Consequently, the experimental nonlinear decoupling framework presented in this paper can effectively decouple nonlinear substructures containing multiple nonlinearities, connected at several DoFs, and having nonlinearities in both substructures, subject to the limitations discussed. The strong performance of NL-FBD in both real experimental application and the continuous system case study is highly promising for practical implementation. In contrast, the less satisfactory results observed in the discrete system case study with multiple connection-DoFs may stem from the arbitrary selection of system parameters, which may not reflect realistic or physically meaningful configurations. The combined case of multiple connection-DoFs with nonlinearities in both substructures remains an important direction for future work, since the Nonlinearity Matrices of both substructures may vary depending on the selected response-controlled DoF.

Authors' Contributions

F.K.Ç., conceptualization, methodology, software, validation, formal analysis, investigation, writing - original draft. M.B.Ö., supervision, conceptualization, methodology, writing - review & editing. H.N.Ö., supervision, conceptualization, methodology, writing - review & editing.

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