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Normal form transformations for structural dynamics: An introduction for linear and nonlinear systems.

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Normal form transformations for structural dynamics: An introduction for linear and nonlinear systems.

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Abstract

The aim of this paper is to provide an introduction to using normal form transformations for linear and nonlinear structural dynamics examples. Starting with linear single-degree-of-freedom systems, a series of examples are presented that eventually lead to the analysis of a system of two coupled nonlinear oscillators. A key part of normal form transformations are the associated coordinate transformations. This review includes topics such as Jordan normal form and modal transformations for linear systems, while for nonlinear systems, near-identity transformations are discussed in detail. For nonlinear oscillators, the classical methods of Poincaré and Birkhoff are covered, alongside more recent approaches to normal form transformations. Other important topics such as nonlinear resonance, bifurcations, frequency detuning and the inclusion of damping are demonstrated using examples. Furthermore, the connection between normal form transformations and Lie series is described for both first and second-order differential equations. The use of normal form transformations to compute backbone curves is described along with an explanation of the relationship to nonlinear normal modes. Lastly, conclusions and possible future directions for research are given.

Keywords: Normal form; coordinate transformation; nonlinear oscillator; nonlinear resonance; frequency detuning

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1 Introduction

Structural dynamics involves modelling the time dependent behaviour of mechanical and structural systems, using differential equations. To give this process meaning, a relevant set of coordinates needs to be defined. If the coordinates are chosen such that the differential equation can be written in its *simplest* (or some other *most natural*) form, then this is an example of a normal form. The process of finding a normal form is undertaken by coordinate transformations, and hence the phrase *normal form transformation* is used to indicate that the process is intended to result in a normal form for the system under consideration.

So why this might be useful for a structural dynamicist? There are two main reasons. Firstly, differential equations can be difficult to solve (exactly or approximately, depending on the context) and transforming to a simpler form typically makes it easier to find potential solutions. Secondly, the transformation process reveals important information about the resonant behaviour of the system. For example, structural dynamicists have used these type of methods extensively in modal analysis [45, 64], and more recently they have been applied to nonlinear normal modes [172, 104, 129], and the understanding of nonlinear coupled oscillators [4, 124, 85, 183]. They are also used in nonlinear dynamics, particularly in the area of bifurcation analysis [55, 100], and more widely in mathematics and physics [120, 22, 116]. We start by considering some simple examples.

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1.1 Motivating examples

1.1.1 Example 1: Unforced, undamped linear oscillator

A common approach in structural dynamics is to use the Newtonian framework to model structures with point masses. In this situation the positions, velocities and accelerations of masses are the natural starting point for defining the relevant coordinates. Consider, for example, the point mass system shown in Fig. 1 (a). The ordinary differential equation representing the motion of this system, obtained by applying Newton's second law[1] is given by

$$m\ddot{x}+kx=0, \quad \rightsquigarrow \quad \ddot{x}+\omega_n^2 x=0, \tag{1}$$

where x is the vertical displacement of the mass, m, and k is the spring constant (also called the stiffness), from which $\omega_n = \sqrt{k/m}$ is the undamped natural frequency. In Eq. (1) a dot indicates differentiation with respect to time t such that $\ddot{x}$ represents acceleration of the mass. Both acceleration and displacement are assumed to act in a single, linear, coordinate direction called a *degree-of-freedom*, and as there is only one, this is a *single degree-of-freedom system*.

Structural dynamicists tend to refer to Eq. (1) by what it doesn't include — the *unforced, undamped, linear oscillator*, where linear in this context means the absence of nonlinear terms of the form $x^2, x^3, x\dot{x}, \sin(x)$ etc. in the equation. But how do we know if Eq. (1) is a normal form? It would seem that it is already quite 'simple', but can it be simplified further?

One possibility is to reduce the order of the differentiation from second-order (meaning $\mathrm{d}^2/\mathrm{d}t^2$) to first-order (meaning $\mathrm{d}/\mathrm{d}t$). To do this we define the following new coordinates $x_1 = x$ and $x_2 = \dot{x}$ such that we can write an equivalent first-order matrix differential equation for Eq. (1) which is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2}$$

which can be written in matrix notation as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \tag{3}$$

where $\boldsymbol{x} = \{x_1, x_2\}^T$ is referred to as the state vector. Now we have an equation that is 'simpler' in the sense that it is first-order in terms of differentiation, but arguably more complicated in that A is a non-diagonal matrix, meaning there is *coupling* between the equations in Eq. (3). However, if we now apply a coordinate transformation $\boldsymbol{x} \to \boldsymbol{q}$ to 'diagonalise' A, then (assuming no other complicating terms arise in the transformation process) we would have a system of two *uncoupled* equations.

This can be done by finding the eigenvalues and eigenvectors[2] of A which for this example are $\lambda_1 = \mathbf{i}\omega_n$, $\lambda_2 = -\mathbf{i}\omega_n$, $\boldsymbol{v}_1 = \{1, \lambda_1\}^T$, and $\boldsymbol{v}_2 = \{1, \lambda_2\}^T$ respectively. These can be used to define the following matrices

$$\Lambda = \begin{bmatrix} \mathbf{i}\omega_n & 0 \\ 0 & -\mathbf{i}\omega_n \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 1 \\ \mathbf{i}\omega_n & -\mathbf{i}\omega_n \end{bmatrix}, \quad \Phi^{-1} = \frac{1}{-2\mathbf{i}\omega_n} \begin{bmatrix} -\mathbf{i}\omega_n & -1 \\ -\mathbf{i}\omega_n & 1 \end{bmatrix}. \tag{4}$$

Then by making the substitution $\boldsymbol{x} = \Phi\boldsymbol{q}$ into Eq. (3) and multiplying by Φ^{-1} we obtain

$$\begin{bmatrix} \dot{q}_p \\ \dot{q}_m \end{bmatrix} = \begin{bmatrix} \mathbf{i}\omega_n & 0 \\ 0 & -\mathbf{i}\omega_n \end{bmatrix} \begin{bmatrix} q_p \\ q_m \end{bmatrix}, \quad \text{or} \quad \dot{\boldsymbol{q}} = \Lambda\boldsymbol{q} \quad \text{where} \quad \boldsymbol{q} = \begin{bmatrix} q_p \\ q_m \end{bmatrix}. \tag{5}$$

These equations in the new coordinates, q_p and q_m, can now be solved via direct integration with initial conditions $q_p(t_0) = q_{p0}$, (and similarly for q_m) such that taking the first row of the first equation in Eq. (5), we can rearrange and integrate both sides to give

$$\int_{q_{p0}}^{q_p} \frac{\mathrm{d}q_p}{q_p} = \mathbf{i}\omega_n \int_{t_0}^{t} \mathrm{d}t \quad \rightsquigarrow \quad \ln(q_p) - \ln(q_{p0}) = \mathbf{i}\omega_n(t-t_0) \quad \rightsquigarrow \quad \ln\left(\frac{q_p}{q_{p0}}\right) = \mathbf{i}\omega_n(t-t_0), \tag{6}$$

[1] An early classical treatment of this example can be found in the Theory of Sound by Lord Rayleigh [145, 146], which sets out a theoretical approach to vibrations that was subsequently used as a template for many others that followed (for example; Den Hartog [32], Timoshenko [167], Bishop and Johnson [12], Warburton [185], Newland [130], Thompson [166], Géradin and Rixen [48]), and is still widely used today (e.g. Ewins [45], Meirovitch [115], Hagedorn and DasGupta [58], Inman [76], Rao and Yap [143], to list just a few).

[2] Note that the eigenvectors are not unique, and a choice needs to be made regarding the relative scaling of these vectors. The choice used here is to scale the leading value of the vector to be one.

so finally we can obtain

$$\frac{q_p}{q_{p0}} = \mathrm{e}^{\mathbf{i}\omega_n(t-t_0)} \quad \rightsquigarrow \quad q_p = \mathrm{e}^{\mathbf{i}\omega_n(t-t_0)} q_{p0}, \tag{7}$$

and if $t_0 = 0$ is assumed[3] (and using a similar approach for q_m) we have the solutions

$$q_p = \mathrm{e}^{\mathbf{i}\omega_n t} q_{p0}, \quad \text{and} \quad q_m = \mathrm{e}^{-\mathbf{i}\omega_n t} q_{m0}. \tag{8}$$

These solutions can also be expressed as

$$\boldsymbol{q} = \mathrm{e}^{\Lambda t} \boldsymbol{q}_0, \tag{9}$$

which, by direct substitution, is the solution to Eq. (5).

In general, the initial values in the $\boldsymbol{q}_0 = \{q_{p0},\, q_{m0}\}^T$ vector are complex, which allows a phase lag to be incorporated into the solution. For simplicity, consider the case where the initial velocity $\dot{x}(0) = 0$, such that the phase lag can also be assumed to be zero, meaning that $q_{p0} = q_{m0} = Q/2$, where $Q/2$ is a real constant[4]. Therefore, using the $\boldsymbol{x} = \Phi\mathbf{q}$ relationship (and Eq. (4)) to regain the expressions in terms of x_1 and x_2 we find

$$\begin{aligned} x = x_1 &= Q\frac{(\mathrm{e}^{\mathbf{i}\omega_n t} + \mathrm{e}^{-\mathbf{i}\omega_n t})}{2} = x(0)\cos(\omega_n t), \quad \text{and} \\ \dot{x} = x_2 &= -Q\omega_n \frac{(\mathrm{e}^{\mathbf{i}\omega_n t} - \mathrm{e}^{-\mathbf{i}\omega_n t})}{2\mathbf{i}} = -\omega_n x(0)\sin(\omega_n t), \end{aligned} \tag{10}$$

where $x(0) = x_0 = Q$ and $\dot{x}(0) = \dot{x}_0 = 0$. So by two steps, (i) writing the second-order system as an equivalent first-order system, and then (ii) using an eigenvalue coordinate transformation to diagonalise the A matrix, we have found a simplified version of the system that can be solved in a straightforward way using direct integration. An example of this solution is shown in Fig. 1 (b).

Solutions of the type in Eq. (8) are called the *base* solutions, and we note that other combinations of the base solutions, such as those in Eq. (10), are also solutions to the original equation. These type of base solutions will play an important role later on for nonlinear oscillators that have 'underlying' linear oscillator(s) — meaning if the nonlinear coefficients are all set to zero, the system reduces to a (set of) linear oscillator(s).

1.1.2 Solving the second-order equation directly

Finally for this example, we note that the same solutions can be found much more directly for the linear oscillator using an operator representation. To see this notice that Eq. (1) can be written as

$$\ddot{x} + \omega_n^2 x = 0 \equiv \left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{i}\omega_n\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{i}\omega_n\right) x = 0, \tag{11}$$

where the terms in brackets are to be thought of as *operators* on x. For Eq. (11) to be satisfied we have that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{i}\omega_n\right) x_p = 0 \qquad \text{or} \qquad \left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{i}\omega_n\right) x_m = 0, \tag{12}$$

where the subscripts p and m have been added to distinguish between the two different solutions. We have just derived the solution to these type of scalar first order differential equations with a solution given by Eq. (8), such that there are two solutions

$$x_p = \mathrm{e}^{\mathbf{i}\omega_n t} x_{p0} \qquad \text{and} \qquad x_m = \mathrm{e}^{-\mathbf{i}\omega_n t} x_{m0}, \tag{13}$$

assuming as before that $t_0 = 0$. In the general case the constants x_{p0} and x_{m0} are a complex conjugate pair

$$x_{p0} = \frac{X}{2}\mathrm{e}^{\mathbf{i}\phi} \qquad \text{and} \qquad x_{m0} = \frac{X}{2}\mathrm{e}^{-\mathbf{i}\phi}, \tag{14}$$

[3] Note that many texts implicitly assume that $t_0 = 0$, but for some nonlinear and piecewise linear systems t_0 must be retained through the analysis.

[4] Note that we choose $Q/2$ rather than Q for convenience in forming the sine and cosine solutions.

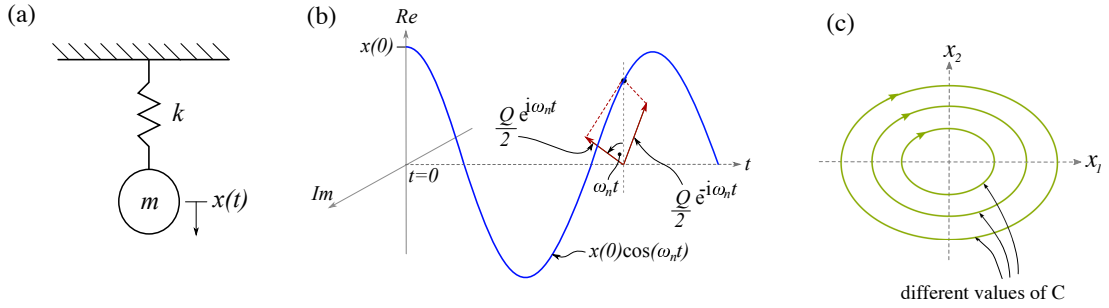

Fig. 1: (a) A schematic representation of a single degree-of-freedom point mass oscillator consisting of mass m kg and spring constant k N/m. (b) The cosine solution formed from the projection onto the real axis of the two complex vectors of the 'base' solutions. Note that in this case, the phase ϕ can only be 0 or $n\pi$ for $n = 1,2,3...$ so we show the case for $\phi = 0$. (c) Solutions in state space (also called the *phase plane* for this type of system). Green arrows indicate time evolution of the trajectories. Expressions for the phase plane solutions for this system can be obtained by taking the ratio of the two velocity expressions in Eq. (2) to give $\frac{\dot{x}_2}{\dot{x}_1} = \frac{\mathrm{d}x_2/\mathrm{d}t}{\mathrm{d}x_1/\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{-\omega_n^2 x_1}{x_2}$. This can be rearranged to give $-\omega_n^2 x_1 \mathrm{d}x_1 = x_2 \mathrm{d}x_2$, which can then be integrated directly and, by incorporating the 1/2 factors into the constant of integration, the following expression is obtained $-\omega_n^2 x_1^2 = x_2^2 - C \rightsquigarrow x_2^2 + \omega_n^2 x_1^2 = C$. This is the equation of an ellipse, which has a size which depends on the constant C, and C in turn depends of the initial conditions, $\boldsymbol{x}_0$. As the constant C is arbitrary, an infinite family of periodic solutions exist which are solutions to the differential equation. In the case where $\omega_n = 1$ the equation governing the motion in the phase plane becomes $x_2^2 + x_1^2 = C$, and in this case instead of ellipses the orbits would be circles.

where X is a real constant representing displacement amplitude and ϕ represents the phase difference between the starting point on the sine wave solution and $t = 0$. The base solutions in Eq. (13) can be added (also using Eq. (14)) to give the general solution to Eq. (1) as

$$x = x_p + x_m = X\frac{(\mathrm{e}^{\mathbf{i}(\omega_n t+\phi)} + \mathrm{e}^{-\mathbf{i}(\omega_n t+\phi)})}{2} = X\cos(\omega_n t + \phi) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t), \tag{15}$$

where C_1 and C_2 are constants to be determined. Now differentiating Eq. (15) and using the initial conditions, we can obtain the values of C_1 and C_2 and the general solution for Eq. (1) as

$$\begin{aligned} x &= x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t), \\ \dot{x} &= -\omega_n x(0)\sin(\omega_n t) + \dot{x}(0)\cos(\omega_n t). \end{aligned} \tag{16}$$

In the case when $\dot{x}(0) = 0$ then this reduces to the same as Eq. (10). An example of the family of periodic orbits that exist in the phase plane for this type of solution (with $\dot{x}(0)$=0) is shown in Fig. 1 (c).

1.1.3 Example 2: Small angle approximation for the simple pendulum

A simple example of a nonlinear oscillator is the equation of motion governing the swing of a simple pendulum, shown in Fig. 2 (a) which has a governing equation of motion given by

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0, \tag{17}$$

where θ is the angle of the pendulum from the downward rest position, ℓ is the length of the pendulum and g is gravitational acceleration. Under certain conditions it is possible to find exact solutions using elliptic functions [132], but in many situations it is helpful to consider an approximate version of Eq. (17)[5].

[5] In 1892 Henri Poincaré published his work relating to the *three-body problem* from celestial mechanics [137, 7]. In this case, the governing differential equations were nonlinear, without any exact solution and Poincaré instead developed new ways to understand the dynamics without exact solutions. The pioneering work started by Poincaré eventually lead to the field of nonlinear dynamics (also called chaos theory, or dynamical systems theory), with a major early contribution from Birkhoff [11]. A short summary of the history of nonlinear dynamics can be found in Holmes [72]. More modern comprehensive overviews of the subject can be found, for example, in; Guckenheimer and Holmes [55], Glendinning [52], Thompson and Stewart [165], and Strogatz [162].

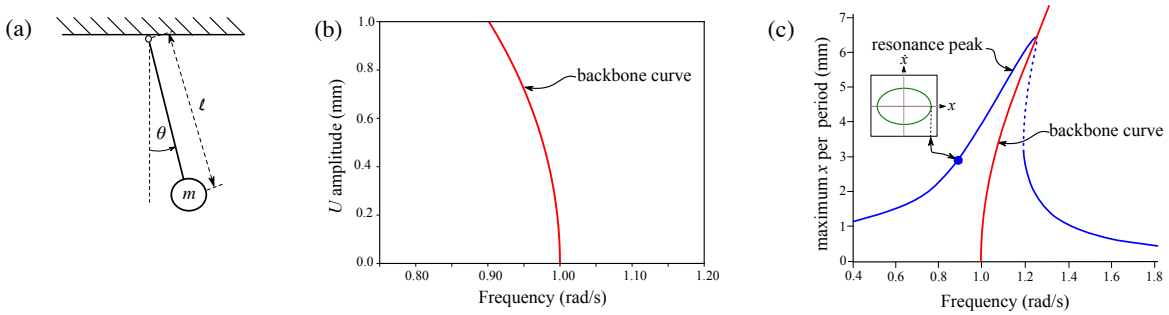

Fig. 2: (a) A simple pendulum of length ℓ m and mass m kg. (b) The conservative backbone curve obtained from Eq. (26) with parameter values $\omega_n = 1$, $\alpha = 0.25$ and $U \in [0, 1]$. Then when $U = 0$, $\omega_r = \omega_n = 1$, and as U increases, $\omega_r < \omega_n$ which corresponds to *softening* behaviour. (c) Showing the forced-damped envelope curve (blue solid and dashed lines) and conservative backbone curve (red solid line) for the Duffing oscillator $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x + \alpha x^3 = F\cos(\Omega t)$. Parameter values are ζ=0.0625, $\omega_n = 4$ rads/s, $\alpha = 0.3$ m^{-2}s^{-2} and $F = 0.016$ N/kg. Note that the blue dashed line corresponds to unstable solutions. For a detailed discussion of the relationship between conservative backbone curves and forced-damped curves, and particularly the crossing point, see Cenedese and Haller [28, 27].

One way to simplify this equation is to consider the case when the angle of swing is small $|\theta| \ll 1$. Then we can approximate $\sin\theta$ as a Taylor series so that $\sin\theta \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$, and then Eq. (17) becomes[6]

$$\ddot{\theta} + \omega_n^2\theta - \alpha\theta^3 \approx 0, \tag{18}$$

where in this case $\omega_n = \sqrt{\frac{g}{\ell}}$ and $\alpha = \frac{g}{6\ell}$. This equation is very similar to Eq. (1), except for the addition of the cubic nonlinear term $-\alpha\theta^3$. Eq. (18) is a version of the *Duffing equation*, named after Georg Duffing, which under certain conditions has exact solutions via elliptic integrals (see for example Chapter 4 in Kovacic and Brennan [96]). Despite the existence of exact solutions, the Duffing oscillator is widely used as an example for normal form analysis and many other approximate methods (as will be the case in this paper) because it is one of the 'simplest' systems to exhibit many of the key characteristics of nonlinear dynamics.

As in the previous example, it is possible to write Eq. (18) in an equivalent *first-order* form which can be achieved by defining $x_1 = \theta$ and $x_2 = \dot{\theta}$ so that[7]

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\omega_n^2 x_1 + \alpha x^3 \end{aligned} \qquad \text{or} \qquad \dot{\boldsymbol{x}} = f(\boldsymbol{x}), \qquad \text{with} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{19}$$

where $f(\cdot)$ on the right-hand side is used to denote a nonlinear function of the states, x_1 and x_2.

Before seeking an (approximate) solution of a nonlinear equation like Eq. (19), an obvious question is: *Is there a coordinate transformation that simplifies this equation?* Ideally a coordinate transformation that eliminates the nonlinear terms would lead to a linear equation, which can be solved exactly, just like the linear oscillator example described above. Although it is possible to eliminate some types of nonlinear terms (see Example 10, Section 4.1.1), for Eq. (19) eliminating the cubic nonlinear terms (as we will describe in Example 11, Section 4.2.2) is not possible. However, the question still remains, as to whether there is a simpler form of the equation, which we call the *normal form*.

As we will describe in Section 4.2.2, the normal form for Eq. (19) can be obtained by first applying a diagonalising transformation $\boldsymbol{x} \to \boldsymbol{q}$ (e.g. as we did above by using Eq. (4) to obtain Eq. (5)) and then a *near-identity* transformation $\boldsymbol{q} \to \boldsymbol{u}$ up to third order (meaning including terms in the series solution up to cubic)[8]

[6] This is based on the assumption that a truncated number of terms in the Taylor series will provide a sufficiently accurate approximation to the full solution when $|\theta| \ll 1$.

[7] If required, at this stage substituting the arc-length relationship $x = \ell\theta$ can be used to transform from angular to arc-length coordinates, and because $|\theta| \ll 1$, the arc-length will be close to the tangential linear displacement. In this case all parameters except α remain the same, and the new value is given by $\alpha = g/6\ell^3$.

[8] Note the classical method of removing nonlinear terms order-by-order in the differential equation and/or pushing them to a higher order can also be used and is described in Section 4.1.1.

which gives the normal form as [80]

$$\begin{aligned} \dot{u}_p = &\quad \mathbf{i}\omega_n u_p - \frac{3\mathbf{i}\alpha}{2\omega_r} u_p^2 u_m, \\ \dot{u}_m = &\quad -\mathbf{i}\omega_n u_m + \frac{3\mathbf{i}\alpha}{2\omega_r} u_p u_m^2, \end{aligned} \tag{20}$$

where $\boldsymbol{u} = [u_p, u_m]^T$ and ω_r is the nonlinear response frequency that is 'close' to[9] the undamped natural frequency ω_n. The associated approximate solution for displacement, x, (again up to third order) is given by

$$x = (U + \frac{3\alpha U^3}{16\omega_r^2})\cos(\omega_r t) - \frac{\alpha U^3}{32\omega_r^2}\cos(3\omega_r t) + \ldots \tag{21}$$

where dots are used to indicate that there are further terms in the series solution, and U is the amplitude of the base solutions[10] in terms of u_p and u_m, namely

$$u_p = \frac{U}{2}\mathrm{e}^{\mathbf{i}\omega_r t}, \quad \text{and} \quad u_m = \frac{U}{2}\mathrm{e}^{-\mathbf{i}\omega_r t}. \tag{22}$$

However, normal form transformations are not unique, and an alternative approach that avoids the use of complex notation, as in Eq. (20), (and is therefore a type of *real* normal form, [104]) gives the normal form equation for the Duffing oscillator as [128]

$$\ddot{u} + \omega_n^2 u - 3\alpha(u_p^2 u_m + u_p u_m^2) = 0, \tag{23}$$

or alternatively, using the fact that $u = u_p + u_m$, this type of normal form can be expressed in terms of the base-solution coordinates

$$\begin{aligned} \ddot{u}_p &+ \omega_n^2 u_p - 3\alpha u_p^2 u_m = 0, \\ \ddot{u}_m &+ \omega_n^2 u_m - 3\alpha u_p u_m^2 = 0. \end{aligned} \tag{24}$$

In this case, the approximate solution for x is given by

$$x = U\cos(\omega_r t) - \frac{\alpha U^3}{32\omega_r^2}\cos(3\omega_r t) + \ldots \tag{25}$$

Several observations can be made about these normal form equations and their approximate solutions:

1. We are claiming that Eq. (20), Eq. (23) and Eq. (24) are all normal form expressions for this Duffing example. Details of the equivalence of these different normal form styles are given in Section 4.2.5. In fact, these are different *styles* of normal form that arise from the different approaches taken, and choices made during the transformation process[11].
2. The concept of a normal form coordinate transformation making things 'simpler' in the nonlinear case is not always very helpful. It could be argued that Eq. (20), Eq. (23) and Eq. (24) are not simpler than the starting equations. However, as will be seen, they are more *useful*.
3. The approximate solutions Eq. (21) and Eq. (25) (the full derivations of which are given in Section 4.2.2 and Section 4.2.4 respectively) are not the same. This again is a consequence of the different approaches taken. In particular, there are *free functions*[12] in the process that can be chosen to give different outcomes. In this example, free function choices can be made so that Eq. (21) reduces to exactly the same as Eq. (25), if so required (an example of this is given in Section 4.4.2).

[9] The expression for ω_r will depend on the type of approximation (often called *detuning*) assumed in the normal form transformation. In many cases $\omega_r \approx \omega_n$ is used in the second terms of the coefficient expressions of Eq. (20) and Eq. (21). Note also that some authors give just one of the normal form equations in Eq. (20) with the implication that the other is the conjugate expression [124, 85]. Detuning will be discussed further in Section 4.6.

[10] Note, as this is an undamped single-degree-of-freedom oscillator phase $\phi = 0$ has been assumed in the base solutions. To keep the formulations as simple as possible, $\phi = 0$ will be used by default where appropriate in this paper. However cases for $\phi \neq 0$ will be discussed in Section 3 and Section 5. Note also, that for nonlinear examples, these base solutions will be used extensively in this paper, but in general higher frequency solution terms could also be included, as is the case in the harmonic balance method.

[11] Here we follow Murdock [120] who defines the final appearance of the normal form the *style*, and the approach taken to achieve it the *format*.

[12] This is the terminology of [85]. In general these are choices of coefficient values in order to reduce certain terms in the final expressions. For example [85] describe some of the choices that reduce Eq. (21) to Eq. (25), and we shall discuss in more detail in Section 4.7.

4. The normal form style based on ordinary differential equations in first-order form (i.e. Eq. (20)) is projected into the complex domain, whereas the styles which operate on ordinary differential equations in second-order form (i.e. Eq. (23) and Eq. (24)) remain real.
5. The formats of Eq. (20) and Eq. (24) use $2N$ equations to represent a system with N degrees-of-freedom. However, the format of Eq. (23) uses N equations. This is useful when there are multiple degrees-of-freedom, and the 'compact' form of Eq. (23) means that there is no increase in the number of equations[13]

1.1.4 Summary and overview of the methodology

Overall, it is reasonable to ask; *What useful information can be obtained from the normal form?*. Well, for linear systems, coordinate transformations have been developed extensively as part of modal analysis, as will be discussed in Section 3. For nonlinear systems there are four things that can be potentially useful in structural dynamics:

I. Firstly, for a limited number of simple nonlinear systems we can obtain an approximate solution for x, such as Eq. (21) and Eq. (25) [14].
II. Secondly, we obtain information on the *nonlinear resonances*, a topic that will be covered in more detail in Section 4.1.2
III. Third we can get a frequency amplitude relationship(s) known as a *backbone curve(s)*[15], which we introduce in Section 4.8, and these are one way to analyse the *nonlinear normal modes of a system*.
IV. Fourth normal form transformations can be used to simplify models of important dynamic phenomena, such as bifurcations. We will describe how this relates to the Hopf bifurcation in Section 4.9.
V. Finally, normal form transformations can be used as an integral part of other methods, notably model-order reduction and system identification. Although these topics are not discussed in detail in this paper, they are mentioned in Section 6.3 where we will discuss future research directions.

It should be noted that, apart from IV, all the other useful outcomes listed above can be (and usually are) obtained using other methods. Some comments giving advantages and disadvantages of normal form techniques compared with other methods are given in Section 6.2. We note here that normal forms are often regarded as being algebraically intense, and some authors have developed symbolic computation methods (particularly using Maple) in order to try and mitigate this problem [10, 190, 193, 122].

In order to obtain the useful outcomes listed in I. to V. above, we will use a two part methodology containing the steps:

A. Use a coordinate transformation to 'simplify' the structure of the governing equations of motion based on a chosen normal form style and format, and
B. Substitute an assumed solution into the normal form equations of motion to obtain some addition insights, such as the backbone curves mentioned in III above.

It is usually the case that we do A followed by B, but in some techniques the steps can be interlinked, as we will describe. It is also the case that the choice of the assumed solution can be significant, and so we will devote quite a lot of space in the paper to discussing this topic.

If we consider how this applies to the pendulum example given above (Section 1.1.3), firstly a coordinate transformation is used to obtain the normal form equations given by Eq. (24) (Step A). Then we can choose the base solutions given in Eq. (22) as the assumed solution, and substitute into either of the normal form expressions in Eq. (24) (Step B) to obtain

$$\omega_r^2 = \omega_n^2 - \frac{3\alpha U^2}{4} + \ldots \tag{26}$$

This gives the backbone curve relationship (which is a truncated asymptotic series) relating the amplitude of the base solutions, U, and the response frequency ω_r. An example is shown in Fig. 2 (b) for the case of softening nonlinearity, i.e. with α negative, as in Eq. (26). One useful property of backbone curves is to represent the

[13]There have also been attempts to develop methods that can be used for simultaneous normal form transformation and model-order reduction, and this topic is the subject of a recent review paper [173].
[14]Where this is possible for nonlinear systems will be discussed in Section 4.
[15]The use of the term "backbone curve" goes back at least as far as the PhD thesis of Rauscher [144]. The term "skeleton curve" is also sometimes used to mean the same thing.

amplitude-frequency dependence of the system, and as such are also a way visualising the *nonlinear normal modes*, a topic we will discuss further in Section 5.

Another useful property of backbone curves arises from the relationship between a backbone curve and the forced-damped response of the oscillator. An example of this is shown for a Duffing oscillator with a hardening nonlinear term (i.e. with α positive) in Fig. 2 (c). Note that in Fig. 2 (both (b) and (c)) the backbone curves shown are so-called *conservative* in the sense that they are obtained from equations of motion without damping or forcing terms. In Fig. 2 (c) the forced-damped response is obtained by taking the maximum displacement value, x, per forcing period, and the blue solid curves show stable solutions, whereas the blue dashed curve relate to unstable solutions. The backbone curve in Fig. 2 (c) captures the hardening effect in the forced-damped response, and the two curves intersect very closely to the peak of the resonance (see [27] for a discussion of the relationship between these curves.).

The topic of backbone curves has been of great interest in the structural dynamics community in recent years — see for example [59, 39, 70, 159] (which is just a small selection of recent papers) and references therein. In fact, the relationship between backbone curves, and nonlinear resonances are arguably the most important application of normal forms (and other relevant methods) for the structural dynamicist. It is worth noting that for systems with damping, *damped backbone curves* have also been defined [97, 15, 5], and the area of identifying backbone curves is another topic of great recent interest [163, 112, 148, 149, 150, 90, 50, 53, 119, 173, 101, 119].

1.1.5 Structure of this paper

It should be noted that this paper is not intended to be a comprehensive review of the topic of normal forms — that can be found in [124, 120, 22, 116, 85], or structural dynamics, for that see the wide range of texts on the subject, such as [145, 146, 32, 167, 12, 185, 130, 166, 48, 189, 45, 64, 115, 123, 58, 76, 143, 183, 102] (to name just a selection) or nonlinear normal modes (the literature on this topic is already very large) see [152, 142, 14, 175, 155, 127, 103, 172, 3, 81, 8, 136, 170, 160, 2, 18, 88, 134, 135, 99, 65, 59, 39, 70, 60, 159, 177] and references therein[16]. Instead, the intention is to give an introductory treatment on using normal form transformations that relate to structural dynamics. As a result there will be selective coverage of topics that may be of interest to structural dynamicists working on nonlinear problems where normal form techniques might be useful. That said, the first part of the review deals with linear vibration problems. The reasons for this are (i) to cover selected normal form topics related to linear problems, (ii) to introduce preliminary material needed for the sections on nonlinear problems, and (iii) show a connection between the linear and nonlinear methods. However, readers that are primarily interested in the nonlinear examples may want to skip the sections on linear topics and go straight to the nonlinear parts.

The intended logic is to progress by starting with single degree-of-freedom linear systems then multi-degree-of-freedom linear systems, followed by single degree-of-freedom nonlinear systems, and finally multi-degree-of-freedom nonlinear systems[17]. Throughout we also make a distinction between equations of motion written in second-order form (the classical approach for structural dynamics) and equations of motion written in first-order form (the classical approach for mathematics and dynamical systems theory).

Because linear systems have exact solutions, they can be tackled without using transformation approaches. Despite this, we hope to show that even if a linear system can be solved using a method exactly, without a transformation, there are still situations where it can be advantageous to use the transformation approach. As a result, the rest of this review paper is organised as follows. Firstly in Section 2 we examine single degree-of-freedom linear systems, including the unforced, undamped and unforced, damped linear oscillators. This Section starts with a review of some exact solutions methods, before considering the transformation approach. Then in Section 3, methods relating to multi-degree-of-freedom linear systems are described, where the main focus is on modal coordinate transformations. The topics relating to nonlinear systems start in Section 4, where near identity transformations are introduced in Section 4.2. In this Section a comparison is given between complex and real normal form styles. In addition, we cover topics such as derivation of the homological equations using Lie series, Section 4.3, the Hamiltonian normal form, Section 4.4, damping in normal form transformations, Section 4.5, frequency detuning Section 4.6, backbone curves, Section 4.8 and normal form transformations

[16] Note that we are also not including control related topics in this review.

[17] As this is a tutorial paper, we devote most space to single degree-of-freedom examples, and only show one multi-degree-of-freedom nonlinear example. For the reader that is most interested in multi-degree-of-freedom nonlinear applications multiple references to other sources are given.

of bifurcation phenomena Section 4.9. Then in Section 5, normal form transformations for coupled nonlinear oscillators are considered. Finally conclusions and future directions for research are given in Section 6.

1.1.6 Notation and terminology

In compiling this type of overview it is very difficult to maintain a consistent set of mathematical notation across all parts of the paper. The methodology used here is to use x, x_i, and $\mathbf{x}$ for physical displacement coordinates (bold denoting a vector and non-bold a scalar). Then q, q_i, and $\mathbf{q}$ for transformed coordinates of some type when the transformation is linear. Following that u, u_i, and $\mathbf{u}$ for transformed coordinates of some type when the transformation is nonlinear.

However, we also need to use the state vector, $\boldsymbol{x}$, which contains both displacement and velocity coordinates. In order to try and differentiate the state vector from the displacement vector we denote it as bold *and* italic $\boldsymbol{x}$. This is a very subtle difference, but we hope it is sufficient to allow readers to distinguish between the two conventions. The same convention will be applied to the transformed coordinates $\boldsymbol{q}$ and $\boldsymbol{u}$, which are both $2N$ long vectors.

Modal matrices will be applied for both the N-degree-of-freedom case and $2N$-degree-of-freedom (state space) case. The convention adopted here is to use Ψ to represent the modal matrix in the N-degree-of-freedom case and Φ for the $2N$-degree-of-freedom case. The ℓ^{th} individual modal vectors will then be given by ψ_ℓ and ϕ_ℓ respectively. Likewise to distinguish between the (diagonal) eigenvector matrices, $\hat{\Lambda}$ is used for the N-degree-of-freedom case and Λ for the $2N$-degree-of-freedom case. Note also that the $\square$ symbol is used to indicate the end point of the examples.

Finally, we will avoid the use of the term *canonical* in this review. This term has numerous meanings and interpretations across science and mathematics. For example, the phrases *canonical coordinates* and *canonical form* are used widely, but can have subtle differences in meaning depending on the exact context.

2 Single degree-of-freedom linear systems

2.1 The exponential matrix solution

In Section 1.1.1 we showed that, for the linear oscillator example, the solution to the equation in the form of $\dot{\boldsymbol{q}} = \Lambda \boldsymbol{q}$, (i.e. Eq. (5)) was given by $\boldsymbol{q} = \mathrm{e}^{\Lambda t}\boldsymbol{q}_0$ (i.e. Eq. (9)). In fact this relationship holds for any system of this type for which Λ is a diagonal matrix.

Furthermore, there is a more direct way of obtaining a solution which avoids the need to diagonalise the system matrix. For linear systems of this type with non-diagonal matrices, A (as in Eq. (2)) for the linear oscillator example), it is possible to show that the associated equation of motion, $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ (i.e. Eq. (3)) has the solution

$$\boldsymbol{x} = \mathrm{e}^{At}\boldsymbol{x}_0 \tag{27}$$

without needing a transformation. This is because the term e^{At} is an *exponential matrix* defined by

$$\mathrm{e}^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k = I + At + \frac{1}{2}A^2t^2 + \ldots \tag{28}$$

Now let's consider how this relates to the undamped linear oscillator example.

Example 3: Undamped linear oscillator example revisited

We start by considering the undamped linear oscillator defined by Eq. (1) from Section 1.1.1. To make things as simple as possible, let us first consider the case when $\omega_n = 1$ so that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{29}$$

then

$$\mathrm{e}^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} t + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{t^3}{3!} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^4}{4!} + \ldots$$

which, becomes

$$e^{At} = \begin{bmatrix} 1-\frac{t^2}{2!}+\frac{t^4}{4!}-\ldots & t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots \\ -t+\frac{t^3}{3!}-\frac{t^5}{5!}+\ldots & 1-\frac{t^2}{2!}+\frac{t^4}{4!}-\ldots \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}. \tag{30}$$

This then gives the solution of Eq. (2) as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \rightsquigarrow \begin{array}{l} x = x(0)\cos(t)+\dot{x}(0)\sin(t), \\ \dot{x} = -x(0)\sin(t)+\dot{x}(0)\cos(t), \end{array} \tag{31}$$

which is familiar as the solution to the undamped, unforced oscillator. □

If we set $\dot{x}(0) = 0$ in Eq. (31), we obtain the same result as derived in Section 1.1.1, Eq. (10) (remembering that $\omega_n = 1$). Of course, this is a highly simplified example in which the series solution for e^{At} is straightforward to compute. In the case of a more complicated A matrix, a more systematic approach can be applied using the Cayley-Hamilton theorem [89]. The Cayley-Hamilton theorem can be used to show that a square matrix satisfies its own characteristic equation. This in turn can be used to express all powers of A^k for $k \geq 2$ as a linear combination of A and I. To demonstrate this, we consider once more the undamped linear oscillator.

Example 4: Undamped linear oscillator solution using the Cayley-Hamilton theorem

Let us consider Eq. (1) but now take the case when $\omega_n \neq 1$ so that the A matrix becomes

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}. \tag{32}$$

To find the characteristic equation we take $|A - \lambda I| = 0$, where λ are the eigenvalues of A, from which we find that

$$\lambda^2 + \omega_n^2 = 0,$$

and so $\lambda_1 = \mathbf{i}\omega_n$ and $\lambda_2 = -\mathbf{i}\omega_n$. From the Cayley-Hamilton theorem we have that

$$A^2 + \omega_n^2 I = 0 \quad \rightsquigarrow \quad A^2 = -\omega_n^2 I.$$

Now we can use this to define the following relationships

$$\begin{aligned} A^3 &= A^2A = -\omega_n^2 IA \\ A^4 &= A^3A = -\omega_n^2 IA^2 = -\omega_n^2 I(-\omega_n^2 I) = \omega_n^4 I \\ A^5 &= A^4A = \omega_n^4 IA \\ A^6 &= A^5A = \omega_n^4 IA^2 = \omega_n^4 I(-\omega_n^2 I) = -\omega_n^6 I \\ &\ldots \\ &\ldots \\ A^{2k} &= (-1)^k \omega_n^{2k} I \\ A^{2k+1} &= (-1)^k \omega_n^{2k} A \end{aligned}$$

Then taking the series solution for e^{At} and separating odd and even powers gives

$$\begin{aligned} e^{At} &= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{A^{2k} t^{2k}}{2k!} + \sum_{k=0}^{\infty} \frac{A^{2k+1} t^{2k+1}}{(2k+1)!}, \\ &= I \sum_{k=0}^{\infty} \frac{(-1)^k \omega_n^{2k} t^{2k}}{2k!} + A \sum_{k=0}^{\infty} \frac{(-1)^k \omega_n^{2k} t^{2k+1}}{(2k+1)!}, \\ &= \cos(\omega_n t) I + \frac{1}{\omega_n} \sin(\omega_n t) A. \end{aligned} \tag{33}$$

From this the solution to the governing equation can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\omega_n t) & \frac{1}{\omega_n}\sin(\omega_n t) \\ -\omega_n \sin(\omega_n t) & \cos(\omega_n t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \tag{34}$$

where $x_1(0) = x(0)$ and $x_2(0) = \dot{x}(0)$. □

However, it is worth noting that there is a more efficient way to find this result. This is achieved by using the fact that the Cayley-Hamilton theorem also implies[18] that if λ is an eigenvalue of A then

$$\mathrm{e}^{\lambda t} = \alpha(t)\lambda t + \beta(t) \quad \Leftrightarrow \quad \mathrm{e}^{At} = \alpha(t)At + \beta(t)I, \tag{35}$$

where $\alpha(t)$ and $\beta(t)$ are time dependent coefficients to be determined. In this example A has two eigenvalues which gives two simultaneous equations

$$\begin{aligned} \mathrm{e}^{\mathbf{i}\omega_n t} &= \alpha(t)\mathbf{i}\omega_n t + \beta(t), \\ \mathrm{e}^{-\mathbf{i}\omega_n t} &= -\alpha(t)\mathbf{i}\omega_n t + \beta(t), \end{aligned} \tag{36}$$

from which we can find directly that

$$\begin{aligned} \alpha(t) &= \frac{\mathrm{e}^{\mathbf{i}\omega_n t} - \mathrm{e}^{-\mathbf{i}\omega_n t}}{2\mathbf{i}\omega_n t} = \frac{1}{\omega_n t}\sin(\omega_n t), \\ \beta(t) &= \frac{\mathrm{e}^{\mathbf{i}\omega_n t} + \mathrm{e}^{-\mathbf{i}\omega_n t}}{2} = \cos(\omega_n t), \end{aligned} \tag{37}$$

and so

$$\mathrm{e}^{At} = \alpha(t)At + \beta(t)I = \frac{1}{\omega_n}\sin(\omega_n t)A + \cos(\omega_n t)I, \tag{38}$$

which is exactly the same as Eq. (33).

We now consider the case when viscous damping is included.

Example 5: Damped, unforced linear oscillator

Now we will consider a damped, unforced linear oscillator given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \tag{39}$$

where $\zeta = \frac{c}{2m\omega_n}$ is the damping ratio, c is the viscous damping coefficient (kg/s) and ω_n is the natural frequency as previously defined[19]. As before we define the state vector $\boldsymbol{x} = \{x_1, x_2\}^T$, where $x_1 = x$ is the displacement and $x_2 = \dot{x}$ is the velocity. Using these definitions notice that $\dot{x}_1 = x_2 = \dot{x}$, and $\dot{x}_2 = \ddot{x}$ which enables the system to be written in first-order form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2\zeta\omega_n x_2 - \omega_n^2 x_1, \end{aligned} \quad \text{or} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which can be written in matrix form as Eq. (3), with solution given by Eq. (27). However, in this case the A matrix is more complicated, as it is given by

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}. \tag{40}$$

However, we can still apply the Cayley-Hamilton theory to this example without too much difficulty using Eq. (35). In this case the eigenvalues are complex conjugate $\lambda_{1,2} = -\zeta\omega_n \pm \mathbf{i}\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm \mathbf{i}\omega_d$, where we are always assuming the underdamped case[20], where $\zeta < 1$. Using Eq. (35) we obtain

$$\begin{aligned} \mathrm{e}^{\lambda_1 t} &= \alpha(t)\lambda_1 t + \beta(t), \\ \mathrm{e}^{\lambda_2 t} &= \alpha(t)\lambda_2 t + \beta(t), \end{aligned} \tag{41}$$

[18] This is because if the characteristic polynomial is $p(\lambda) = \det(A - \lambda I) = 0$, then the C-H theorem states that $p(A) = 0$. So an analytic function represented as a series expansion $f(\lambda) = \sum_{k=0}^{\infty}\beta_k\lambda^k$ can be re-expressed as $f(\lambda) = q(\lambda)p(\lambda) + r(\lambda)$, where $r(\lambda)$ is a residual polynomial of degree $N-1$, and $q(\lambda)p(\lambda)$ represents the rest of the terms in the series. Because $p(\lambda) = 0$ this reduces to $f(\lambda) = r(\lambda)$. In this example $f(\lambda t) = r(\lambda t) = \alpha(t)\lambda t + \beta(t)$.

[19] This can also be derived by starting with the equation of motion in the form $m\ddot{x} + c\dot{x} + kx = 0$ and dividing through by m.

[20] This is because we are interested in oscillations.

from which we can find directly that

$$\alpha(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{2\mathbf{i}\omega_d t} = \frac{1}{\omega_d t} e^{-\zeta\omega_n t}\sin(\omega_d t),$$
$$\beta(t) = \frac{e^{\lambda_1 t} + e^{\lambda_2 t} + \alpha 2\zeta\omega_n t}{2} = e^{-\zeta\omega_n t}(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)), \tag{42}$$

and so

$$e^{At} = \alpha(t)At + \beta(t)I = e^{-\zeta\omega_n t}\frac{1}{\omega_d}\sin(\omega_d t)A + e^{-\zeta\omega_n t}(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t))I,$$

$$= e^{-\zeta\omega_n t}\begin{bmatrix} \cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t) & \frac{1}{\omega_d}\sin(\omega_d t) \\ \frac{-\omega_n^2}{\omega_d}\sin(\omega_d t) & \cos(\omega_d t) + \frac{-\zeta\omega_n}{\omega_d}\sin(\omega_d t) \end{bmatrix} \tag{43}$$

which when substituted into Eq. (27) gives the well known transient solution to the unforced, damped linear oscillator. □

Note that for complex eigenvalues $\lambda_{1,2} = a \pm \mathbf{i}b$, Eq. (43) can be written more compactly as

$$e^{At} = e^{at}\left(\frac{\sin(bt)}{b}(A - aI) + \cos(bt)I\right),$$

which is known as Putzer's spectral formula [140].

An alternative approach to dealing with the A matrix in the damped oscillator, Eq. (40) is to split[21] it into two giving

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -2\zeta\omega_n \end{bmatrix}, \tag{44}$$

which we will write as $A = W + D$. We might then hope to use the fact that

$$e^{At} = e^{(W+D)t} = e^{Wt}e^{Dt}, \tag{45}$$

as we have already found an expression for e^{Wt} and e^{Dt} has a simple diagonal form. Unfortunately, this is only possible if $WD - DW = 0$. This is usually referred to as the commutator operation defined (for matrices) as

$$[W, D] = WD - DW, \tag{46}$$

and if $[W, D] = 0$ then W and D are said to commute. It is left for the reader to convince themselves that for this example $[W, D] \neq 0$ and therefore W and D do not commute.

However there is a solution in the case when W and D do not commute, which is provided by the Zassenhaus formula that is derived from the Baker–Campbell–Hausdorff formula [113, 25]

$$e^{At} = e^{(W+D)t} = e^{Wt}e^{Dt}e^{\frac{-t^2}{2}[W,D]}e^{\frac{t^3}{6}(2[W,[W,D]]+[D,[W,D]])}\ldots \tag{47}$$

which is an infinite product of successive exponential terms with repeated commutator brackets. It should be clear that when $[W, D] = 0$ and all higher order brackets are also zero, then Eq. (47) reduces to Eq. (45).

Note that up until now we have not transformed A in order to find e^{At}. Of course if A is diagonal, then it is very easy to find e^{At} because if

$$A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \quad \text{then} \quad A^k = \begin{bmatrix} a_{11}^k & 0 \\ 0 & a_{22}^k \end{bmatrix}, \quad \text{and} \quad e^A = \begin{bmatrix} e^{a_{11}} & 0 \\ 0 & e^{a_{22}} \end{bmatrix}. \tag{48}$$

So an often used approach in structural dynamics analysis is to try and diagonalise A, just as we did in both the motivating examples in Section 1.1. Notice that the approach to diagonalise A in Section 1.1 led to complex diagonal matrices. This is natural for oscillatory solutions, but also somewhat more difficult to deal with than real matrices. As a result an alternative for oscillatory solutions that retains a real matrix structure is to use *Jordan normal form*, which will be considered next.

[21] We use the term "splitting" some what loosely here, a more formal discussion of splitting methods can be found in [120].

2.2 Jordan normal form

For linear systems of the type defined by $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ (i.e. Eq. (3)), a common method used to simplify the system is Jordan normal form[22]. Depending on the problem at hand, the simplified matrix will be either diagonal, upper triangular, or real. In the analysis of structural dynamics systems we are typically only interested in vibrations (i.e. underdamped systems) where the eigenvalues of A will be complex $\lambda_{1,2} = a \pm \mathbf{i}b$, and therefore we only consider this case. We have already seen examples of the diagonal form with complex eigenvalues, e.g. in Example 1 Section 1.1.1, and so we will not consider that case again here. Instead we consider the other Jordan normal form that is relevant to underdamped linear systems with complex eigenvalues[23].

Firstly, we assume that there is a transformation matrix, P, where $\boldsymbol{x} = P\boldsymbol{y}$ such that

$$\dot{\boldsymbol{y}} = P^{-1}\dot{\boldsymbol{x}} = P^{-1}Ax = P^{-1}AP\boldsymbol{y} \tag{49}$$

and it can be shown, [52], that if $\lambda_{1,2} = a \pm \mathbf{i}b$ we can find P such that

$$P^{-1}AP = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = B, \tag{50}$$

such that

$$\dot{\boldsymbol{y}} = B\boldsymbol{y} \tag{51}$$

is the final transformed equation. Notice that B is not diagonal, but it is a real matrix.

This form allows us to avoid complex matrices, which would be the result if we transformed A to a diagonal eigenvalue matrix (as we did in Section 1.1.1). Notice that we can split B to give

$$B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \tag{52}$$

or as we used before $B = W + D$. Checking that $[W, D] = 0$ in this case, we see that D can be exponentiated using Eq. (48) and e^{W} can be found in the same way as for Eq. (32), which leads to

$$\mathrm{e}^{Bt} = \mathrm{e}^{Wt}\mathrm{e}^{Dt} = \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix} \begin{bmatrix} \mathrm{e}^{at} & 0 \\ 0 & \mathrm{e}^{at} \end{bmatrix} = \mathrm{e}^{at} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}. \tag{53}$$

We note that the solution to the transformed equation is $\boldsymbol{y} = \mathrm{e}^{Bt}\boldsymbol{y}_0$, where $\boldsymbol{y}_0 = P^{-1}\boldsymbol{x}_0$. Because $\boldsymbol{x} = P\boldsymbol{y}$ then $\boldsymbol{x} = P\mathrm{e}^{Bt}\boldsymbol{y}_0 = P\mathrm{e}^{Bt}P^{-1}\boldsymbol{x}_0$ with the result that by comparison with Eq. (27) we see that

$$\mathrm{e}^{At} = P\mathrm{e}^{Bt}P^{-1}. \tag{54}$$

But what should we choose as P? Well we know that $P^{-1}AP = B$ so that means that $AP = PB$ and we can also exploit the fact that for a complex eigenvalue $a + \mathbf{i}b$ there is a corresponding complex eigenvector $\boldsymbol{v}$ such that $A\boldsymbol{v} = (a + \mathbf{i}b)\boldsymbol{v}$ and because A is real $A\mathrm{Re}(\boldsymbol{v}) = \mathrm{Re}((a + \mathbf{i}b)\boldsymbol{v})$ and $A\mathrm{Im}(\boldsymbol{v}) = \mathrm{Im}((a + \mathbf{i}b)\boldsymbol{v})$. So if we take $P = [\mathrm{Im}(\boldsymbol{v}), \mathrm{Re}(\boldsymbol{v})]$ then we have

$$\begin{aligned} A[\mathrm{Im}(\boldsymbol{v}), \mathrm{Re}(\boldsymbol{v})] &= [\mathrm{Im}((a + \mathbf{i}b)\boldsymbol{v}), \mathrm{Re}((a + \mathbf{i}b)\boldsymbol{v})], \\ &= [a\mathrm{Im}(\boldsymbol{v}) + b\mathrm{Re}(\boldsymbol{v}), a\mathrm{Re}(\boldsymbol{v}) - b\mathrm{Im}(\boldsymbol{v})], \\ &= [\mathrm{Im}(\boldsymbol{v}), \mathrm{Re}(\boldsymbol{v})] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = PB. \end{aligned} \tag{55}$$

Now we consider how this analysis can be applied to the damped linear oscillator example.

[22] Here we follow the approach taken by [52], but only discuss oscillatory systems that have complex eigenvalues.

[23] Throughout this paper we primarily use the diagonal case with complex eigenvalues. An example of the real Jordan normal form is shown in Section 4.9.1. For those interested in other Jordan normal forms applied in the nonlinear case, discussions can be found in [16, 120, 124, 85].

Example 6: Damped, unforced linear oscillator revisited

In the example of the damped linear oscillator we know that the eigenvalues are $\lambda_{1,2} = -\zeta\omega_n \pm \mathbf{i}\omega_d$, where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. Therefore we have that $a = -\zeta\omega_n$ and $b = \omega_d$. We also know that the eigenvectors are given by $\boldsymbol{v}_1 = \{1, \lambda_1\}^T$, and $\boldsymbol{v}_2 = \{1, \lambda_2\}^T$. The analysis above corresponds to a choice for P based on $\boldsymbol{v}_1$, and so

$$P = [\mathrm{Im}(\boldsymbol{v}_1), \mathrm{Re}(\boldsymbol{v}_1)] = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega_d & -\zeta\omega_n \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} \frac{\zeta\omega_n}{\omega_d} & \frac{1}{\omega_d} \\ 1 & 0 \end{bmatrix}. \tag{56}$$

Next we can write down the B matrix

$$B = P^{-1}AP = \begin{bmatrix} -\zeta\omega_n & -\omega_d \\ \omega_d & -\zeta\omega_n \end{bmatrix}. \tag{57}$$

Now from Eq. (53) we have

$$\mathrm{e}^{Bt} = \mathrm{e}^{-\zeta\omega_n t}\begin{bmatrix} \cos(\omega_d t) & -\sin(\omega_d t) \\ \sin(\omega_d t) & \cos(\omega_d t) \end{bmatrix}. \tag{58}$$

Finally substituting Eq. (56) and Eq. (58) into Eq. (54) gives exactly the same solution as we derived using the Cayley-Hamilton methods, namely Eq. (43). □

2.3 The method of "reduction of order"

At the end of the first motivating example of Section 1.1 we showed a method of solving the undamped, unforced linear oscillator equation ($\ddot{x} + \omega_n^2 x = 0$ i.e. Eq. (1)) directly using the idea of an operator formulation. Now we show how that can be combined with another normal form technique called *the method of reduction of order*[24]. To do this we revisit the damped linear oscillator again

Example 7: Damped, unforced linear oscillator revisited again

In this method we apply a transformation where $x = q(t)\mathrm{e}^{-\zeta\omega_n t}$. This leads to the relationships

$$\begin{aligned} x &= q(t)\mathrm{e}^{-\zeta\omega_n t}, \\ \dot{x} &= \dot{q}\mathrm{e}^{-\zeta\omega_n t} - \zeta\omega_n q\mathrm{e}^{-\zeta\omega_n t}, \\ \ddot{x} &= \ddot{q}\mathrm{e}^{-\zeta\omega_n t} - 2\zeta\omega_n\dot{q}\mathrm{e}^{-\zeta\omega_n t} + \zeta^2\omega_n^2 q\mathrm{e}^{-\zeta\omega_n t}. \end{aligned} \tag{59}$$

Substituting Eq. (59) into the equation of motion for the damped linear oscillator (Eq. (39)) gives the following outcome.

$$\begin{aligned} &\text{Start with Eq. (39):} \quad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \quad \text{substituting Eq. (59)} \quad \rightsquigarrow \\ &\ddot{q}\mathrm{e}^{-\zeta\omega_n t} - 2\zeta\omega_n\dot{q}\mathrm{e}^{-\zeta\omega_n t} + \zeta^2\omega_n^2 q\mathrm{e}^{-\zeta\omega_n t} + 2\zeta\omega_n(\dot{q}\mathrm{e}^{-\zeta\omega_n t} - \zeta\omega_n q\mathrm{e}^{-\zeta\omega_n t}) + \omega_n^2 q(t)\mathrm{e}^{-\zeta\omega_n t} = 0, \rightsquigarrow \\ &\ddot{q}\mathrm{e}^{-\zeta\omega_n t} + q(\zeta^2\omega_n^2 - 2\zeta^2\omega_n^2 + \omega_n^2)\mathrm{e}^{-\zeta\omega_n t} = 0, \rightsquigarrow \\ &\ddot{q} + \omega_d^2 q = 0, \end{aligned} \tag{60}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. So now we see that the last equation in Eq. (60) is the transformed version of Eq. (39) and appears to be an "undamped" linear oscillator, but with a damped natural frequency, ω_d.

Now applying the operator method from Section 1.1.2 to $\ddot{q} + \omega_d^2 q = 0$, we can obtain a solution for $q(t)$ as

$$q(t) = q(0)\cos(\omega_d t) + \frac{\dot{q}(0)}{\omega_d}\sin(\omega_d t), \tag{61}$$

[24] There is potential for confusion with the idea of reducing the order from $\mathrm{d}^2/\mathrm{d}t^2$ terms to $\mathrm{d}/\mathrm{d}t$ terms. In fact, the method in the form presented here can be interpreted as a variation on the WKB (Wentzel, Kramers & Brillouin — see for example [89, 9]) method, and is sometimes just called the "normal form" for example [161].

and so the complete solution, $x = q(t)\mathrm{e}^{-\zeta\omega_n t}$ is given by

$$\begin{aligned} x(t) &= \mathrm{e}^{-\zeta\omega_n t}(q(0)\cos(\omega_d t) + \frac{\dot{q}(0)}{\omega_d}\sin(\omega_d t)), \rightsquigarrow \\ x(t) &= \mathrm{e}^{-\zeta\omega_n t}(x(0)\cos(\omega_d t) + \frac{\dot{x}(0)+\zeta\omega_n x(0)}{\omega_d}\sin(\omega_d t)), \end{aligned} \tag{62}$$

once the initial conditions are taken into consideration (i.e. by using the first equation in Eq. (62) and its derivative with $t = 0$). □

This method gives a solution that is equivalent to the previous solutions for the damped, linear oscillator, but may not be particularly familiar to many structural dynamicists. It is notably shorter and more direct than the previous methods for this example.

2.4 Summary

The exponential matrix solution provides a direct way to develop solutions for linear single-degree-of-freedom oscillators, as described in Section 2.1. Alternatives, such as the Jordan normal form, can be used to obtain oscillatory solutions whilst maintaining real matrix expressions. A key part of this later method (and also required for some of the other methods discussed) is the computation of eigenvalues and eigenvectors. As is well known in structural dynamics, these can be related to resonant frequencies and mode shapes, and they form a key part of the discussion as we move on to talk about multi-degree-of-freedom linear systems.

3 Multi-degree-of-freedom linear systems

One of the most important topics in structural dynamics is the analysis of linear multi-degree-of-freedom systems. For a vector of discrete displacement coordinates, $\mathbf{x} = [x_1, x_2, x_3, ...x_N]^T$ the equation of motion governing the (linear) dynamic behaviour is commonly written as

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}_e, \tag{63}$$

where $\mathbf{F}_e$ is the external forcing vector. Note that $\mathbf{x}$ *is not* the state vector[25]. The matrices M, C and K represent the mass, damping and stiffness properties of the structure respectively[26].

The matrices M, C and K are not usually diagonal[27], so the equations are coupled, and as a result it is useful to see if the system of equations can be simplified by applying a coordinate transformation. This transformation is called by several names, often a *modal transformation*, to signify the role that the *mode shapes* play[28]. The aim of the modal transformation is (if possible) to simultaneously diagonalise M, C and K resulting in N uncoupled second-order differential equations. All though it is not normally discussed in terms of normal form transformations, the modal transformation is a way of using a coordinate transformation to simplify the governing equations, and therefore fits exactly our definition of a normal form, assuming that we arrive at a system that can be considered to be in it's "simplest" or normal form.

As we noted in Section 1.1.3, coordinate transformations can be carried out with the governing differential equations in either second-order form or in first-order form. We will consider both cases.

3.1 The modal transformation for multi-degree-of-freedom linear systems

First consider the case when the governing differential equations are in second-order form (i.e. containing $\mathrm{d}^2/\mathrm{d}t^2$ terms).

[25] The state vector is bold *and* italic.

[26] There are a large number of texts that deal with these type of linear vibration problems, see for example [115, 76]

[27] In the analysis of vibration problems there are many specific structures of these matrices that arise, e.g., for gyroscopic systems, for which the total stiffness matrix is not symmetric but is a sum of symmetric and skew-symmetric matrices. See for example, [45]. In this paper we will consider only simple examples where the mass matrix is diagonal.

[28] For a detailed description of mode shapes and related topics the interested reader is referred to specialist books on this, such as [45, 64].

3.1.1 The undamped, unforced case

As we are only interested in lightly damped systems it is useful to consider the case of the undamped, unforced system, first. When $C = 0$ and $\mathbf{F}_e = 0$, the governing equation becomes

$$M\ddot{\mathbf{x}} + K\mathbf{x} = 0 \quad \rightsquigarrow \quad \ddot{\mathbf{x}} + M^{-1}K\mathbf{x} = 0, \tag{64}$$

which can be treated as a linear matrix eigenvalue problem with an assumed solution of $\mathbf{x} = \psi_j e^{i\omega_{nj}t}$ where ψ_j is the eigenvector (related to the mode-shape) and ω_{nj} is the natural frequency (related to the eigenvalue) of the j^{th} mode for the undamped, unforced system[29].

Substituting the assumed solution into Eq. (64) leads to

$$\omega_{nj}^2 \psi_j = M^{-1}K\psi_j, \tag{65}$$

where ψ_j is the jth eigenvector and ω_{nj}^2 the corresponding jth eigenvalue of $M^{-1}K$. These eigenvectors and eigenvalues can be used to derive a coordinate transformation that replaces M and K with diagonal matrices. To do this, premultiplying by M and the ℓ^{th} mode-shape, Eq. (65) can be rewritten as

$$\omega_{nj}^2 \psi_\ell^T M\psi_j = \psi_\ell^T K\psi_j \quad \text{or the equivalent expression} \quad \omega_{n\ell}^2 \psi_j^T M\psi_\ell = \psi_j^T K\psi_\ell. \tag{66}$$

Taking the transpose of this last equation and assuming that M and K are symmetric (such that $M^T = M$ and $K^T = K$) allows this expression to be rewritten in the following steps

$$\omega_{n\ell}^2 \psi_j^T M^T \psi_\ell = \psi_j^T K^T \psi_\ell, \rightsquigarrow \omega_{n\ell}^2 \psi_\ell^T (\psi_j^T M^T)^T = \psi_\ell^T (\psi_j K^T)^T, \rightsquigarrow \omega_{n\ell}^2 \psi_\ell^T M\psi_j = \psi_\ell^T K\psi_j, \tag{67}$$

by using the fact that $M\psi_j = (\psi_j^T M^T)^T$. Now subtracting the last of Eq. (67) from the first of Eq. (66) gives

$$(\omega_{nj}^2 - \omega_{n\ell}^2)\psi_\ell^T M\psi_j = \psi_\ell^T K\psi_j - \psi_\ell^T K\psi_j = 0. \tag{68}$$

From this it can be seen that the system has the property of *orthogonality*, meaning that when $\omega_{nj}^2 \neq \omega_{n\ell}^2$, then $\psi_\ell^T M\psi_j = 0$ for $\ell \neq j$ must be true for Eq. (68) to hold.

This property is crucial to modal analysis for linear systems. It means that if we form a modal matrix $\Psi = [\psi_1, \psi_2, ...\psi_N]$, then substituting $\mathbf{x} = \Psi\mathbf{q}$ into Eq. (64) and pre-multiplying by Ψ^T leads to a transformed set of *decoupled* governing equations given by

$$\Psi^T\Psi\ddot{\mathbf{q}} + \Psi^T M^{-1}K\Psi\mathbf{q} = 0 \quad \rightsquigarrow \quad I\ddot{\mathbf{q}} + \hat{\Lambda}\mathbf{q} = 0 \tag{69}$$

where the diagonal matrix, $\hat{\Lambda}$ contains the system eigenvalues, given by

$$\hat{\Lambda} = \begin{bmatrix} \omega_{n1}^2 & 0 & ... & 0 & 0 \\ 0 & \omega_{n2}^2 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & ... & 0 & \omega_{nN}^2 \end{bmatrix}, \tag{70}$$

where ω_{nj} for $j = 1, 2, 3..., N$ are the natural frequencies of the system, and $\Psi^T\Psi = I$ because Ψ is an orthogonal matrix. Note that the eigenvectors, ψ_j are not unique, and can be scaled — see for example the discussion in [13, 45]. For the purposes of our analysis, the scaling chosen must be such that $\Psi^T M^{-1}K\Psi$ gives the result in Eq. (70).

3.1.2 Example 8: Modal transformation of an unforced, undamped three degree-of-freedom linear system

Consider the three degree-of-freedom oscillator shown in Fig. 3 (a) (with dampers $c_\ell = 0$) for the case where the springs and masses are equal $k_\ell = k$ and $m_\ell = m$ for $\ell = 1, 2, 3, 4$.

[29] Although in general both the eigenvalues and eigenvectors can be complex, in classical vibration theory, the nature of the underlying assumptions leads to *real* eigenvectors, which is why they are directly relatable to the mode-shapes of the system under consideration. If the eigenvectors are not real, then the system is said to have “complex modes".

The equation of motion of the system is given by

$$M\ddot{\mathbf{x}}+K\mathbf{x}=0,\quad \text{where}\quad M=\begin{bmatrix} m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & m\end{bmatrix}=mI,\quad K=\begin{bmatrix} 2k & -k & 0\\ -k & 2k & -k\\ 0 & -k & 2k\end{bmatrix}=k\begin{bmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix},$$

where I is a 3×3 identity matrix. The eigenvalues and eigenvectors of $M^{-1}K$ can be computed either numerically or by solving the characteristic equation. First we note that

$$M^{-1}K=\frac{k}{m}\begin{bmatrix} 2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}\rightsquigarrow\hat{\Lambda}=\frac{k}{m}\begin{bmatrix} 2-\sqrt{2} & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2+\sqrt{2}\end{bmatrix},\quad\text{and}\quad\Psi=\begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{1}{2}\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{2} & \frac{1}{\sqrt{2}} & -\frac{1}{2}\end{bmatrix},\tag{71}$$

for given values of m and k, where $\hat{\Lambda}$ is the matrix of eigenvalues and $\Psi=[v_1,v_2,v_1]$ is the matrix of eigenvectors, v_j for $j=1,2,3$.[30].

Making the transformation $\mathbf{x}=\Psi\mathbf{q}$ decouples the system into

$$\ddot{\mathbf{q}}+\hat{\Lambda}\mathbf{q}=0\quad\rightsquigarrow\quad\begin{aligned}&\ddot{q}_1+\omega_{n1}^2q_1=0,\\ &\ddot{q}_2+\omega_{n2}^2q_2=0,\\ &\ddot{q}_3+\omega_{n3}^2q_2=0,\end{aligned}\tag{72}$$

where $\omega_{n1}=\sqrt{(k/m)(2-\sqrt{2})}$, $\omega_{n2}=\sqrt{(2k/m)}$, and $\omega_{n3}=\sqrt{(k/m)(2+\sqrt{2})}$ are the natural frequencies of the system from $\hat{\Lambda}$. Each of these expressions can be solved exactly using (for example) Eq. (10). So, assuming the initial conditions are $\mathbf{x}=[1\,\text{mm},0,1\,\text{mm}]^T$ and $\dot{\mathbf{x}}=[0,0,0]^T$ using $\mathbf{q}=\Psi^{-1}\mathbf{x}$ gives $q_1(0)=1$, $q_2(0)=0$ and $q_3(0)=-1$ and $\dot{q}_1(0)=\dot{q}_2(0)=\dot{q}_3(0)=0$. This leads to

$$\begin{aligned}&q_1=q_1(0)\cos(\omega_{n1}t)\\ &q_2=q_2(0)\cos(\omega_{n2}t)\\ &q_3=q_3(0)\cos(\omega_{n3}t)\end{aligned}\quad\text{then using}\quad\mathbf{x}=\Psi\mathbf{q}\quad\rightsquigarrow\quad\begin{aligned}&x_1=\tfrac{1}{2}\cos(\omega_{n1}t)+\tfrac{1}{2}\cos(\omega_{n3}t),\\ &x_2=\tfrac{1}{\sqrt{2}}\cos(\omega_{n1}t)-\tfrac{1}{\sqrt{2}}\cos(\omega_{n3}t),\\ &x_3=\tfrac{1}{2}\cos(\omega_{n1}t)+\tfrac{1}{2}\cos(\omega_{n3}t).\end{aligned}\qquad\square$$

A similar approach can be applied to the case when damping is included (e.g. from Eq. (63) use $M\ddot{\mathbf{x}}+C\dot{\mathbf{x}}+K\mathbf{x}=0$). Here a simultaneous diagonalisation of the three matrices M, C and K is required, which puts addition restrictions onto the structure of the matrices, which is usually satisfied in a structural dynamics context by using *Rayleigh damping* so that $C=\alpha M+\beta K$ is assumed[31]. In this case, the concept of *lambda matrices* also applies [105]. We now consider an alternative state space approach where damping is included.

3.2 State Space Form for Discrete Linear Systems

Now consider the case where the discrete system is written in first-order form. In this case, Eq. (63) (with $\mathbf{F}_e=0$) may be written in terms of the state vector $\boldsymbol{x}=[\mathbf{x}^T,\dot{\mathbf{x}}^T]^T$ as[32]

$$\dot{\boldsymbol{x}}=A\boldsymbol{x}+\begin{bmatrix} 0_N\\ M^{-1}\mathbf{F}_e\end{bmatrix},\quad A=\begin{bmatrix} 0_{NN} & I\\ -M^{-1}K & -M^{-1}C\end{bmatrix},\tag{73}$$

where 0_{NN} is an $N\times N$ matrix of zeros, 0_N is a $N\times 1$ zero vector and I is an $N\times N$ identity matrix where N is the number of degrees-of-freedom. A coordinate transform for $\boldsymbol{x}$ is now required such that A can be replaced by a diagonal matrix such that the states become decoupled[33]. As was done in Section 3.1, we consider using the eigenvectors of A to create such a transformation.

[30] Note that the eigenvectors can be scaled. Here, for simplicity, they are left unscaled. For a discussion on this type of scaling, such as mass normalised modes, see [51, 45].

[31] In fact, there is a more general concept called *extended Rayleigh damping* in which this is just the first term of a series expansion [30]. There is also the related classical approach of Caughey and O'Kelly [26]. In addition, the whole subject of damping in structural dynamics is a large one, including many other types of damping models, see for example [106, 121, 82, 1].

[32] There are alternative formulations of first-order state space that are used for nonlinear normal form transformation as described by Jain and Haller [79].

[33] There are situations where Jordan normal form is preferable to diagonal matrices, but these are not considered here.

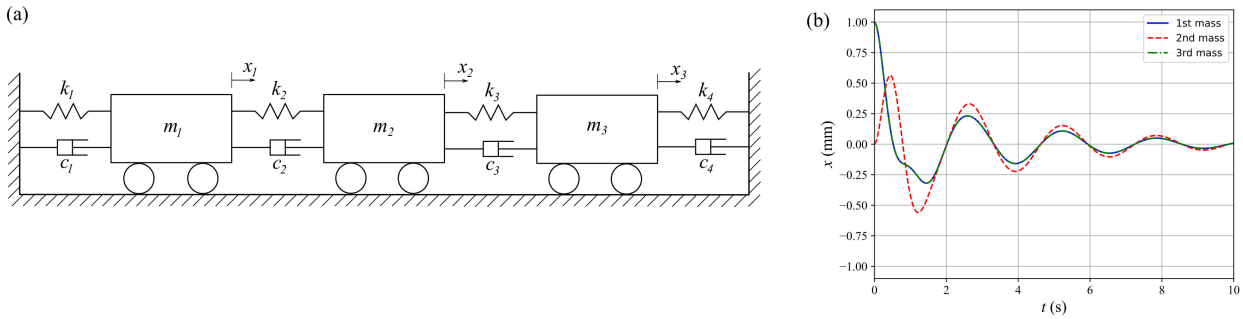

Fig. 3: (a) A three degree-of-freedom point mass oscillator consisting of masses m_ℓ, dampers c_ℓ and springs k_ℓ. (Note that the rollers are assumed to be frictionless). (b) The exact solutions for the damped state space example (Section 3.2.1) given by equation Eq. (78). Parameter values $m_\ell = 1.0$ kg, dampers $c_\ell = 1.0$ kg/s and springs $k_\ell = 10$ N/m. The initial conditions are given by $\boldsymbol{x} = [1\,\text{mm}, 0, 1\,\text{mm}, 0, 0, 0]^T$. Note that the response of masses 1 and 3 are identical.

Using an assumed solution of $\boldsymbol{x} = \boldsymbol{\phi}_\ell e^{\lambda_\ell t}$ where $\boldsymbol{\phi}_\ell$ is the eigenvector and λ_ℓ is the associated eigenvalue, the damped, unforced ($2N$ state space) system may be written as[34]

$$\lambda_\ell \boldsymbol{\phi}_\ell = A\boldsymbol{\phi}_\ell. \tag{74}$$

Using this expression, a modal matrix, Φ, can be defined in which $\boldsymbol{\phi}_\ell$ forms the ℓ^{th} column. The corresponding diagonal eigenvalue matrix is Λ where the ℓ^{th} diagonal value is λ_ℓ. Assuming that the eigenvectors are orthogonal, this allows the matrix relationship of Eq. (74) to be written as

$$\Phi\Lambda = A\Phi \quad \rightsquigarrow \quad \Lambda = \Phi^T A \Phi. \tag{75}$$

Now applying the coordinate transformation $\boldsymbol{x} = \Phi\boldsymbol{q}$ to the *unforced* version of Eq. (73) gives

$$\Phi\dot{\boldsymbol{q}} = A\Phi\boldsymbol{q},$$

which when multiplied by Φ^T combined with Eq. (75)) and the fact that $\Phi^T\Phi = I$ (assuming orthogonal eigenvectors) gives

$$\dot{\boldsymbol{q}} = \Lambda\boldsymbol{q} \quad \rightsquigarrow \quad \boldsymbol{q} = e^{\Lambda t}\boldsymbol{q}_0. \tag{76}$$

Because Λ is a diagonal matrix, the system is now decoupled with respect to the states (which include the displacements and velocities). Notice that damping is still included in the matrix, A, which will lead to complex eigenvalues and eigenvectors for underdamped vibrations[35]. An example is considered next.

3.2.1 Example 9: State space eigenvector transformation of an unforced, damped three degree-of-freedom linear system

We now consider the same example as in Section 3.1.2, the three degree-of-freedom oscillator shown in Fig. 3 (a), but with viscous damping now included. It is assumed that the springs, dampers and masses are equal $k_\ell = k$, $c_\ell = c$ and $m_\ell = m$ for $\ell = 1,2,3,4$, and that the damping is proportional to the stiffness via $C = 0.1K$. This leads to an unforced state space equation of motion of

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \quad A = \frac{k}{m}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0.2 & -0.1 & 0 \\ -1 & 2 & -1 & -0.1 & 0.2 & -0.1 \\ 0 & -1 & 2 & 0 & -0.1 & 0.2 \end{bmatrix},$$

[34] We note that it is also possible to extend methods developed in Section 2 to solve this problem.

[35] Note this should not be confused with *complex modes* which arise for systems with non-proportional damping. See, [45] for a more detailed discussion.

where $\boldsymbol{x} = [\mathbf{x}^T, \dot{\mathbf{x}}^T]^T$. The state space formulation is decoupled using the eigenvalues and eigenvectors of A which are typically computed using numerical methods, and leads to expressions of the form

$$\Phi\Lambda = A\Phi, \;\; \Phi = \begin{bmatrix} v_1 & \overline{v}_1 & v_2 & \overline{v}_2 & v_3 & \overline{v}_3 \\ \lambda_1 v_1 & \overline{\lambda}_1\overline{v}_1 & \lambda_2 v_2 & \overline{\lambda}_2\overline{v}_2 & \lambda_3 v_3 & \overline{\lambda}_3\overline{v}_3 \end{bmatrix}, \;\; \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{\lambda}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\lambda}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{\lambda}_3 \end{bmatrix}, \quad (77)$$

where the eigenvalues λ_ℓ appear in complex conjugate pairs and the eigenvectors are related to those from Section 3.1.2, namely v_j for $j = 1,2,3.$ from Eq. (71). Each of the λ_ℓ eigenvalues and the conjugate can be expressed as

$$\begin{aligned} \lambda_\ell &= -\zeta_\ell \omega_{n\ell} + \mathbf{i}\,\omega_{d\ell} \\ \overline{\lambda}_\ell &= -\zeta_\ell \omega_{n\ell} - \mathbf{i}\,\omega_{d\ell} \end{aligned}$$

where ζ_ℓ is the ℓth modal damping ratio, and $\omega_{d\ell}$ is the associated damped modal frequency[36]. For example, the eigenvalues and eigenvectors obtained when $k/m = 10$ are listed in Table 1[37].

Applying the transformation $\boldsymbol{x} = \Phi\boldsymbol{q}$ (where $\boldsymbol{q}$ is a 4×1 vector) results in the decoupled equation $\dot{\boldsymbol{q}} = \Lambda\boldsymbol{q}$ using Eq. (76). The initial conditions are $\boldsymbol{x} = [1\,\text{mm}, 0, 1\,\text{mm}, 0, 0, 0]^T$ which can be transformed into the modal coordinates using $\boldsymbol{q} = \Phi^{-1}\boldsymbol{x}$ resulting in

$$\boldsymbol{q}(0) = \begin{bmatrix} 0.5001 - \mathbf{i}0.0610 \\ 0.5001 + \mathbf{i}0.0610 \\ 0.0000 - \mathbf{i}0.0000 \\ 0.0000 + \mathbf{i}0.0000 \\ -0.5005 + \mathbf{i}0.1529 \\ -0.5005 - \mathbf{i}0.1529 \end{bmatrix} = \begin{bmatrix} \frac{Q_1}{2}\mathrm{e}^{\mathbf{i}\varphi_1} \\ \frac{Q_1}{2}\mathrm{e}^{-\mathbf{i}\varphi_1} \\ 0 \\ 0 \\ \frac{Q_3}{2}\mathrm{e}^{\mathbf{i}\varphi_3} \\ \frac{Q_3}{2}\mathrm{e}^{-\mathbf{i}\varphi_3} \end{bmatrix},$$

where $Q_1 = 1.0076$, $Q_3 = 1.0468$, $\varphi_1 = -0.1213$ and $\varphi_3 = 2.8451$. Now using Eq. (76) with the initial conditions gives

$$\begin{aligned} q_1 &= q_1(0)\mathrm{e}^{\lambda_1 t} = \tfrac{Q_1}{2}\mathrm{e}^{\mathbf{i}\varphi_1}\mathrm{e}^{\lambda_1 t} = \tfrac{Q_1}{2}\mathrm{e}^{-\zeta_1\omega_{n1}t}\mathrm{e}^{\mathbf{i}(\omega_{d1}+\varphi_1)}, \\ q_2 &= q_2(0)\mathrm{e}^{\overline{\lambda}_1 t} = \tfrac{Q_1}{2}\mathrm{e}^{-\mathbf{i}\varphi_1}\mathrm{e}^{\overline{\lambda} t} = \tfrac{Q_1}{2}\mathrm{e}^{-\zeta_1\omega_{n1}t}\mathrm{e}^{-\mathbf{i}(\omega_{d1}+\varphi_1)}, \\ q_3 &= q_3(0)\mathrm{e}^{\lambda_2 t} = 0, \\ q_4 &= q_4(0)\mathrm{e}^{\overline{\lambda}_2 t} = 0, \\ q_5 &= q_5(0)\mathrm{e}^{\lambda_3 t} = \tfrac{Q_3}{2}\mathrm{e}^{\mathbf{i}\varphi_3}\mathrm{e}^{\lambda_3 t} = Q_3\mathrm{e}^{-\zeta_3\omega_{n3}t}\mathrm{e}^{\mathbf{i}(\omega_{d3}+\varphi_3)}, \\ q_6 &= q_6(0)\mathrm{e}^{\overline{\lambda}_3 t} = \tfrac{Q_3}{2}\mathrm{e}^{\mathbf{i}\varphi_1}\mathrm{e}^{\overline{\lambda}_3 t} = Q_3\mathrm{e}^{-\zeta_3\omega_{n3}t}\mathrm{e}^{-\mathbf{i}(\omega_{d3}+\varphi_3)}. \end{aligned}$$

Then using the modal matrix relationship $\boldsymbol{x} = \Phi\boldsymbol{q}$ gives the displacements as

$$\begin{aligned} x_1 &= v_{11}Q_1\mathrm{e}^{-\zeta_1\omega_{n1}t}\cos(\omega_{d1}t + \varphi_1) + v_{31}Q_3\mathrm{e}^{-\zeta_3\omega_{n3}t}\cos(\omega_{d3}t + \varphi_3), \\ x_2 &= v_{12}Q_1\mathrm{e}^{-\zeta_1\omega_{n1}t}\cos(\omega_{d1}t + \varphi_1) + v_{32}Q_3\mathrm{e}^{-\zeta_3\omega_{n3}t}\cos(\omega_{d3}t + \varphi_3), \\ x_3 &= v_{13}Q_1\mathrm{e}^{-\zeta_1\omega_{n1}t}\cos(\omega_{d1}t + \varphi_1) + v_{33}Q_3\mathrm{e}^{-\zeta_3\omega_{n3}t}\cos(\omega_{d3}t + \varphi_3). \end{aligned} \quad \square \quad (78)$$

[36] Note that these expressions can be used to show that $\zeta_\ell = 1/((\mathrm{Re}\,\lambda_\ell/\mathrm{Im}\,\lambda_\ell)^2 + 1)$.

[37] Note that most numerical routines do not automatically return the eigenvalues or eigenvectors in the format of Eq. (77). The ordering convention is that the eigenvalues are sorted based on the $\omega_{d\ell}$ values with the smallest first. The eigenvectors are ordered accordingly, but also typically need scaling to obtain the format of Eq. (77). Each eigenvector needs to be scaled by the appropriate complex and/or real constant(s) in order to achieve this.

ℓ	λ_ℓ	$v_{1\ell}$	$v_{2\ell}$	$v_{3\ell}$
1	$-0.293+\mathbf{i}2.403$	$0.4999+\mathbf{i}0.0000$	$-0.7070+\mathbf{i}0.0000$	$-0.4995+\mathbf{i}0.0000$
2	$-1.000+\mathbf{i}4.359$	$0.7070+\mathbf{i}0.0000$	$0.0000+\mathbf{i}0.0000$	$0.7064-\mathbf{i}0.0000$
3	$-1.707+\mathbf{i}5.588$	$0.4999+\mathbf{i}0.0000$	$0.7070-\mathbf{i}0.0000$	$-0.4995+\mathbf{i}0.0000$
4		$-0.1464+\mathbf{i}1.2011$	$0.7070-\mathbf{i}3.0819$	$0.8527-\mathbf{i}2.7912$
5		$-0.2071+\mathbf{i}1.6986$	$-0.0000+\mathbf{i}0.0000$	$-1.2058+\mathbf{i}3.9473$
6		$-0.1464+\mathbf{i}1.2011$	$-0.7070+\mathbf{i}3.0819$	$0.8527-\mathbf{i}2.7912$

Table 1: Eigenvalues and eigenvectors for damped state space example with $k/m = 10$, and $C = 0.1K$.

An example set of simulation results are shown in Fig. 3 (b). Here the exponential decay of the damped, unforced solutions can be seen for the displacements of the three masses.

3.3 Summary

Examples 8 and 9 show several important (and well known) features that are worth commenting on:

1. Both transformations have reduced the equations of motion to a series of uncoupled linear oscillators for which the exact solutions were derived in Section 1.1.1 and Section 2.
2. The exact modal solutions found in $\mathbf{q}$ coordinates can be transformed into $\mathbf{x}$ coordinates using $\mathbf{x} = \Psi\mathbf{q}$ or $\boldsymbol{x} = \Phi\boldsymbol{q}$, such that the $\mathbf{x}$ displacements can be found as the sum of the modal contributions, which is sometimes referred to as *superposition* of solutions.
3. The modal matrix is independent of time, and so it is straightforward to find the inverse relationship $\mathbf{q} = \Psi^{-1}\mathbf{x}$ or $\boldsymbol{q} = \Phi^{-1}\boldsymbol{x}$ when required (or if the eigenvectors are orthogonal $\mathbf{q} = \Psi^T\mathbf{x}$ or $\boldsymbol{q} = \Phi^T\boldsymbol{x}$). For example, if initial conditions are given in terms of the $\mathbf{x}$ need to be transformed into the modal domain.

Although these are well known results, they will be helpful in formulating an approach to nonlinear coordinate transformations in Section 4.

4 Normal form transformations of nonlinear systems

4.1 Introduction

The ultimate aim of this paper is to use normal form transformations to help understand the dynamics of multi-degree-of-freedom nonlinear oscillators[38] defined by

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} + \mathcal{N}(\mathbf{x}, \dot{\mathbf{x}}) = 0, \tag{79}$$

where M, C and K are mass, damping and stiffness matrices and the nonlinear terms $\mathcal{N}$ are assumed (in general) to be a function of the displacement and velocity vectors. Using the linear modes $\mathbf{x} = \Psi\mathbf{q}$, as defined in Section 3.1, a modal transformation can be applied to this expression which leads to

$$\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + \hat{\Lambda}\mathbf{q} + \mathbf{N}_q(\mathbf{q}, \dot{\mathbf{q}}) = 0, \tag{80}$$

where $\mathbf{N}_q(\mathbf{q}, \dot{\mathbf{q}}) = \Psi^T\mathcal{N}(\Psi\mathbf{q}, \Psi\dot{\mathbf{q}})$ and $D = \alpha I + \beta\hat{\Lambda}$ is the diagonal proportional damping matrix (see Section 3). Eq. (79) can also be written as an equivalent state space system of first-order differential equations

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \tilde{\mathcal{N}}(\boldsymbol{x}) + \boldsymbol{F},$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad \tilde{\mathcal{N}} = \begin{Bmatrix} 0 \\ -M^{-1}\mathcal{N}(\boldsymbol{x}) \end{Bmatrix},$$

[38] Note also that we do not consider normal form transformations of forced nonlinear systems in this paper. For an introduction to that topic see [85, 128].

where $\tilde{\mathcal{N}}(\boldsymbol{x})$ is obtained from $\mathcal{N}(\mathbf{x},\dot{\mathbf{x}})$ via the relationship $\boldsymbol{x}=[\mathbf{x},\dot{\mathbf{x}}]^T$. Now transforming the state space equation using the linear modal transformation $\boldsymbol{x}=\Phi\boldsymbol{q}$ (as defined in Section 3.2) leads to

$$\dot{\boldsymbol{q}}=\Lambda\boldsymbol{q}+\boldsymbol{N}(\boldsymbol{q}), \tag{81}$$

where $\boldsymbol{N}(\boldsymbol{q})=\Phi^{-1}\tilde{\mathcal{N}}(\Phi\boldsymbol{q})$.

In this Section we will introduce a general approach to multi-degree-of-freedom nonlinear oscillators, but we will limit the examples to just single degree-of-freedom for now[39]. Examples of coupled (i.e. more than one-degree-of-freedom) nonlinear oscillators are algebraically intense and lengthy, and for this reason we delay discussion of them until Section 5. To get started we discuss some of the key topics relating to normal form transformations, which are motivated with the following example.

4.1.1 Example 10: the real normal form for an undamped oscillator that has both quadratic and cubic nonlinear terms

Consider an undamped oscillator that has both quadratic and cubic nonlinear terms written as[40]

$$\ddot{q}+\omega_n^2 q+\beta q^2+\alpha q^3=0, \quad \text{or} \quad \ddot{q}+\omega_n^2 q+n^{(2)}(q)+n^{(3)}(q)=0, \tag{82}$$

where ω_n is the natural frequency, $n^{(2)}(q)=\beta q^2$ denotes the quadratic nonlinear terms and $n^{(3)}(q)=\alpha q^3$ denotes the cubic nonlinear terms. In addition, β and α are nonlinear coefficients (and in general we write $n^{(k)}(q)$ for the kth oder term). We want to find a nonlinear coordinate transformation from coordinate q to a new coordinate u, and if possible obtain a simplified system of the form

$$\ddot{u}+\omega_n^2 u+n_u^{(2)}(u)+n_u^{(3)}(u)=0, \tag{83}$$

where $n_u^{(2)}(u)$ denotes the simplified quadratic nonlinear terms and $n_u^{(3)}(u)$ denotes the simplified cubic nonlinear terms. In addition, the transformed system has linear part equivalent to

$$\begin{bmatrix}\dot{u}_p\\ \dot{u}_m\end{bmatrix}=\begin{bmatrix}\mathbf{i}\omega_n & 0\\ 0 & -\mathbf{i}\omega_n\end{bmatrix}\begin{bmatrix}u_p\\ u_m\end{bmatrix}+\mathscr{O}(2), \tag{84}$$

where $u=u_p+u_m$. Next we make a coordinate transformation of the form

$$q=u+h^{(2)}(u,\dot{u})+h^{(3)}(u,\dot{u})+.... \quad \rightsquigarrow \quad q=u_p+u_m+h^{(2)}(u_p,u_m)+h^{(3)}(u_p,u_m)+.... \tag{85}$$

where the dots are used to denote that the asymptotic series expansion has higher order terms, and we have included up to quadratic and cubic terms to match the nonlinear terms in Eq. (82)[41]. Substituting the first of Eq. (85) into the second of Eq. (82) and replacing $\ddot{u}$ using Eq. (83) gives

$$\omega_n^2 u+n_u^{(2)}+n_u^{(3)}=\frac{\mathrm{d}^2}{\mathrm{d}t^2}(h^{(2)}+h^{(3)})+\omega_n^2 u+\omega_n^2(h^{(2)}+h^{(3)})+n^{(2)}(u+h^{(2)}+h^{(3)})+n^{(3)}(u+h^{(2)}+h^{(3)}), \tag{86}$$

so that taking the u terms order by order

$$\begin{aligned}
&\text{Order } k=1: \quad \omega_n^2 u=\omega_n^2 u,\\
&\text{Order } k=2: \quad n_u^{(2)}(u)=\frac{\mathrm{d}^2 h^{(2)}(u)}{\mathrm{d}t^2}+\omega_n^2 h^{(2)}(u)+n^{(2)}(u),\\
&\text{Order } k=3: \quad n_u^{(3)}(u)=\frac{\mathrm{d}^2 h^{(3)}(u)}{\mathrm{d}t^2}+\omega_n^2 h^{(3)}(u)+n_{(3)}^{(2)}(u+...)+n^{(3)}(u+...)^{(3)},\\
&\text{Order } k=4: ...\\
&\vdots
\end{aligned} \tag{87}$$

[39] This allows us to neglect phase lag in this Section, and thus keep the examples as simple as possible. It should be noted however, that in many applications, including a phase lag may be required. We will introduce phase lag in Section 5.

[40] For single-degree-of-freedom examples in second order form there is no modal transformation so we take $x=q$ at the start.

[41] Note also that the $h^{(k)}$ terms are functions of both u and $\dot{u}$ in general, where $\dot{u}=\mathbf{i}\omega_n(u_p-u_m)$.

where the $n^{(2)}_{(3)}(u+...)$ term represents the cubic (order $k=3$ terms) generated when $q=u+h^{(2)}+...$ is substituted into the $n^{(2)}$ nonlinear terms, and $n^{(3)}(u+...)^{(3)}$ represents the cubic (order $k=3$) terms generated when $q=u+h^{(2)}+...$ is substituted into the $n^{(3)}$ term. So now we can rearrange Eq. (87) for the order $k=2$ and order $k=3$ expressions to obtain *homological equations* as

$$\begin{aligned}\text{Order } k=2: &\quad -\frac{\mathrm{d}^2h^{(2)}}{\mathrm{d}t^2}-\omega_n^2h^{(2)}=n^{(2)}-n^{(2)}_u,\\ \text{Order } k=3: &\quad -\frac{\mathrm{d}^2h^{(3)}}{\mathrm{d}t^2}-\omega_n^2h^{(3)}=n^{(2)}_{(3)}+n^{(3)}-n^{(3)}_u,\end{aligned} \tag{88}$$

which can be considered in turn.

So, let's consider the order $k=2$ homological equation first. We anticipate from previous studies that the quadratic terms in this example can be eliminated, so we first try setting $n^{(2)}_u=0$ in the $k=2$ homological equation in Eq. (88). Then we take $u=u_p+u_m$ and expand the $h^{(2)}$ term to mirror the expansion of the $n^{(2)}$ nonlinear term which gives

$$n^{(2)}(u_p,u_m)=\beta(u_p^2+2u_pu_m+u_m^2),\quad \text{and}\quad h^{(2)}(u_p,u_m)=b_1u_p^2+b_2u_pu_m+b_3u_m^2, \tag{89}$$

where the b_j are coefficients that are yet to be identified. Substituting this into the order $k=2$ homological equation with $n^{(2)}_u=0$ gives

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(b_1u_p^2+b_2u_pu_m+b_3u_m^2)-\omega_n^2(b_1u_p^2+b_2u_pu_m+b_3u_m^2)=\beta(u_p^2+2u_pu_m+u_m^2)+\mathscr{O}(3). \tag{90}$$

In order to compute the $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ derivatives we use the chain rule and expressions in Eq. (84)[42]. For example for the u_p^2 term

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(u_p^2)=\frac{\partial\left(u_p^2\right)}{\partial u_p}\dot{u}_p=2u_p\mathbf{i}\omega_nu_p=2\mathbf{i}\omega_nu_p^2,\quad \rightsquigarrow \frac{\mathrm{d}^2}{\mathrm{d}t^2}(u_p^2)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}(u_p^2)\right)=2\mathbf{i}\omega_n\frac{\partial\left(u_p^2\right)}{\partial u_p}\dot{u}_p=-4\omega_n^2u_p^2,\\ &\text{and similarly}\quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}(u_pu_m)=\frac{\partial\left(u_pu_m\right)}{\partial u_p}\dot{u}_p+\frac{\partial\left(u_pu_m\right)}{\partial u_m}\dot{u}_m=0\quad\text{and}\quad\frac{\mathrm{d}^2}{\mathrm{d}t^2}(u_m^2)=-4\omega_n^2u_m^2,\end{aligned} \tag{91}$$

which when substituted into Eq. (90) leads to

$$4b_1\omega_n^2u_p^2+4b_3\omega_n^2u_p^2-\omega_n^2b_1u_p^2-\omega_n^2b_2u_pu_m-\omega_n^2b_3u_p^2=\beta u_p^2+\beta 2u_pu_m+\beta u_m^2, \tag{92}$$

and by comparing coefficients we obtain the relationships

$$\begin{aligned}&b_13\omega_n^2=\beta\\ &-\omega_n^2b_2=2\beta\\ &b_33\omega_n^2=\beta\end{aligned}\quad\rightsquigarrow\quad\begin{aligned}b_1&=\frac{\beta}{3\omega_n^2}\\ b_2&=-2\frac{\beta}{\omega_n^2}\\ b_3&=\frac{\beta}{3\omega_n^2}\end{aligned}\quad\text{with}\quad n^{(2)}_u=0 \tag{93}$$

which confirms that the assumption of $n^{(2)}_u=0$ is correct. Substituting the coefficient values from Eq. (93) into Eq. (85) gives the coordinate transformation for linear and quadratic terms as

$$q=u_p+u_m+\frac{\beta}{3\omega_n^2}u_p^2-2\frac{\beta}{\omega_n^2}u_pu_m+\frac{\beta}{3\omega_n^2}u_m^2+\mathscr{O}(3). \tag{94}$$

Next we need to compute the $k=3$ homological equation to deal with the cubic terms. As before we take $u=u_p+u_m$ and expand the $h^{(3)}$ term to mirror the expansion of the $n^{(3)}$ nonlinear term so that

$$n^{(3)}(u_p,u_m)=\alpha(u_p^3+3u_p^2u_m+3u_pu_m^2+u_m^2),\quad\text{and}\quad h^{(3)}(u_p,u_m)=b_4u_p^3+b_5u_p^2u_m+b_6u_pu_m^2+b_7u_m^3, \tag{95}$$

[42]This will be formalised in Section 4.2 as a computation of the *Lie derivative*. Note also, that the use of Eq. (84) truncated at $\mathscr{O}(2)$ means that this is a first order approximation of the time derivatives.

where the b_j are coefficients that are yet to be identified. For the cubic terms we cannot (at this stage) assume that the $n_u^{(3)}$ is zero, and we also have to account for the $n_{(3)}^{(2)}$ in Eq. (88). Both of these terms are also expanded to mirror the $n^{(3)}$ nonlinear term which gives

$$n_{(3)}^{(2)}(u_p,u_m)=\frac{2\beta^2}{3\omega_n^2}(u_p^3-5u_p^2u_m-5u_pu_m^2+u_m^2),\quad\text{and}\quad n_u^{(3)}(u_p,u_m)=n_{u1}^*u_p^3+n_{u2}^*u_p^2u_m+n_{u3}^*u_pu_m^2+n_{u4}^*u_m^3, \tag{96}$$

where the n_{uj}^* are coefficients that are yet to be identified. Now we substitute all of these relationships into the $k=3$ homological equation from Eq. (88) to give

$$\begin{aligned}&-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(b_4u_p^3+b_5u_p^2u_m+b_6u_pu_m^2+b_7u_m^3)-\omega_n^2(b_4u_p^3+b_5u_p^2u_m+b_6u_pu_m^2+b_7u_m^3)\\&=\frac{2\beta^2}{3\omega_n^2}(u_p^3-5u_p^2u_m-5u_pu_m^2+u_m^2)+\alpha(u_p^3+3u_p^2u_m+3u_pu_m^2+u_m^2)-(n_{u1}^*u_p^3+n_{u2}^*u_p^2u_m+n_{u3}^*u_pu_m^2+n_{u4}^*u_m^3).\end{aligned} \tag{97}$$

Using the chain rule and expressions in Eq. (84) gives

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2}(b_4u_p^3+b_5u_p^2u_m+b_6u_pu_m^2+b_7u_m^3)=-9\omega_n^2b_4u_p^3-\omega_n^2b_5u_p^2u_m-\omega_n^2b_6u_pu_m^2-9\omega_n^2b_7u_m^3, \tag{98}$$

which when substituted into Eq. (97) gives

$$\begin{aligned}(8\omega_n^2)b_4u_p^3+(8\omega_n^2)b_7u_m^3=\frac{2\beta^2}{3\omega_n^2}(u_p^3-5u_p^2u_m-5u_pu_m^2+u_m^2)+\alpha(u_p^3+3u_p^2u_m+3u_pu_m^2+u_m^2)\\-(n_{u1}^*u_p^3+n_{u2}^*u_p^2u_m+n_{u3}^*u_pu_m^2+n_{u4}^*u_m^3).\end{aligned} \tag{99}$$

We want to satisfy this equation with as many of the n_{uj}^* set to zero as possible. Note that on the left-hand side of Eq. (99), the $u_p^2u_m$ and $u_pu_m^2$ no longer exists, and so on the right-hand side n_{u2}^* and n_{u3}^* cannot be zero. Conversely n_{u1}^* and n_{u4}^* can be set to zero, because the b_4 and b_7 coefficients can be chosen as required to satisfy Eq. (99). This gives relationships

$$\begin{aligned}&8\omega_n^2b_4=\frac{2\beta^2}{3\omega_n^2}+\alpha\\&\frac{-10\beta^2}{3\omega_n^2}+3\alpha-n_{u2}^*=0\\&\frac{-10\beta^2}{3\omega_n^2}+3\alpha-n_{u3}^*=0\\&8\omega_n^2b_7=\frac{2\beta^2}{3\omega_n^2}+\alpha\end{aligned}\quad\rightsquigarrow\quad\begin{aligned}&b_4=b_7=\frac{\beta^2}{12\omega_n^4}+\frac{\alpha}{8\omega_n^2}\\&n_{u2}^*=n_{u3}^*=-\frac{10\beta^2}{3\omega_n^2}+3\alpha\end{aligned} \tag{100}$$

Substituting these coefficient values into Eq. (85) gives the coordinate transformation for linear, quadratic and cubic terms as

$$q=u_p+u_m+\frac{\beta}{3\omega_n^2}u_p^2-2\frac{\beta}{\omega_n^2}u_pu_m+\frac{\beta}{3\omega_n^2}u_m^2+\left(\frac{\beta^2}{12\omega_n^4}+\frac{\alpha}{8\omega_n^2}\right)u_p^3+\left(\frac{\beta^2}{12\omega_n^4}+\frac{\alpha}{8\omega_n^2}\right)u_m^3+\mathscr{O}(4), \tag{101}$$

and substituting the n_{u2}^* and n_{u3}^* values into Eq. (83) gives

$$\ddot{u}+\omega_n^2u+\left(3\alpha-\frac{10\beta^2}{3\omega_n^2}\right)(u_p^2u_m+u_pu_m^2)=0, \tag{102}$$

or alternatively, using the fact that $u=u_p+u_m$, we can write a pair of normal form equations as

$$\begin{aligned}&\ddot{u}_p+\omega_n^2u_p+\left(3\alpha-\frac{10\beta^2}{3\omega_n^2}\right)u_p^2u_m=0,\\&\ddot{u}_m+\omega_n^2u_m+\left(3\alpha-\frac{10\beta^2}{3\omega_n^2}\right)u_pu_m^2=0.\end{aligned} \tag{103}$$ □

This example shows how a near-identity transformation can be used to eliminate the quadratic term in this type of nonlinear oscillator equation. However, for the cubic terms this was not possible and we shall develop a more complete formulation for dealing with this process in Section 4.2. We note also that:

1. We have only considered terms up to cubic order, and the expansions in Eq. (87) can be continued to higher orders. To deal with these higher order terms, order $k=4$ terms must be added to the assumed solutions in Eq. (85), which in this case would mean

$$q = u + h^{(2)}(u,\dot{u}) + h^{(3)}(u,\dot{u}) + h^{(4)}(u,\dot{u}) + ... \quad \rightsquigarrow \quad q = u_p + u_m + h^{(2)}(u_p,u_m) + h^{(3)}(u_p,u_m) + h^{(4)}(u_p,u_m) + ... \tag{104}$$

   which in turn lead to more higher order terms being generated in the expansion of the nonlinear term. This process can be undertaken as many times as required, and is one way to analyse the convergence of the series solution — see for an example [104]. However, specific proofs have been developed to show that the coefficients can be found at any order in the near identity transform providing certain conditions are met, and this will be discussed in Section 4.1.2.
2. The fact that we could solve the order $k=2$ homological equation with $n^{(2)}(u)=0$ means that there are no nonlinear resonances from the quadratic terms. However, the order $k=3$ homological equation had some $n_u^{(3)}$ terms that could not be eliminated, and these correspond to resonant terms. This will be discussed in more detail in Section 4.1.2 and methods for dealing with these terms are formalised in Section 4.2.
3. The quadratic terms in this example generated terms at cubic order (i.e. the $n_{(2)}^{(3)}$ terms) which then affected the result of the cubic order ($k=3$ order) homological equation. This is actually a generic property of these types of oscillators, and as a result, the normal form computation will need to be carried out to at least the first k odd order that appears.

As a numerical example we show the solutions for x up to $\mathscr{O}(3)$ that can be obtained directly from Eq. (94) because $x=q$ in this case. Taking the assumed solutions for u_p and u_m to be the expressions given by Eq. (22) with $\omega_r=\omega_n$ leads to

$$x = U\left(\frac{e^{i\omega_n t}+e^{-i\omega_n t}}{2}\right) + \frac{\beta}{3\omega_n^2}\frac{U^2}{2}\left(\frac{e^{2i\omega_n t}+e^{-2i\omega_n t}}{2}\right) - 2\frac{\beta}{\omega_n^2}\frac{U^2}{4} \quad \rightsquigarrow = U\cos(\omega_n t) + \frac{\beta U^2}{6\omega_n^2}\cos(2\omega_n t) - \frac{\beta U^2}{2\omega_n^2}. \tag{105}$$

Considering the case at the initial conditions when $t=0$, then

$$x(0) = U_0 - \frac{\beta U_0^2}{3\omega_n^2} \quad \rightsquigarrow \quad U_0 = \frac{3\omega_n^2}{2\beta}\left(1 - \sqrt{1 - \frac{4\beta x(0)}{3\omega_n^2}}\right). \tag{106}$$

Notice that the quadratic term, which arises from asymmetry, generates an amplitude-dependent offset from zero in the response in Eq. (105). It also always tends to create a softening nonlinear effect in the frequency-amplitude response —- a phenomena that was introduced in Example 2, see Fig. 2, in the context of backbone curves, and will also be described in Example 18. Note also, that if the cubic coefficients in the normal form are zero, then additional higher-orders should be computed in order to determine the type of nonlinear behaviour in the system. A numerical example is shown in Fig. 4 (b) where the normal form solutions from Eq. (105) with Eq. (106) is compared to a direct numerical computation of the equations of motion.

4.1.2 Nonlinear resonance

Now let us consider in more detail the resonant terms that occurred in the order $k=3$ homological equation from Example 10 (Section 4.1.1). First we assume that any terms of the type $n_{(2)}^{(3)}$ (or in general $n_{(k-1)}^{(k)}$) are combined with the $n^{(3)}$ (or in general $n^{(k)}$) on the right hand side of the order $k=3$ homological equation in Eq. (88). As a result the kth order homological equation (for a single-degree-of-freedom oscillator) can be written as

$$\text{Order } k: \quad -\frac{d^2 h^{(k)}}{dt^2} - \omega_n^2 h^{(k)} = n^{(k)} - n_u^{(k)}, \tag{107}$$

where $n^{(k)}$ is now understood to include the k order terms from order k and order $k-1$. Then we will generalise the $h^{(k)}$, $n^{(k)}$ and $n_u^{(k)}$ terms in the following way

$$h^{(k)} = \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m}, \quad n^{(k)} = \sum_{\mathbf{m}_k} n^*_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m}, \quad \text{and} \quad n_u^{(k)} = \sum_{\mathbf{m}_k} n^*_{u,\mathbf{m}_k} u_p^{m_p} u_m^{m_m}, \tag{108}$$

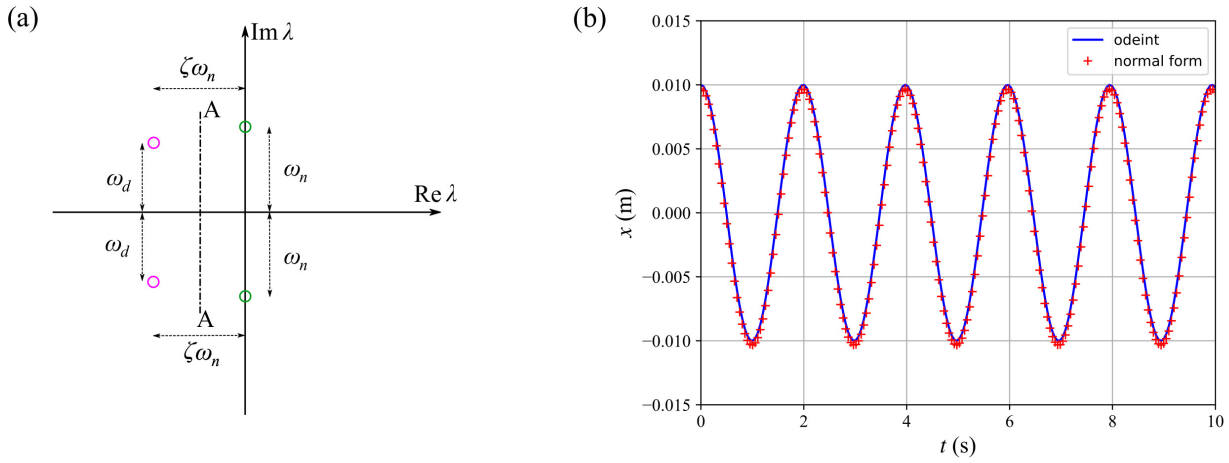

Fig. 4: (a) Eigenvalues for a damped (purple circles) and undamped (green circles) single degree-of-freedom oscillator. The straight line AA divides the damped eigenvalues (purple) from the origin and therefore these eigenvalues can be said to be in the Poincaré domain. No such line can be drawn for the undamped (green) eigenvalues, and these are therefore in the Siegel domain. (b) The solution, x, (where $x = q$) to the oscillator from Example 10 with $\alpha = 0$. The solution is computed using Eq. (105) with Eq. (106) and the result shown with red crosses. This normal form solution is compared with a reference solution generated from a fourth-order Runge-Kutta integration (the 'odeint' numerical integration routine from the Python numerical library — blue solid line). Parameter values are: $x(0) = 0.01$, $\omega_n = \sqrt{10}$ rad/s and $\beta = 1$ N/(kgm^2), $\alpha = 0$.

where $\mathbf{m}_k$ is based on a vector notation for multi-indices such that $\mathbf{m}_k = (m_p, m_m)$, where m_p and m_m are whole numbers in the range $0 \leq (m_p, m_m) \leq k$ with the additional condition that $m_p + m_m = k$ and the sum in (108) is over the set of all such indices [129]. Examples of $\mathbf{m}_k$ and $h^{(k)}$ for $k = 2, 3, 4$ are given in Table 2 along with the associated vector $\mathbf{u}^*$ which we will use later in the generalisations of this process.

k	$\mathbf{m}_k$	$h^{(k)}$	$\mathbf{u}^*$	L_k
2	$(2,0)$, $(1,1)$, $(0,2)$	$b_{(2,0)}u_p^2 + b_{(1,1)}u_p u_m + b_{(0,2)}u_m^2$	$[u_p^2 \;\; u_p u_m \;\; u_m^2]^T$	3
3	$(3,0)$, $(2,1)$, $(1,2)$ $(0,3)$	$b_{(3,0)}u_p^3 + b_{(2,1)}u_p^2 u_m + b_{(1,2)}u_p u_m^2 + b_{(0,3)}u_m^3$	$[u_p^3 \;\; u_p^2 u_m \;\; u_p u_m^2 \;\; u_m^3]^T$	4
4	$(4,0)$, $(3,1)$, $(2,2)$, $(1,3)$ $(0,4)$	$b_{(4,0)}u_p^4 + b_{(3,1)}u_p^3 u_m^2 + b_{(2,2)}u_p^2 u_m^2 + b_{(1,3)}u_p^2 u_m^3 + b_{(0,4)}u_m^4$	$[u_p^4 \;\; u_p^3 u_m^2 \;\; u_p^2 u_m^2 \;\; u_p^2 u_m^3 \;\; u_m^4]^T$	5

Table 2: Examples of $\mathbf{m}_k$, $h^{(k)}$, and $\mathbf{u}^*$ for $k = 2, 3, 4$. Note that we will only use this notation for the b_j coefficients in this subsection. The $\mathbf{m}_k$ notation is useful to motivate the discussion, and show how the coefficients relate to the $\mathbf{u}^*$ vector terms, but is too complex for use in general. L_k is the length of the $\mathbf{u}^*$ vector.

If we substitute the relationships in Eq. (108) into Eq. (107), then the order k homological equation becomes

$$-\sum_{\mathbf{m}_k} b_{\mathbf{m}_k} \frac{\mathrm{d}^2 u_p^{m_p} u_m^{m_m}}{\mathrm{d}t^2} - \omega_n^2 \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m} = \sum_{\mathbf{m}_k} n^*_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m} - \sum_{\mathbf{m}_k} n^*_{u,\mathbf{m}_k} u_p^{m_p} u_m^{m_m}. \tag{109}$$

Then, we need to evaluate the $\frac{\mathrm{d}^2 u_p^{m_p} u_m^{m_m}}{\mathrm{d}t^2}$ derivative term using the chain rule and expressions in Eq. (84), as we did for Example 10, only this time we write these latter relationships as $\dot{u}_p = \lambda_p u_p$ and $\dot{u}_m = \lambda_m u_m$, where λ_p and λ_m are the eigenvalues, which for this example are $\lambda_p = \mathrm{i}\omega_n$ and $\lambda_m = -\mathrm{i}\omega_n$. Using these relationships, Eq. (109) becomes

$$
\begin{aligned}
-\sum_{\mathbf{m}_k} b_{\mathbf{m}_k} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u_p^{m_p}u_m^{m_m}}{\mathrm{d}t}\right) - \omega_n^2 \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m} &= -\sum_{\mathbf{m}_k} b_{\mathbf{m}_k} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u_p^{m_p}u_m^{m_m}}{\partial u_p}\dot{u}_p + \frac{\partial u_p^{m_p}u_m^{m_m}}{\partial u_m}\dot{u}_m\right) - \omega_n^2 \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m} \\
&= -\sum_{\mathbf{m}_k} b_{\mathbf{m}_k} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u_p^{m_p}u_m^{m_m}}{\partial u_p}\lambda_p u_p + \frac{\partial u_p^{m_p}u_m^{m_m}}{\partial u_m}\lambda_m u_m\right) - \omega_n^2 \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m} \\
&= -\sum_{\mathbf{m}_k} b_{\mathbf{m}_k} \frac{\mathrm{d}}{\mathrm{d}t}\left([m_p\lambda_p + m_m\lambda_m]u_p^{m_p}u_m^{m_m}\right) - \omega_n^2 \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m} \\
&= -\sum_{\mathbf{m}_k} b_{\mathbf{m}_k} \left([m_p\lambda_p + m_m\lambda_m]^2 u_p^{m_p}u_m^{m_m}\right) - \omega_n^2 \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} u_p^{m_p} u_m^{m_m} \\
&= \sum_{\mathbf{m}_k} b_{\mathbf{m}_k} \left(-[m_p\lambda_p + m_m\lambda_m]^2 - \omega_n^2\right) u_p^{m_p}u_m^{m_m} \\
&= \sum_{\mathbf{m}_k} n^*_{\mathbf{m}_k} u_p^{m_p}u_m^{m_m} - \sum_{\mathbf{m}_k} n^*_{u,\mathbf{m}_k} u_p^{m_p}u_m^{m_m}.
\end{aligned}
\tag{110}
$$

From this we can see that the expression $(-[m_p\lambda_p + m_m\lambda_m]^2 - \omega_n^2)$ plays an important role in relation to the homological equation. Considering each of the $\mathbf{m}_k$ terms we can set $n^*_{u,\mathbf{m}_k} = 0$ and solve for

$$
b_{\mathbf{m}_k} = \frac{n^*_{\mathbf{m}_k}}{(-[m_p\lambda_p + m_m\lambda_m]^2 - \omega_n^2)} \quad \text{for} \quad (-[m_p\lambda_p + m_m\lambda_m]^2 - \omega_n^2) \neq 0, \tag{111}
$$

and because in this example, $\lambda_p^2 = \lambda_m^2 = -\omega_n^2$, the relationship becomes

$$
b_{\mathbf{m}_k} = \frac{n^*_{\mathbf{m}_k}}{([m_p - m_m]^2 - 1)\omega_n^2} \quad \text{for} \quad ([m_p - m_m]^2 - 1) \neq 0. \tag{112}
$$

In the case when $(-[m_p\lambda_p + m_m\lambda_m]^2 - \omega_n^2) = 0$ then the corresponding $n^*_{u,\mathbf{m}_k}$ coefficient in Eq. (110) cannot be set to zero, and the term is said to be resonant. So for the resonant case we have

$$
\begin{aligned}
-[m_p\lambda_p + m_m\lambda_m]^2 - \omega_n^2 = 0 \quad &\rightsquigarrow \quad m_p\lambda_p + m_m\lambda_m = +\mathbf{i}\omega_n \quad \text{and} \quad m_p\lambda_p + m_m\lambda_m = -\mathbf{i}\omega_n \\
&\rightsquigarrow \quad m_p\mathbf{i}\omega_n - m_m\mathbf{i}\omega_n = \mathbf{i}\omega_n \quad \text{and} \quad m_p\mathbf{i}\omega_n - m_m\mathbf{i}\omega_n = -\mathbf{i}\omega_n,
\end{aligned}
\tag{113}
$$

which for the cubic terms in Example 10 is true when $\mathbf{m}_k = (2,1)$ and $\mathbf{m}_k = (1,2)$. Notice from Table 2, Eq. (112) and Eq. (113) that $m_p - m_m = \pm 1$ results in a resonant terms for the types of nonlinear examples considered here. As a result, by inspecting Table 2 it can be noticed that the k even nonlinearities will not have resonant terms of this type, whereas the k odd terms will have them.

Notice also that the key relationship in Eq. (113) depends only on the eigenvalues of the system. More formally, undamped oscillator equations that have pairs of complex conjugate eigenvalues that lie on the imaginary axis, and are said to be in the *Siegel domain* [4, 120, 85] (defined by the fact that no straight line can be drawn which divides the eigenvalues from the origin — see Fig. 4 (a)). On the contrary, eigenvalues for damped oscillators, that typically both have negative real parts, are said to be in the *Poincaré domain*, which is a case that will be considered in more detail in Section 4.5. An example showing both cases is shown schematically in Fig. 4 (a).

For eigenvalues in the Poincaré domain, there are only a finite number of resonant terms, and theories developed by Poincaré and Dulac give specific conditions for the existence of near-identity transformation in this case — see [4, 16, 85] for details. For eigenvalues in the Siegel domain, the situation is more complex, because there are an infinite number of resonant terms. In Example 10, for example the eigenvalues are $\lambda_{p,m} = \pm\mathbf{i}\omega_n$, and then the following relationship holds [85]

$$
\lambda_p = (n+1)\lambda_p + n\lambda_m \quad \text{for} \quad n = 1,2,3,4.... \tag{114}
$$

which means there are an infinite number of potential resonant combinations, and therefore potential resonant terms, and these are defined as *trivial* resonance terms in [169, 172] and *unconditionally* resonant terms in [183]. Siegel and others (discussions can be found in [4, 16, 85]), have studied this case in detail and present

an number of ways of developing appropriate near-identity transformations[43]. In this work, we consider only systems relevant to structural dynamics which are essentially “well behaved" and so the behaviour of the asymptotic series expansions is assumed to be a reasonable approximation to the complete solution throughout the paper[44]. That said, the trivial resonances serve an important role in the normal form process for undamped oscillators, as they highlight which terms cannot be removed by the transformation (and therefore they are unconditional in the sense that they do not need a specific condition to be true in order to occur).

This is in contrast to the concept of a nonlinear resonant phenomena called *internal resonance* which occurs for systems with more than a single degree-of-freedom. For example consider a system with N degrees-of-freedom, and N pairs of complex conjugate eigenvalues λ_{pj}, λ_{mj} for $j = 1,2,3...,N$. Then an internal resonance[45] exists for an eigenvalue, $\lambda_{p\ell}$, if

$$\lambda_{p\ell} = r_{p1}\lambda_{p1} + r_{p2}\lambda_{p2} + r_{p3}\lambda_{p3} + ... \quad \rightsquigarrow \quad \lambda_{\ell} = \sum_{j=1}^{N} r_{pj}\lambda_{pj} \quad \text{for} \quad j = 1,2,3...,N, \tag{115}$$

where r_{pj} are integers. The r_{pj} values define the condition on the righthand side of Eq. (115) for this type of resonance to occur, and therefore this can also be called a *conditional* resonance.

Nonlinear resonance is a key topic for normal form transformations, and we will show multiple examples in the rest of this paper. Next however, we consider a subtle but important point regarding the notation in normal form transformations.

4.1.3 k-order and ε-order notation in normal form transformations

So far in introducing the method of normal forms for nonlinear oscillator systems we have shown how the analysis can be separated into k-order homological equations, and then solved one order after another until the required level of accuracy is obtained for the specific problem being considered [104]. However, there is an alternative form of notation used in the normal form literature, which we will call the ε-order notation for normal form transformations. This notation is used in a variety of ways including:

1. In exactly the same way as the k-order notation, to keep track of the polynomial nonlinear terms. In this case ε is often called a *book-keeping* parameter, and will often be attributed the value $\varepsilon = 1$.
2. For cases when the nonlinear terms are not separated into an ascending sum of polynomial nonlinear terms. Here ε-order notation can be used instead. As with point 1. in this case ε is often called a *book-keeping* parameter.
3. As part of a series expansion used in the normal form method, as will be used in the method of Lie series in Section 4.3. This has been used in particular in connection with the Hamiltonian normal form, first introduced by Birkhoff [11]. The use of the ε-order notation in this context goes back at least as far as the work of Hori [74]. In this case ε is a Lie series expansion parameter and is generally considered to be small $\varepsilon \ll 1$.
4. Within a Taylor series expansion when dealing with nonlinear functions that are not already in the form of polynomial nonlinearities [183]. In this case ε is a Taylor series expansion parameter and is generally considered to be small $\varepsilon \ll 1$.
5. Related to points 3. and 4. normal form methods can also be formulated in terms of perturbation theory (see for example Nayfeh [124]), where ε is the perturbation parameter and is generally considered to be small $\varepsilon \ll 1$.
6. Related to point 5., in some normal form analysis, ε has been used to link the size of the nonlinear term to the size the viscous damping and external forcing terms, which are all assumed to be small [183]. This is a case of practical importance in structural dynamics, where large resonant responses can occur, for small nonlinearity, damping and forcing [183].
7. To include parameter variation close to the equilibrium point around which the normal form expansion is being carried out. This will be discussed further in Section 4.6.

[43] These and other authors also discuss the use of Jordan normal form for the different eigenvalue cases including those with degenerate behaviour [4, 16, 120]. This is beyond the scope of this current paper.

[44] Formally we consider only systems that are diagonalisable with full rank diagonal matrices of distinct eigenvalues. Often referred to as the semisimple case [4, 120]. For cases where this assumption regarding asymptotic series expansions does not necessarily hold, see Murdock [120]. For an example of the non-semisimple case see [43].

[45] There are also other names for this phenomena, such as "modal coupling", and "autoparametric resonance", see for example [164]. In terms of recent applications the area of micro- and nano-mechanical structures offers some very nice examples [157, 31].

Murdock [120] gives a clear explanation of the difference between the k-order and ε-order notation in terms of five specific formats. The k-order notation arises primarily in the so-called *direct* formats (e.g. Example 10, Section 4.1.1), and the ε-order notation primarily in the *generated* formats (as will be described in Section 4.3). As noted by Murdock, in some methods, such as those by Hori [74] and Deprit [34], ε is treated as an independent variable, and so cannot be eliminated. In order to try and retain 'the best of both worlds' we will develop a generalised theory of near-identity transformations based on ε-order notation, that can in certain circumstances use $\varepsilon = 1$ and revert to the k-order approach. Next we consider the near-identity transformation process in more detail.

4.2 Near-identity transformations of differential equations

In this Section we consider near-identity transformations for both first-order and second-order sets of differential equations. It is helpful to maintain the distinction in variables between these two classes of differential equation. So we introduce the following notation: FO will denote the case for first-order sets of differential equations and SO will denote the case for second-order sets of differential equations. In both cases, t is the scalar independent variable and remains untransformed (or its transform is the identity). In addition for the SO case $\mathbf{q} = \mathbf{q}(t)$ is an $N \times 1$ vector containing the dependent (displacement) variables, while $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ are $N \times 1$ vectors containing the velocities and accelerations, respectively. For the FO case, $\boldsymbol{q} = [\mathbf{q}, \dot{\mathbf{q}}]^T$ is the $2N \times 1$ state vector, and $\dot{\boldsymbol{q}}$ is the time derivative of the state vector.

The functional form of the near-identity transformation will be defined as the following two mappings

$$\begin{aligned} &T_{FO} : (t, \boldsymbol{q}) \to (t, \boldsymbol{u}) : \qquad t = t \qquad \text{and} \qquad \boldsymbol{q} = \boldsymbol{T}(\boldsymbol{u}, \varepsilon) = \boldsymbol{u} + \varepsilon \boldsymbol{h}_1(\boldsymbol{u}) + \varepsilon^2 \boldsymbol{h}_2(\boldsymbol{u}) + \ldots \\ &T_{SO} : (t, \mathbf{q}) \to (t, \mathbf{u}) : \qquad t = t \qquad \text{and} \qquad \mathbf{q} = \mathbf{T}(\mathbf{u}, \dot{\mathbf{u}}, \varepsilon) = \mathbf{u} + \varepsilon \mathbf{h}_1(\mathbf{u}, \dot{\mathbf{u}}) + \varepsilon^2 \mathbf{h}_2(\mathbf{u}, \dot{\mathbf{u}}) + \ldots \end{aligned} \tag{116}$$

Notice also that $\mathbf{h}_j$ is assumed in general to be a function of both $\mathbf{u}$ and $\dot{\mathbf{u}}$[46]. The nonlinear functions $\boldsymbol{h}_j(\boldsymbol{u})$ and $\mathbf{h}_j(\mathbf{u}, \dot{\mathbf{u}})$ are defined as

$$\boldsymbol{h}_1 = \boldsymbol{b}_1 \mathbf{u}^*, \quad \boldsymbol{h}_2 = \boldsymbol{b}_2 \mathbf{u}^+ \qquad \text{and} \qquad \mathbf{h}_1 = \mathbf{b}_1 \mathbf{u}^*, \quad \mathbf{h}_2 = \mathbf{b}_2 \mathbf{u}^+ \tag{117}$$

where $\boldsymbol{b}_1$ is an $2N \times L_k$ coefficient matrix, $\mathbf{b}_1$ is an $N \times L_k$ coefficient matrix, and $\mathbf{u}^*$ is an $L_k \times 1$ vector of polynomial functions of the elements of $\boldsymbol{u}$ or $\mathbf{u}$, which for a single-degree-of-freedom oscillator are u_p and u_m as can be seen in Table 2. Similarly, $\boldsymbol{b}_2$ is an $2N \times L_k$ coefficient matrix, $\mathbf{b}_2$ is an $N \times L_k$ coefficient matrix, and $\mathbf{u}^+$ is an $L_k \times 1$ vector of polynomial functions of the elements of $\boldsymbol{u}$ or $\mathbf{u}$. For example, if we consider the oscillator system from Example 10 and rewrite the near-identity transformation in the form of Eq. (116), then Eq. (101) becomes

$$\begin{aligned} q = u_p + u_m + \left[\frac{\beta}{3\omega_n^2} \quad -2\frac{\beta}{\omega_n^2} \quad \frac{\beta}{3\omega_n^2}\right] \begin{bmatrix} u_p^2 \\ u_p u_m \\ u_m^2 \end{bmatrix} + \left[\left(\frac{\beta^2}{12\omega_n^4} + \frac{\alpha}{8\omega_n^2}\right) \;\; 0 \;\; 0 \;\; \left(\frac{\beta^2}{12\omega_n^4} + \frac{\alpha}{8\omega_n^2}\right)\right] \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \end{bmatrix} + \ldots, \\ \text{so taking} \quad \mathbf{b}_1 = \left[\frac{\beta}{3\omega_n^2} \quad -2\frac{\beta}{\omega_n^2} \quad \frac{\beta}{3\omega_n^2}\right], \quad \text{and} \quad \mathbf{b}_2 = \left[\left(\frac{\beta^2}{12\omega_n^4} + \frac{\alpha}{8\omega_n^2}\right) \;\; 0 \;\; 0 \;\; \left(\frac{\beta^2}{12\omega_n^4} + \frac{\alpha}{8\omega_n^2}\right)\right] \\ \rightsquigarrow q = u + \mathbf{b}_1 \mathbf{u}^* + \mathbf{b}_2 \mathbf{u}^+ + \ldots \quad \text{and with } \varepsilon \text{ notation} \quad q = u + \varepsilon \mathbf{b}_1 \mathbf{u}^* + \varepsilon^2 \mathbf{b}_2 \mathbf{u}^+ + \ldots \end{aligned} \tag{118}$$

where $u = u_p + u_m$, $N = 1$, $L_{k=2} = 3$, $L_{k=3} = 4$ and $\varepsilon = 1$.

4.2.1 First-order equations

First let us consider first-order differential equations of the form

$$\dot{\boldsymbol{q}} = \mathscr{F}(\boldsymbol{q}, \varepsilon) = \Lambda \boldsymbol{q} + \boldsymbol{N}(\boldsymbol{q}) = \Lambda \boldsymbol{q} + \varepsilon \boldsymbol{n}_1(\boldsymbol{q}) + \varepsilon^2 \boldsymbol{n}_2(\boldsymbol{q}) + \ldots \tag{119}$$

[46] Of course the state vector already includes the velocities because $\boldsymbol{q} = [\mathbf{q}, \dot{\mathbf{q}}]^T$, so this comment applies only to second-order sets of differential equations.

where $\mathscr{F}(\cdot)$ represents the general expression for the nonlinear function of $\boldsymbol{q}$, Λ is a diagonal matrix containing the system eigenvalues, $\boldsymbol{N}(\boldsymbol{q})$ contains the nonlinear terms (i.e. as defined in Eq. (81)), which are approximated by the $\boldsymbol{n}_j(\boldsymbol{q})$ terms. Define the structure of the transformed differential equation to be

$$\dot{\boldsymbol{u}} = \mathscr{G}(\boldsymbol{u},\varepsilon) = \Lambda\boldsymbol{u} + \varepsilon\boldsymbol{n}_{u1}(\boldsymbol{u}) + \varepsilon^2\boldsymbol{n}_{u2}(\boldsymbol{u}) + ..., \tag{120}$$

where $\mathscr{G}(\cdot)$ is the transformed nonlinear expression, and $\boldsymbol{n}_{uj}$ are the transformed vectors of nonlinear terms. Here the objective of the transformation process is to simplify, as much as possible, the transformed vectors of nonlinear terms $\boldsymbol{n}_{uj}$. The standard approach to this problem is to substitute (the upper equation in) Eq. (116) into Eq. (119) to give

$$\dot{\boldsymbol{u}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\boldsymbol{h}_1 + \varepsilon^2\boldsymbol{h}_2 + ...\right) = \Lambda(\boldsymbol{u} + \varepsilon\boldsymbol{h}_1 + \varepsilon^2\boldsymbol{h}_2 + ...) + \varepsilon\boldsymbol{n}_1(\boldsymbol{u} + \varepsilon\boldsymbol{h}_1 + \varepsilon^2\boldsymbol{h}_2 + ...) + \varepsilon^2\boldsymbol{n}_2(\boldsymbol{u} + \varepsilon\boldsymbol{h}_1 + \varepsilon^2\boldsymbol{h}_2 + ...)... \tag{121}$$

and by substituting for $\dot{\boldsymbol{u}}$ using Eq. (120) gives

$$\begin{aligned}\Lambda\boldsymbol{u} + \varepsilon\boldsymbol{n}_{u1}(\boldsymbol{u}) + \varepsilon^2\boldsymbol{n}_{u2}(\boldsymbol{u}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon\boldsymbol{h}_1 + \varepsilon^2\boldsymbol{h}_2 + ...\right) = \\ \Lambda\boldsymbol{u} + \varepsilon\Lambda\boldsymbol{h}_1 + \varepsilon^2\Lambda\boldsymbol{h}_2 + \varepsilon\boldsymbol{n}_1(\boldsymbol{u} + \varepsilon\boldsymbol{h}_1 + \varepsilon^2\boldsymbol{h}_2 + ...) + \varepsilon^2\boldsymbol{n}_2(\boldsymbol{u} + \varepsilon\boldsymbol{h}_1 + \varepsilon^2\boldsymbol{h}_2 + ...)...\end{aligned} \tag{122}$$

For cases when $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are relatively simple, the substituted terms in Eq. (122) can be expanded directly. More generally the nonlinear terms $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ can be expanded as Taylor series using

$$\boldsymbol{n}_i(\boldsymbol{u} + \varepsilon\boldsymbol{h}_1(\boldsymbol{u}) + \dots) = \boldsymbol{n}_i(\boldsymbol{u}) + \varepsilon D\{\boldsymbol{n}_i(\boldsymbol{u})\}\boldsymbol{h}_1(\boldsymbol{u}) + \dots, \tag{123}$$

where $D\{\boldsymbol{n}_i\}$ is the Jacobian of $\boldsymbol{n}_i$. Using this Taylor series expansion and then equating ε terms in Eq. (122) gives

$$\varepsilon^0: \quad \Lambda\boldsymbol{u} = \Lambda\boldsymbol{u}, \tag{124}$$

$$\varepsilon^1: \quad \frac{\mathrm{d}\boldsymbol{h}_1}{\mathrm{d}t} - \Lambda\boldsymbol{h}_1 = \boldsymbol{n}_1 - \boldsymbol{n}_{u1}, \tag{125}$$

$$\varepsilon^2: \quad \frac{\mathrm{d}\boldsymbol{h}_2}{\mathrm{d}t} - \Lambda\boldsymbol{h}_2 = D\{\boldsymbol{n}_1\}\boldsymbol{h}_1 + \boldsymbol{n}_2 - \boldsymbol{n}_{u2}, \tag{126}$$

To deal with the time derivatives in Eq. (125) and Eq. (126) we use the Lie derivative. For example, consider computing the Lie derivate of the term $\boldsymbol{h}_1 = \boldsymbol{b}_1\mathbf{u}^*$ in Eq. (125) to order ε^0. First $\boldsymbol{b}_1$ is a matrix of constants, so we are interested only in the Lie derivative of $\mathbf{u}^*$, which is given by

$$\left.\frac{\mathrm{d}\mathbf{u}^*}{\mathrm{d}t}\right|_{\varepsilon=0} = \left.\frac{\partial\mathbf{u}^*}{\partial\boldsymbol{u}}\dot{\boldsymbol{u}}\right|_{\varepsilon=0} = \frac{\partial\mathbf{u}^*}{\partial\boldsymbol{u}}\Lambda\boldsymbol{u} \equiv \Lambda^*\mathbf{u}^*, \tag{127}$$

where Λ^* is a constant $L_k \times L_k$ diagonal matrix and L_k is the length of the vector $\mathbf{u}^*$. Substituting this, and relations for $\boldsymbol{h}_1$ from Eq. (117) into the homological equations Eq. (125) and Eq. (126) gives

$$\varepsilon^1: \quad \frac{\partial\boldsymbol{h}_1}{\partial\boldsymbol{u}}\Lambda\boldsymbol{u} - \Lambda\boldsymbol{h}_1 = \boldsymbol{n}_1 - \boldsymbol{n}_{u1}, \qquad \rightsquigarrow \qquad \boldsymbol{b}_1\Lambda^*\mathbf{u}^* - \Lambda\boldsymbol{b}_1\mathbf{u}^* = \boldsymbol{n}_1 - \boldsymbol{n}_{u1}, \tag{128}$$

$$\varepsilon^2: \quad \frac{\partial\boldsymbol{h}_2}{\partial\boldsymbol{u}}\Lambda\boldsymbol{u} - \Lambda\boldsymbol{h}_2 = D\{\boldsymbol{n}_1\}\boldsymbol{h}_1 + \boldsymbol{n}_2 - \boldsymbol{n}_{u2}, \qquad \rightsquigarrow \qquad \boldsymbol{b}_2\Lambda^+\mathbf{u}^+ - \Lambda\boldsymbol{b}_2\mathbf{u}^+ = D\{\boldsymbol{n}_1\}\boldsymbol{b}_1\mathbf{u}^* + \boldsymbol{n}_2 - \boldsymbol{n}_{u2}, \tag{129}$$

where Λ^+ is a diagonal matrix obtained from taking the Lie derivative of $\mathbf{u}^+$ using Eq. (127), and we have truncated the expressions at order-ε^2.

These relationships are called the order-ε^1 and order-ε^2 *homological equations* (without frequency detuning which will be added in Section 4.6). Notice that the left-hand side of the homological equations in Eq. (128) and Eq. (129) can be reformulated as a Lie bracket for the two vector fields involved in the transformation process such that

$$\frac{\partial\boldsymbol{h}_\ell}{\partial\boldsymbol{u}}\mathscr{G}_0 - \frac{\partial\mathscr{G}_0}{\partial\boldsymbol{u}}\boldsymbol{h}_\ell = [\boldsymbol{h}_\ell, \mathscr{G}_0], \quad \text{where} \quad \mathscr{G}_0 = \Lambda\boldsymbol{u} = \dot{\boldsymbol{u}}|_{\varepsilon=0} \quad \text{for} \quad \ell = 1,2,3... \tag{130}$$

where $[\cdot,\cdot]$ indicates the bracketing operation, and it should be understood that as all functions in Eq. (130) are functions of $\boldsymbol{u}$, the bracket is an operator $[\cdot,\cdot](\boldsymbol{u})$. As a result for the order-ε^1 homological operator we can write[47]

$$[\boldsymbol{h}_1,\mathscr{G}_0]=\boldsymbol{n}-\boldsymbol{n}_u=\mathscr{H}_1. \tag{131}$$

Notice also the similarity of the structure of these homological equations and those derived in Example 10, particularly in Eq. (88), although is should be noted that in this Section we are dealing with first-order equations. A direct comparison can be made with the results in Section 4.3.2. Next we consider an example of how to compute the order-ε^1 homological equation in practice.

It is important to note, that in the next example we use only the order-ε^1 homological equation because the example contains only a single odd polynomial nonlinear term. As we saw in Example 10, if there are both even and odd polynomial terms, the even terms will lead to terms at the next order, and so in that case the order-ε^2 homological equation also needs to be included in the analysis. An example of this type will be considered in Section 4.8.1.

4.2.2 Example 11: the undamped, unforced Duffing oscillator

The undamped, unforced Duffing oscillator can be written as

$$\ddot{x}+\omega_n^2 x+\varepsilon\alpha x^3=0, \tag{132}$$

where ω_n is the linear natural frequency, ε is the book-keeping parameter and α is a nonlinear coefficient.

For the classical method of treating the Duffing oscillator, a first-order representation of Eq. (132) is defined in terms of the state vector $\boldsymbol{x}=\{x_1\ \ x_2\}^T=\{x\ \ \dot{x}\}^T$. Then the matrix and vector form of Eq. (132) is given by

$$\dot{\boldsymbol{x}}=A\boldsymbol{x}+\tilde{\mathscr{N}}(\boldsymbol{x}):\quad A=\begin{bmatrix}0 & 1\\ -\omega_n^2 & 0\end{bmatrix},\quad \tilde{\mathscr{N}}(\boldsymbol{x})=\begin{Bmatrix}0\\ -\varepsilon\alpha x_1^3\end{Bmatrix}.$$

Transforming using the diagonalising modal transform from Eq. (75), where $\boldsymbol{x}=\Phi\boldsymbol{q}$ can be used to give an equation of the form of Eq. (119) as

$$\dot{\boldsymbol{q}}=\Lambda\boldsymbol{q}+\varepsilon\boldsymbol{n}_1(\boldsymbol{q}):\quad \Lambda=\begin{bmatrix}\mathbf{i}\omega_n & 0\\ 0 & -\mathbf{i}\omega_n\end{bmatrix},\quad \Phi=\begin{bmatrix}1 & 1\\ \mathbf{i}\omega_n & -\mathbf{i}\omega_n\end{bmatrix},\quad \Phi^{-1}=\frac{1}{-2\mathbf{i}\omega_n}\begin{bmatrix}-\mathbf{i}\omega_n & -1\\ -\mathbf{i}\omega_n & 1\end{bmatrix}, \tag{133}$$

where the nonlinear function $\boldsymbol{n}_1(\boldsymbol{q})=\Phi^{-1}\tilde{\mathscr{N}}(\Phi\boldsymbol{q})$, and $\boldsymbol{q}=[q_p\ \ q_m]^T$. Note there is just one nonlinear term, and so we set $\boldsymbol{n}_2=0,\ldots$ etc. and then as there is just a single term we set $\boldsymbol{n}_1(\boldsymbol{q})=\boldsymbol{n}(\boldsymbol{q})$ which can be expressed as

$$\boldsymbol{n}(\boldsymbol{q})=\Phi^{-1}\tilde{\mathscr{N}}(\Phi\boldsymbol{q})=\frac{\varepsilon\alpha}{2\mathbf{i}\omega_n}\begin{bmatrix}-(q_p+q_m)^3\\ (q_p+q_m)^3\end{bmatrix}=\frac{\alpha}{2\mathbf{i}\omega_n}\begin{bmatrix}-1 & -3 & -3 & -1\\ 1 & 3 & 3 & 1\end{bmatrix}\begin{bmatrix}q_p^3\\ q_p^2q_m\\ q_pq_m^2\\ q_m^3\end{bmatrix}=\boldsymbol{n}^*\mathbf{q}^*, \tag{134}$$

where

$$\boldsymbol{n}^*=\frac{\alpha}{2\mathbf{i}\omega_n}\begin{bmatrix}-1 & -3 & -3 & -1\\ 1 & 3 & 3 & 1\end{bmatrix},\quad \text{and}\quad \mathbf{q}^*=\begin{bmatrix}q_p^3\\ q_p^2q_m\\ q_pq_m^2\\ q_m^3\end{bmatrix}. \tag{135}$$

[47] It can be shown that because of the structure of the basis functions in $\mathbf{u}^*$, the Lie derivative of $\mathbf{u}^*$ is a constant matrix times $\mathbf{u}^*$. In other words, the derivative of $\mathbf{u}^*$ with respect to time has the same characteristics as the exponential function, which is of course the underlying solution to all these differential equations. It is also important to realise that this is the classical Lie derivative of a scalar function along the vector field $\mathscr{G}_0$. In fact, we use the format defined by [151], where there are a set of scalar functions, $u_1^*,u_2^*,u_3^*,\ldots,u_L^*$, grouped together in a single vector $\mathbf{u}^*$.

Substituting the near-identity transformation, Eq. (116), into this expression is required as part of the process in deriving the homological equations, as we described above. However, if the analysis is carried out retaining only terms up to order ε^1 this will result in

$$\varepsilon \boldsymbol{n}(\boldsymbol{q}) \to \varepsilon \boldsymbol{n}(\boldsymbol{u} + \varepsilon \boldsymbol{h}_1 + ...) = \varepsilon \boldsymbol{n}(\boldsymbol{u}) + \mathscr{O}(\varepsilon^2),$$

and therefore to order ε^1

$$\boldsymbol{n}(\boldsymbol{q}) = \boldsymbol{n}^* \mathbf{q}^* \to \boldsymbol{n}^* \mathbf{u}^* \quad \text{where for this example} \quad \mathbf{u}^* = \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \end{bmatrix}, \tag{136}$$

and $\boldsymbol{n}^*$ represents the coefficient matrix and $\mathbf{u}^*$ is the vector of nonlinear basis terms.

Now the order-ε^1 homological equation, (Eq. (128)) can most easily be evaluated if the structure of $\boldsymbol{n}_u$ (from Eq. (120) with $\boldsymbol{n}_u = \boldsymbol{n}_{u1}$) and $\boldsymbol{h}$ (from Eq. (117) with $\boldsymbol{h} = \boldsymbol{h}_1$) are set to be the same as $\boldsymbol{n}$, which in turn is defined by the problem at hand. Therefore we set $\boldsymbol{h}(\boldsymbol{u}) = \boldsymbol{b}\mathbf{u}^*$ and $\boldsymbol{n}_u(\boldsymbol{u}) = \boldsymbol{n}_u^* \mathbf{u}^*$, where $\boldsymbol{b}$ and $\boldsymbol{n}_u^*$ are coefficient matrices of the same dimensions as $\boldsymbol{n}^*$. In fact, as $\mathbf{u}^*$ follows the structure of the nonlinear terms in $\boldsymbol{n}(\boldsymbol{q})$, the coefficient matrices $\boldsymbol{b}$ and $\boldsymbol{n}_u^*$ are the only remaining unknowns that are to be determined from the homological equation. Having defined $\boldsymbol{h}(\boldsymbol{u})$ we see that $\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}} = \boldsymbol{b}\frac{\partial \mathbf{u}^*}{\partial \boldsymbol{u}}$ and in this example $\boldsymbol{u} = [u_p \;\; u_m]^T$, then[48]

$$\frac{\partial \mathbf{u}^*}{\partial \boldsymbol{u}} = \begin{bmatrix} 3u_p^2 & 0 \\ 2u_p u_m & u_p^2 \\ u_m^2 & 2u_p u_m \\ 0 & 3u_m^2 \end{bmatrix}, \quad \text{and} \quad \frac{\partial \mathbf{u}^*}{\partial \mathbf{u}} \Lambda \mathbf{u} \equiv \begin{bmatrix} 3\mathbf{i}\omega_n & 0 & 0 & 0 \\ 0 & \mathbf{i}\omega_n & 0 & 0 \\ 0 & 0 & -\mathbf{i}\omega_n & 0 \\ 0 & 0 & 0 & -3\mathbf{i}\omega_n \end{bmatrix} \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \end{bmatrix} = \Lambda^* \mathbf{u}^*. \tag{138}$$

Now using the expressions developed above, we obtain the Lie bracket expression (Eq. (130)) for the Duffing example as

$$\begin{aligned} [\boldsymbol{h}, \mathscr{G}_0] = & \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{bmatrix} \begin{bmatrix} 3\mathbf{i}\omega_n & 0 & 0 & 0 \\ 0 & \mathbf{i}\omega_n & 0 & 0 \\ 0 & 0 & -\mathbf{i}\omega_n & 0 \\ 0 & 0 & 0 & -3\mathbf{i}\omega_n \end{bmatrix} \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \end{bmatrix} \\ & - \begin{bmatrix} \mathbf{i}\omega_n & 0 \\ 0 & -\mathbf{i}\omega_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{bmatrix} \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \end{bmatrix}. \end{aligned} \tag{139}$$

We know from the order-ε^1 homological equation (Eq. (131)) that $\boldsymbol{n} - \boldsymbol{n}_u = [\boldsymbol{h}, \mathscr{G}_0]$ and so using Eq. (134), Eq. (136) and Eq. (139) gives,

$$\begin{aligned} & \left(\frac{\alpha}{2\mathbf{i}\omega_n} \begin{bmatrix} -1 & -3 & -3 & -1 \\ 1 & 3 & 3 & 1 \end{bmatrix} - \begin{bmatrix} n_{u1}^* & n_{u2}^* & n_{u3}^* & n_{u4}^* \\ n_{u5}^* & n_{u6}^* & n_{u7}^* & n_{u8}^* \end{bmatrix} \right) \mathbf{u}^* = \\ & \qquad \begin{bmatrix} b_1(2\mathbf{i}\omega_n) & b_2(0) & b_3(-2\mathbf{i}\omega_n) & b_4(-4\mathbf{i}\omega_n) \\ b_5(4\mathbf{i}\omega_n) & b_6(2\mathbf{i}\omega_n) & b_7(0) & b_8(-2\mathbf{i}\omega_n) \end{bmatrix} \mathbf{u}^*. \end{aligned} \tag{140}$$

[48] Note that the second equation in Eq. (138) gives an $L \times 1$ vector that has been re-expressed as an $L \times L$ matrix, Λ^* multiplied by $\mathbf{u}^*$ (which is an $L \times 1$ vector). We note that Λ^* can be obtained directly from

$$\Lambda^* = I_L \odot \left[\mathbf{1}_L^T \otimes \left(\left[\frac{\partial \mathbf{u}^*}{\partial \mathbf{u}} \Lambda \mathbf{u} \right] \oslash \mathbf{u}^* \right) \right] \tag{137}$$

where I_L is an $L \times L$ identity matrix, $\mathbf{1}_L$ is a $L \times 1$ vector of ones, $\odot$ denotes Hadamard product, $\otimes$ denotes outer product, and $\oslash$ denotes Hadamard division. Note that Neild and Wagg [128] derive an alternative method for computing Λ^* which based on the indices of the terms in $\mathbf{u}^*$. This method may be preferred in practice, particularly for multi-degree-of-freedom nonlinear oscillators.

To solve for the unknown coefficients, b_i and n^*_{ui} for $i = 1, 2..8$ we use the following logic. If there is a zero on the right hand-side of Eq. (140), then $n^*_{ui} = n^*_i$ from solving the lefthand side of Eq. (140). If there is not a zero on the right hand-side of Eq. (140), then $n^*_{ui} = 0$ is assumed, and the coefficient b_i can be solved for. Note that in this example this choice means that b_2 and b_7 *can have any value*, and following [85] we refer to these coefficients as the *free functions*, which will be discussed in Section 4.7. Using this approach the coefficient matrices can be found as

$$\boldsymbol{b} = \begin{bmatrix} \frac{\alpha}{4\omega_n^2} & 0 & \frac{-3\alpha}{4\omega_n^2} & \frac{-\alpha}{8\omega_n^2} \\ \frac{-\alpha}{8\omega_n^2} & \frac{-3\alpha}{4\omega_n^2} & 0 & \frac{\alpha}{4\omega_n^2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{n}^*_u = \begin{bmatrix} 0 & \frac{-3\alpha}{2\mathbf{i}\omega_n} & 0 & 0 \\ 0 & 0 & \frac{3\alpha}{2\mathbf{i}\omega_n} & 0 \end{bmatrix}.$$

Now the normal form (Eq. (120)), up to order ε^1, can be written as

$$\dot{\boldsymbol{u}} = \Lambda \boldsymbol{u} + \varepsilon \boldsymbol{n}^*_u \mathbf{u}^* \quad \rightsquigarrow \quad \begin{aligned} \dot{u}_p &= \mathbf{i}\omega_n u_p + \varepsilon \frac{3\mathbf{i}\alpha}{2\omega_n} u_p^2 u_m, \\ \dot{u}_m &= -\mathbf{i}\omega_n u_m - \varepsilon \frac{3\mathbf{i}\alpha}{2\omega_n} u_p u_m^2, \end{aligned} \tag{141}$$

and because of the occurrence of imaginary parts in these expressions, this is called the *complex* normal form format. The near-identity transform (Eq. (117)) becomes

$$\boldsymbol{q} = \boldsymbol{u} + \varepsilon \boldsymbol{b} \mathbf{u}^* \quad \rightsquigarrow \quad \begin{bmatrix} q_p \\ q_m \end{bmatrix} = \begin{bmatrix} u_p \\ u_m \end{bmatrix} + \frac{\varepsilon\alpha}{\omega_n^2} \begin{bmatrix} \frac{u_p^3}{4} - \frac{3u_p u_m^2}{4} - \frac{u_m^3}{8} \\ -\frac{u_p^3}{8} - \frac{3u_p^2 u_m}{4} + \frac{u_m^3}{4} \end{bmatrix} \tag{142}$$

and because $x = q_p + q_m$ from Eq. (142) we can express x as

$$x = u_p + u_m + \frac{\varepsilon\alpha}{\omega_n^2}\left[\frac{u_p^3 + u_m^3}{8} - \frac{3}{4}(u_p^2 u_m + u_p u_m^2)\right] = (U - \varepsilon\frac{3\alpha U^3}{16\omega_n^2})\cos(\omega_r t) + \varepsilon\frac{\alpha U^3}{32\omega_n^2}\cos(3\omega_r t) + \mathscr{O}(\varepsilon^2), \tag{143}$$

once the base solutions from Eq. (22) are substituted[49]. □

Note that Eq. (141) can be compared with Eq. (20) from Example 2 (Section 1.1.3) in the case where $\varepsilon = 1$ (note also the sign difference because of the $-\alpha$ term in Eq. (19)). Likewise, Eq. (143) can be compared with Eq. (21) noting again the $-\alpha$ sign difference from Eq. (19). Note also the subtle differences between these expressions in terms of when ω_r appears compared to ω_n. In order to formalise this we will introduce a detuning in Section 4.6.

The zeros in the homological equation, Eq. (140) relate to the resonant terms, that occur in this example. As a result, the resulting set of normal form equations in Eq. (141) is not linear, the terms relating to the resonances must be retained.

In order to actually compute the x approximation from Eq. (143), we first need to find the initial condition value of U, which we denote U_0 at time $t = 0$. Substituting $t = 0$ into Eq. (143), and using Cardano's formula leads to

$$x(0) = U_0 - \varepsilon\frac{5\alpha U^3}{32\omega_n^2}, \rightsquigarrow U_0 = \sqrt[3]{-\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \frac{\Pi^3}{27}}} + \sqrt[3]{-\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} + \frac{\Pi^3}{27}}}, \text{ where } \Pi = -\frac{32\omega_n^2}{5\varepsilon\alpha}, \tag{144}$$

and $\Gamma = -\Pi x(0)$. Note that Cardano's formula applies when the discriminant is positive, meaning $4\Pi^3 + 27\Gamma^2 > 0$. A numerical example is shown in Fig. 5.

Next we consider the approach to near-identity transformations for second-order equations.

[49] Note that the base solutions in Eq. (22) are defined in terms of a response frequency ω_r which may be taken as equal to or not equal to the natural frequencies ω_n and ω_d depending on the specific context of the problem at hand.

4.2.3 Second-order equations

Now let us consider second-order differential equations of the form

$$\ddot{\mathbf{q}} = \hat{\mathscr{F}}(\mathbf{q},\varepsilon) = -\hat{\Lambda}\mathbf{q} - \varepsilon\hat{\mathbf{n}}_{\mathbf{1}}(\mathbf{q}) - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{2}}(\mathbf{q}) + \dots \tag{145}$$

where $\hat{\Lambda}$ is defined in Eq. (70) and the $\hat{\mathbf{n}}_j(\mathbf{q})$ contain the nonlinear terms. In this case, we define the structure of the desired transformed differential equation to be

$$\ddot{\mathbf{u}} = \hat{\mathscr{G}}(\mathbf{u},\varepsilon) = -\hat{\Lambda}\mathbf{u} - \varepsilon\hat{\mathbf{n}}_{\mathbf{u1}}(\mathbf{u}) - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{u2}}(\mathbf{u}) + \dots \tag{146}$$

Now substituting (the lower of) Eq. (116) into Eq. (145) gives

$$\ddot{\mathbf{u}} + \varepsilon\frac{\mathrm{d}^2\mathbf{h}_1}{\mathrm{d}t^2} + \varepsilon^2\frac{\mathrm{d}^2\mathbf{h}_2}{\mathrm{d}t^2} = -\hat{\Lambda}\mathbf{u} - \varepsilon\hat{\Lambda}\mathbf{h}_1 - \varepsilon^2\hat{\Lambda}\mathbf{h}_2 - \varepsilon\hat{\mathbf{n}}_{\mathbf{1}}(\mathbf{u}+\varepsilon\mathbf{h}_1+\varepsilon^2\mathbf{h}+\dots) - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{2}}(\mathbf{u}+\varepsilon\mathbf{h}_1+\varepsilon^2\mathbf{h}+\dots) + \dots \tag{147}$$

Then substituting for $\ddot{\mathbf{u}}$ from Eq. (146) gives

$$\begin{aligned} -\hat{\Lambda}\mathbf{u} - \varepsilon\hat{\mathbf{n}}_{\mathbf{u1}}(\mathbf{u}) - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{u2}}(\mathbf{u}) + \varepsilon\frac{\mathrm{d}^2\mathbf{h}_1}{\mathrm{d}t^2} + \varepsilon^2\frac{\mathrm{d}^2\mathbf{h}_2}{\mathrm{d}t^2} \\ = -\hat{\Lambda}\mathbf{u} - \varepsilon\hat{\Lambda}\mathbf{h}_1 - \varepsilon\hat{\Lambda}\mathbf{h}_2 - \varepsilon\hat{\mathbf{n}}_{\mathbf{1}}(\mathbf{u}+\varepsilon\mathbf{h}_1+\varepsilon^2\mathbf{h}+\dots) - \varepsilon\hat{\mathbf{n}}_{\mathbf{2}}(\mathbf{u}+\varepsilon\mathbf{h}_1+\varepsilon^2\mathbf{h}+\dots) + \dots \end{aligned} \tag{148}$$

Using a Taylor series expansion in the same way as Eq. (123), we obtain

$$\hat{\mathbf{n}}_i(\mathbf{u}+\varepsilon\mathbf{h}_1(\mathbf{u})+\dots) = \hat{\mathbf{n}}_i(\mathbf{u}) + \varepsilon D\{\hat{\mathbf{n}}_i(\mathbf{u})\}\mathbf{h}_1(\mathbf{u}) + \dots, \tag{149}$$

where $D\{\hat{\mathbf{n}}_i\}$ is the Jacobian matrix (or gradient vector) depending on the dimensions of $\mathbf{u}$. Then equating ε terms in Eq. (148) gives

$$\varepsilon^0: \quad -\hat{\Lambda}\mathbf{u} = -\hat{\Lambda}\mathbf{u} \tag{150}$$

$$\varepsilon^1: \quad -\frac{\mathrm{d}^2\mathbf{h}_1}{\mathrm{d}t^2} - \hat{\Lambda}\mathbf{h}_1 = \hat{\mathbf{n}}_{\mathbf{1}} - \hat{\mathbf{n}}_{\mathbf{u1}}, \tag{151}$$

$$\varepsilon^2: \quad -\frac{\mathrm{d}^2\mathbf{h}_2}{\mathrm{d}t^2} - \hat{\Lambda}\mathbf{h}_2 = D\{\hat{\mathbf{n}}_1\}\mathbf{h}_1 + \hat{\mathbf{n}}_{\mathbf{2}} - \hat{\mathbf{n}}_{\mathbf{u2}}, \tag{152}$$

Obtaining an expression for $\frac{\mathrm{d}^2\mathbf{h}_j}{\mathrm{d}t^2}$ will be achieved by differentiating the first Lie derivative from Eq. (127) a second time. The second Lie derivative of $\mathbf{u}^*$ to order ε^0 is then defined by[50]

$$\left.\frac{\mathrm{d}^2\mathbf{u}^*}{\mathrm{d}t^2}\right|_{\varepsilon=0} = \left.\frac{\partial}{\partial\boldsymbol{u}}\left(\Lambda^*\mathbf{u}^*\right)\dot{\boldsymbol{u}}\right|_{\varepsilon=0} = \Lambda^*\frac{\partial\mathbf{u}^*}{\partial\boldsymbol{u}}\Lambda\boldsymbol{u} = \left(\Lambda^*\right)^2\mathbf{u}^*. \tag{153}$$

Substituting this, and relations for $\boldsymbol{h}$ from Eq. (117) into the homological equations Eq. (151) and Eq. (152) gives

$$\varepsilon^1: \quad -\frac{\partial}{\partial\boldsymbol{u}}\left(\frac{\partial\mathbf{h}_1}{\partial\boldsymbol{u}}\Lambda\boldsymbol{u}\right)\Lambda\boldsymbol{u} - \hat{\Lambda}\mathbf{h}_1 = \hat{\mathbf{n}}_{\mathbf{1}} - \hat{\mathbf{n}}_{\mathbf{u1}}, \qquad \rightsquigarrow \qquad -\mathbf{b}_1\left(\Lambda^*\right)^2\mathbf{u}^* - \hat{\Lambda}\mathbf{b}_1\mathbf{u}^* = \hat{\mathbf{n}}_{\mathbf{1}} - \hat{\mathbf{n}}_{\mathbf{u1}}, \tag{154}$$

$$\varepsilon^2: \quad -\frac{\partial}{\partial\boldsymbol{u}}\left(\frac{\partial\mathbf{h}_2}{\partial\boldsymbol{u}}\Lambda\boldsymbol{u}\right)\Lambda\boldsymbol{u} - \hat{\Lambda}\mathbf{h}_2 = D\{\hat{\mathbf{n}}_1\}\mathbf{h}_1 + \hat{\mathbf{n}}_{\mathbf{2}} - \hat{\mathbf{n}}_{\mathbf{u2}}, \rightsquigarrow \qquad -\mathbf{b}_2\left(\Lambda^+\right)^2\mathbf{u}^+ - \hat{\Lambda}\mathbf{b}_2\mathbf{u}^+ = D\{\hat{\mathbf{n}}_1\}\mathbf{b}_1\mathbf{u}^* + \hat{\mathbf{n}}_{\mathbf{2}} - \hat{\mathbf{n}}_{\mathbf{u2}}, \tag{155}$$

As a result, the order-ε^1 homological equation (without frequency detuning which will be added in Section 4.6) for second-order equations is

$$\hat{\mathbf{n}}_{\mathbf{1}} - \hat{\mathbf{n}}_{\mathbf{u1}} = -\mathbf{b}_1\left(\Lambda^*\right)^2\mathbf{u}^* - \hat{\Lambda}\mathbf{b}_1\mathbf{u}^* = \hat{\mathscr{H}}_1(\mathbf{u}), \tag{156}$$

which is equivalent to the homological equation defined by [128] when they first introduced this type of transformation. An example is considered next.

[50] Notice that the derivative is with respect to $\boldsymbol{u}$ just as for first-order differential equations. The relationship is that for for each coordinate u_j in the vector $\mathbf{u}$ there is a corresponding u_{pj} and u_{mj} in the vector $\boldsymbol{u}$, and $\mathbf{u}^*$ is a function of u_{pj} and u_{mj}, for $j = 1,2,3,\dots N$. In addition, there is a relationship between $\hat{\Lambda}$ and Λ for undamped oscillators that can be inferred from Eq. (70) and Eq. (77).

4.2.4 Example 12: The real normal form for the undamped cubic-quintic oscillator

In this example, we follow the approach proposed by [128] which leads to a type of real normal form transformation[51]. First we note that the undamped cubic-quintic oscillator is the same as the Duffing equation (Eq. (132)) with an additional quintic polynomial term, that we write in the form of Eq. (145), and because this is a single degree-of-freedom example, $x = q$, therefore

$$\begin{aligned} \ddot{x} + \omega_n^2 x + \varepsilon\alpha_1 x^3 + \varepsilon\alpha_2 x^5 = 0 \quad \text{with } x = q \rightsquigarrow \quad & \ddot{q} + \omega_n^2 q + \varepsilon\alpha_1 q^3 + \varepsilon\alpha_2 q^5 = 0, \\ \rightsquigarrow \quad & \ddot{q} = -\omega_n^2 q - \varepsilon\hat{\mathbf{n}}_{\mathbf{1}}(q_p, q_m) = 0, \end{aligned} \tag{157}$$

where

$$\begin{aligned} \hat{\mathbf{n}}_{\mathbf{1}}(q_p, q_m) &= \alpha_1(q_p + q_m)^3 + \alpha_2(q_p + q_m)^5 \\ &= \alpha_1(q_p^3 + 3q_p^2 q_m + 3q_p q_m^2 + q_m^3) + \alpha_2(q_p^5 + 5q_p^4 q_m + 10q_p^3 q_m^2 + 10q_p^2 q_m^3 + 5q_p q_m^4 + q_m^5) \\ &= [\alpha_1 \;\; 3\alpha_1 \;\; 3\alpha_1 \;\; \alpha_1 \;\; \alpha_2 \;\; 5\alpha_2 \;\; 10\alpha_2 \;\; 10\alpha_2 \;\; 5\alpha_2 \;\; \alpha_2]\mathbf{q}^*, \end{aligned} \tag{158}$$

where $\mathbf{q} = [q_p, q_m]^T$ and $\mathbf{q}^* = [q_p^3 \;\; q_p^2 q_m \;\; q_p q_m^2 \;\; q_m^3 \;\; q_p^5 \;\; q_p^4 q_m \;\; q_p^3 q_m^2 \;\; q_p^2 q_m^3 \;\; q_p q_m^4 \;\; q_m^5]^T$. Note that we are treating the combination of both the cubic and quintic nonlinear terms as just a single nonlinear term, $\hat{\mathbf{n}}_{\mathbf{1}}$. This is possible in the ε-order approach, but not in the k-order approach, where instead we would take $\hat{\mathbf{n}}_{\mathbf{1}} = \alpha_1 q^3$ and $\hat{\mathbf{n}}_{\mathbf{2}} = \alpha_2 q^5$. However, there are potential drawbacks to be aware of. Specifically, for this example there will be 5th order terms generated when the assumed solution is substituted into the cubic terms in $\hat{\mathbf{n}}_{\mathbf{1}}$ (in the terminology of Example 10 these are $n^{(3)}_{(5)}$ terms), and these will only be captured in the ε^2-order analysis which we will show in Example 18 (Section 4.8.1) — see also Example 17 (Section 4.7.1).

Using this approach, we can set $\mathbf{n}_2 = 0, \ldots$ etc. and so $\mathbf{n}_1 = \mathbf{n}$, and correspondingly $\mathbf{b}_1 = \mathbf{b}$ and $\hat{\mathbf{n}}_{\mathbf{u1}} = \hat{\mathbf{n}}_{\mathbf{u}}$ throughout the rest of this example. Next we apply a near-identity nonlinear transform, Eq. (116), of the form

$$q = u + \varepsilon\mathbf{b}\mathbf{u}^* \qquad \text{where} \qquad \mathbf{b} = [b_1 \;\; b_2 \;\; b_3 \;\; b_4 \;\; b_5 \;\; b_6 \;\; b_7 \;\; b_8 \;\; b_9 \;\; b_{10}], \tag{159}$$

and $\mathbf{u}^*$ is defined as $\mathbf{u}^* = [u_p^3 \;\; u_p^2 u_m \;\; u_p u_m^2 \;\; u_m^3 \;\; u_p^5 \;\; u_p^4 u_m \;\; u_p^3 u_m^2 \;\; u_p^2 u_m^3 \;\; u_p u_m^4 \;\; u_m^5]^T$. The transformed dynamic equation, Eq. (146), is expressed in the new coordinates as

$$\ddot{u} + \omega_n^2 u + \varepsilon\hat{\mathbf{n}}_u(u_p, u_m) = 0 \quad \rightsquigarrow \quad \ddot{u} = -\omega_n^2 u - \varepsilon\hat{\mathbf{n}}_u(u_p, u_m). \tag{160}$$

Substituting Eq. (159) and Eq. (160) into Eq. (157) leads the order-ε^1 homological equation $\mathscr{H}_1(\mathbf{u})$ given in Eq. (156), which in this example (remembering that we have dropped the subscript 1 from $\mathbf{n}$, $\mathbf{b}$ and $\hat{\mathbf{n}}_{\mathbf{u}}$) can be written as

$$\varepsilon^1: \quad -\mathbf{b}\frac{\mathrm{d}^2\mathbf{u}^*}{\mathrm{d}t^2} - \omega_n^2\mathbf{b}\mathbf{u}^* = \hat{\mathbf{n}} - \hat{\mathbf{n}}_{\mathbf{u}} = -\mathbf{b}\left(\Lambda^*\right)^2\mathbf{u}^* - \omega_n^2\mathbf{b}\mathbf{u}^*. \tag{161}$$

The Lie derivative term can be determined from Eq. (153), with $\boldsymbol{u} = [u_p, u_m]^T$ and Λ as defined in Eq. (133). For this example we obtain

$$\frac{\partial\mathbf{u}^*}{\partial\boldsymbol{u}} = \begin{bmatrix} 3u_p^2 & 0 \\ 2u_p u_m & u_p^2 \\ u_m^2 & 2u_p u_m \\ 0 & 3u_m^2 \\ 5u_p^4 & 0 \\ 4u_p^3 u_m & u_p^4 \\ 3u_p^2 u_m^2 & 2u_p^3 u_m \\ 2u_p u_m^3 & 3u_p^2 u_m^2 \\ u_m^4 & 4u_p u_m^3 \\ 0 & 5u_p^4 \end{bmatrix} \text{ and } \frac{\partial\mathbf{u}^*}{\partial\boldsymbol{u}}\Lambda\boldsymbol{u} \equiv \mathbf{i}\boldsymbol{\omega}_n \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \\ u_p^5 \\ u_p^4 u_m \\ u_p^3 u_m^2 \\ u_p^2 u_m^3 \\ u_p u_m^4 \\ u_m^5 \end{bmatrix} = \Lambda^*\mathbf{u}^*.$$

[51] In classifying this as a real normal form, we are following the definitions of [104] who introduced this terminology to distinguish between the *complex* normal form format such as [80] and the *real* normal form formats such as [128, 168, 169].

(162)

Then using $\hat{\mathbf{n}}_\mathbf{u} = [\hat{n}_{u1}\ \hat{n}_{u2}\ \hat{n}_{u3}\ \hat{n}_{u4}\ \hat{n}_{u5}\ \hat{n}_{u6}\ \hat{n}_{u7}\ \hat{n}_{u8}\ \hat{n}_{u9}\ \hat{n}_{u10}]\mathbf{u}^*$ and substituting Eq. (158), with $q_i \to u_i$ (to order ε^0, similar to Eq. (136)) into Eq. (161) gives

$$([\alpha_1\ 3\alpha_1\ 3\alpha_1\ \alpha_1\ \alpha_2\ 5\alpha_2\ 10\alpha_2\ 10\alpha_2\ 5\alpha_2\ \alpha_2] - [\hat{n}_{u1}\ \hat{n}_{u2}\ \hat{n}_{u3}\ \hat{n}_{u4}\ \hat{n}_{u5}\ \hat{n}_{u6}\ \hat{n}_{u7}\ \hat{n}_{u8}\ \hat{n}_{u9}\ \hat{n}_{u10}])\mathbf{u}^* =$$
$$\left(-(\mathbf{i}\omega_n)^2[b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ b_8\ b_9\ b_{10}] \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}^2 -$$
$$\omega_n^2[b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7\ b_8\ b_9\ b_{10}]\Bigg)\mathbf{u}^*. \tag{163}$$

Evaluating the right hand side gives four zero values which allows the $\hat{\mathbf{n}}_\mathbf{u}$ coefficients to be identified such that

$$([\alpha_1\ 3\alpha_1\ 3\alpha_1\ \alpha_1\ \alpha_2\ 5\alpha_2\ 10\alpha_2\ 10\alpha_2\ 5\alpha_2\ \alpha_2] - [0\ 3\alpha_1\ 3\alpha_1\ 0\ 0\ 0\ 10\alpha_2\ 10\alpha_2\ 0\ 0])\mathbf{u}^* =$$
$$[b_1(8\omega_n^2)\ b_2(0)\ b_3(0)\ b_4(8\omega_n^2)\ b_5(24\omega_n^2)\ b_6(8\omega_n^2)\ b_7(0)\ b_8(0)\ b_9(8\omega_n^2)\ b_{10}(24\omega_n^2)]\mathbf{u}^*. \tag{164}$$

From which the transform coefficient values are $b_1 = b_4 = \alpha_1/8\omega_n^2$, $b_6 = b_9 = 5\alpha_2/8\omega_n^2$, $b_5 = b_{10} = \alpha_2/24\omega_n^2$, and $b_2 = b_3 = b_7 = b_8 = 0$. Note that b_2, b_3 b_7 and b_8 are the free functions in this example, as any value can be selected without affecting the outcome of the homological equation. Here we choose to set the free functions to zero. Other choices will be discussed in Section 4.7. Next, using the fact that

$$\varepsilon\hat{\mathbf{n}}_\mathbf{u}(u_p, u_m) = \varepsilon[0\ 3\alpha_1\ 3\alpha_1\ 0\ 0\ 0\ 10\alpha_2\ 10\alpha_2\ 0\ 0]\mathbf{u}^* = \varepsilon 3\alpha_1(u_p^2u_m + u_pu_m^2) + \varepsilon 10\alpha_2(u_p^3u_m^2 + u_p^2u_m^3)$$

gives the ε^1 normal form as

$$\ddot{u} + \omega_n^2 u + \varepsilon 3\alpha_1(u_p^2u_m + u_pu_m^2) + \varepsilon 10\alpha_2(u_p^3u_m^2 + u_p^2u_m^3) = 0, \tag{165}$$

where $u = u_p + u_m$, and this type of normal form can be expressed in terms of the u_p and u_m coordinates separately as

$$\begin{aligned} \ddot{u}_p &+ \omega_n^2 u_p + \varepsilon 3\alpha_1 u_p^2 u_m + \varepsilon 10\alpha_2(u_p^3u_m^2 + u_p^2u_m^3) = 0, \\ \ddot{u}_m &+ \omega_n^2 u_m + \varepsilon 3\alpha_1 u_p u_m^2 + \varepsilon 10\alpha_2(u_p^3u_m^2 + u_p^2u_m^3) = 0. \end{aligned} \tag{166}$$

Note that the analysis needs to be extended to ε^2 to capture the effect of the $5th$-order term generated by the cubic nonlinearity — see Example 18 and Eq. (303). The near-identity transform, to order ε^1 may now be written as

$$x = u + \varepsilon\mathbf{b}\mathbf{u}^* = \frac{U}{2}(\mathrm{e}^{\mathbf{i}(\omega_r t)} + \mathrm{e}^{-\mathbf{i}(\omega_r t)}) + \varepsilon\left[\frac{\alpha_1}{8\omega_n^2}\ 0\ 0\ \frac{\alpha_1}{8\omega_n^2}\ \frac{\alpha_2}{24\omega_n^2}\ \frac{5\alpha_2}{8\omega_n^2}\ 0\ 0\ \frac{5\alpha_2}{8\omega_n^2}\ \frac{\alpha_2}{24\omega_n^2}\right]\mathbf{u}^* \tag{167}$$

using the base solutions from Eq. (22) in the $u = u_p + u_m$ term. Then substituting the base solutions from Eq. (22) into the $\mathbf{u}^*$ vector, the solution for x becomes

$$x = U\cos(\omega_r t) + \varepsilon\left(\frac{\alpha_1 U^3}{32\omega_n^2} + \frac{5\alpha_2 U^5}{128\omega_n^2}\right)\cos(3\omega_r t) + \varepsilon\frac{\alpha_2 U^5}{384\omega_n^2}\cos(5\omega_r t) + \mathscr{O}(\varepsilon^2). \qquad \square \tag{168}$$

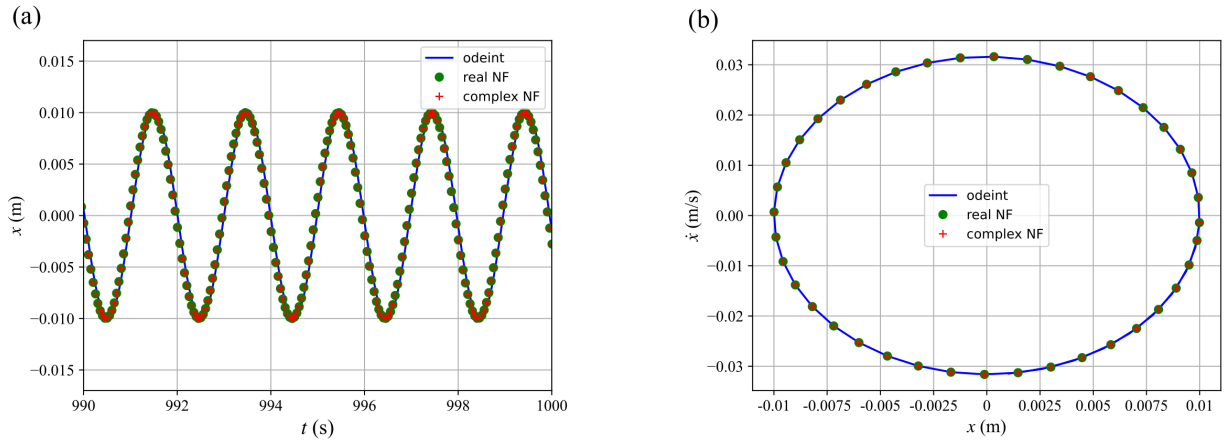

Fig. 5: A simulation of the undamped Duffing oscillator showing (a) the solution, x, and (b) the phase portrait. The normal form solutions are from Example 12 (complex normal form shown with red crosses), and Example 13 (real normal form shown with green circles). These normal form solutions are compared with a reference solution generated from a fourth-order Runge-Kutta integration (the 'odeint' numerical integration routine from the Python numerical library — blue solid line). The complex normal form solution is computed using Eq. (143) with Eq. (144) (red crosses) and the real normal form solution is computed with Eq. (168) with Eq. (169) with $\alpha_2 = 0$ and $\alpha_1 = \alpha$ (green circles). These solutions are differentiated to get $\dot{x}$ values. Parameter values are: $x(0) = 0.01$, $\omega_n = \sqrt{10}$ rad/s, and $\alpha_1 = \alpha = -10$ N/(kgm^3) $\alpha_2 = 0$.

In order to make a comparison with Example 11, we set $\alpha_2 = 0$ and $\alpha_1 = \alpha$. Then if this solution for x, Eq. (168), is compared to Eq. (143) from Example 11, it can be noted that there is no correction to the $\cos(\omega_n t)$ term (sometimes also called the fundamental term or the ε^0 term). In this normal form, each order of ε adds an additional term in the series without adding a correction to the fundamental term (although corrections are added to other terms), a feature called *killing the fundamental* meaning that the higher-order corrections are *orthogonal* to the fundamental part of the solution [85].

Next, in order to compute the x approximation from Eq. (168), we first need to find the initial condition value of U (which we denote U_0 at time $t = 0$). Substituting $t = 0$, $\alpha_2 = 0$ and $\alpha_1 = \alpha$ into Eq. (168) and using Cardano's formula leads to

$$x(0) = U_0 + \varepsilon \frac{\alpha U^3}{32\omega_n^2}, \rightsquigarrow U_0 = \sqrt[3]{-\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \frac{\Pi^3}{27}}} + \sqrt[3]{-\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} + \frac{\Pi^3}{27}}}, \text{ where } \Pi = \frac{32\omega_n^2}{\varepsilon\alpha}, \tag{169}$$

and $\Gamma = -\Pi x(0)$. Note that Cardano's formula applies when the discriminant is positive meaning $4\Pi^3 + 27\Gamma^2 > 0$. A numerical example is shown in Fig. 5.

It can be noted from Fig. 5 that the complex and real normal form solutions give very similar results (note that the comparison is made with $\alpha_2 = 0$ and $\alpha_1 = \alpha$ in Fig. 5). The equivalence of these two solutions in discussed in Section 4.2.5. In addition, the ε^2-order analysis for this example (with frequency detuning) is given in Examples 17 (Section 4.7.1) and 18 (Section 4.8.1). For example, the solutions for x from this example, Eq. (168), can be compared to Eq. (294) (minimal normal form) and Eq. (305) (non-minimal normal form).

4.2.5 Equivalence of complex and real normal form

In order to make a comparison between the complex and real normal form, we consider the example of the Duffing oscillator. In order to do this, we make a comparison between Eq. (166) with $\alpha_2 = 0$ and $\alpha_1 = \alpha$ (so the result reduces to that of the Duffing oscillator) and the results from Example 11 (Section 4.2.2). To do this, first notice that differentiating the normal form equations for the complex Duffing normal form, (Example 11, Eq. (141)) with respect to time gives

$$\begin{aligned} \ddot{u}_p &= \mathbf{i}\omega_n \dot{u}_p + \varepsilon \frac{3\mathbf{i}\alpha}{2\omega_n} \frac{\mathrm{d}}{\mathrm{d}t}(u_p^2 u_m), \\ \ddot{u}_m &= -\mathbf{i}\omega_n \dot{u}_m - \varepsilon \frac{3\mathbf{i}\alpha}{2\omega_n} \frac{\mathrm{d}}{\mathrm{d}t}(u_p u_m^2). \end{aligned} \tag{170}$$

Now eliminating the $\dot{u}_p$ and $\dot{u}_m$ by substituting from Eq. (141) (and using operator representation Section 1.1.2) leads to

$$\begin{aligned} &\ddot{u}_p + \omega_n^2 u_p - \varepsilon \frac{3\mathbf{i}\alpha}{2\omega_n}\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{i}\omega_n\right)(u_p^2 u_m) = 0 \\ &\ddot{u}_m + \omega_n^2 u_m + \varepsilon \frac{3\mathbf{i}\alpha}{2\omega_n}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathbf{i}\omega_n\right)(u_p u_m^2) = 0, \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} &\ddot{u}_p + \omega_n^2 u_p + \varepsilon 3\alpha u_p^2 u_m = 0, \\ &\ddot{u}_m + \omega_n^2 u_m + \varepsilon 3\alpha u_p u_m^2 = 0. \end{aligned} \tag{171}$$

Finally it can be seen that adding the two expressions on the right of Eq. (171) gives

$$\ddot{u}_p + \ddot{u}_m + \omega_n^2(u_p + u_m) + \varepsilon 3\alpha(u_p^2 u_m + u_p u_m^2) = 0 \quad \rightsquigarrow \quad \ddot{u} + \omega_n^2 u + \varepsilon 3\alpha(u_p^2 u_m + u_p u_m^2) = 0 \tag{172}$$

which is the same as the real normal form expression from Eq. (165) when $\alpha_2 = 0$ and $\alpha_1 = \alpha$ are introduced.

The solutions for x between the complex and real normal form for the Duffing oscillator can be made the same using a specific choice of the free functions. This will be described in more detail in Section 4.4.2. Having shown the equivalence of the complex and real normal form, we will focus on the real, and Hamiltonian normal form for the remainder of this paper.

4.3 Derivation of the homological equations using Lie series

In the notation of this paper[52] the Lie series can be defined in the following way (see [22, 120] for detailed derivations). Let $\Theta[\mathbf{q}]$ be an arbitrary function of $\mathbf{q}$, then the Lie series is

$$\Theta[\mathbf{q}] = \Theta[\mathbf{u}] + \varepsilon(\mathrm{G}\Theta) + \frac{\varepsilon^2}{2!}(\mathrm{G}(\mathrm{G}\Theta)) + \frac{\varepsilon^3}{3!}(\mathrm{G}(\mathrm{G}(\mathrm{G}\Theta))) + \ldots \tag{173}$$

with group operator

$$\mathrm{G} = \frac{\partial \cdot}{\partial \mathbf{u}}\mathbf{h}\bigg|_{\varepsilon=0}. \tag{174}$$

Notice that from Eq. (173) that the order-ε^1 terms satisfy

$$(\Theta[\mathbf{q}] - \Theta[\mathbf{u}])_{\varepsilon^1} \approx \mathrm{G}\Theta \tag{175}$$

which when $\mathbf{q}$ is substituted, using the near-identity transformation, Eq. (116), is the order ε^1 homological equation from the method of normal forms discussed in detail in Section 4.2. Furthermore, the term $\mathrm{G}\Theta$ corresponds to either $\mathscr{H}_1$ for first-order differential equations or $\mathscr{H}_1$ for second-order differential equations, as will be demonstrated later in this Section.

In order to consider Lie series transformations of both first and second-order differential equations, the functions Θ will need to include the variables t, $\boldsymbol{q}$, and $\dot{\boldsymbol{q}}$, in the first-order case, and t, $\mathbf{q}$, $\dot{\mathbf{q}}$ and $\ddot{\mathbf{q}}$ in the second-order case. The method for *prolonging* the group transformation to include these variables is described in detail by [22]. We now use this analysis to derive the homological equations for both first- and second-order differential equations.

4.3.1 First-order equations

First let us consider first-order equations in the format of Eq. (119), i.e.

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \mathscr{F}(\boldsymbol{q}, \varepsilon) \quad \text{which can be redefined as} \quad \dot{\boldsymbol{q}} - \mathscr{F}(\boldsymbol{q}, \varepsilon) = \Theta_1(\boldsymbol{q}, \dot{\boldsymbol{q}}, \varepsilon) = 0. \tag{176}$$

Now we can consider a transformation of this first-order differential equation using the method of Lie series[53]. To do this we need to revisit the near-identity transformation given in Eq. (116). First we write the differentials of t and $\boldsymbol{q}$ as

$$\mathrm{d}t = \mathrm{d}t, \quad \text{and} \quad \mathrm{d}\boldsymbol{q} = \frac{\partial \boldsymbol{T}}{\partial t}\mathrm{d}t + \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{u}}\mathrm{d}\boldsymbol{u}, \quad \rightsquigarrow \quad \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial \boldsymbol{T}}{\partial t} + \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{u}}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \tag{177}$$

[52] We use $\mathbf{q}$ etc. in this expression, but the same relationships apply for $\boldsymbol{q}$ and so on.
[53] We follow the analysis from [22].

Noting that $\boldsymbol{T}$ is defined in Eq. (116), and is not an explicit function of t (so that $\partial \boldsymbol{T}/\partial t = 0$) and substituting $\boldsymbol{T}(\boldsymbol{u},\varepsilon) = \boldsymbol{u} + \varepsilon \boldsymbol{h}(\boldsymbol{u})$ means that Eq. (177) becomes[54]

$$\dot{\boldsymbol{q}} = \frac{\partial}{\partial \boldsymbol{u}}(\boldsymbol{u} + \varepsilon \boldsymbol{h})\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}} + \varepsilon \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}}\dot{\boldsymbol{u}} = \dot{\boldsymbol{u}} + \varepsilon \boldsymbol{b}\Lambda^* \mathbf{u}^* = \boldsymbol{T}^\dagger(\boldsymbol{u},\dot{\boldsymbol{u}},\varepsilon), \tag{178}$$

where we have substituted $\boldsymbol{h} = \boldsymbol{b}\mathbf{u}^*$ and used the Lie derivative from Eq. (127) to evaluate the partial derivative term. Now Eq. (178) can be added to Eq. (116) as an extension or *prolongation* of the variables being transformed, so that

$$\begin{aligned} t &= t, \\ \boldsymbol{q} &= \boldsymbol{T}(\boldsymbol{u},\varepsilon) = \boldsymbol{u} + \varepsilon \boldsymbol{b}\mathbf{u}^*, \\ \dot{\boldsymbol{q}} &= \boldsymbol{T}^\dagger(\boldsymbol{u},\dot{\boldsymbol{u}},\varepsilon) = \dot{\boldsymbol{u}} + \varepsilon \boldsymbol{b}\Lambda^*\mathbf{u}^*. \end{aligned} \tag{179}$$

Now we will define new augmented vectors as

$$\tilde{\boldsymbol{q}} = \begin{bmatrix} \boldsymbol{q} \\ \dot{\boldsymbol{q}} \end{bmatrix}, \quad \tilde{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u} \\ \dot{\boldsymbol{u}} \end{bmatrix}, \quad \tilde{\boldsymbol{h}} = \begin{bmatrix} \boldsymbol{h} \\ \boldsymbol{h}^\dagger \end{bmatrix}, \quad \tilde{\boldsymbol{T}} = \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{T}^\dagger \end{bmatrix}, \tag{180}$$

to give a near-identity transformation in the new variables as

$$\tilde{\boldsymbol{q}} = \tilde{\boldsymbol{T}}(\tilde{\boldsymbol{u}},\varepsilon) = \tilde{\boldsymbol{u}} + \varepsilon \tilde{\boldsymbol{h}} \tag{181}$$

where $\boldsymbol{h} = \boldsymbol{b}\mathbf{u}^*$ and from Eq. (127)

$$\boldsymbol{h}^\dagger = \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \boldsymbol{b}\frac{\mathrm{d}\mathbf{u}^*}{\mathrm{d}t} = \boldsymbol{b}\Lambda^*\mathbf{u}^*. \tag{182}$$

Next the augmented variables can be substituted into Eq. (174), so that the group operator becomes

$$G_{FO}\Theta_1 = \frac{\partial \Theta_1}{\partial \tilde{\boldsymbol{u}}}\tilde{\boldsymbol{h}} = \frac{\partial \Theta_1}{\partial \boldsymbol{u}}\boldsymbol{h} + \frac{\partial \Theta_1}{\partial \dot{\boldsymbol{u}}}\boldsymbol{h}^\dagger, \tag{183}$$

where G_{FO} is the Lie group operator for first-order equations and $\frac{\partial \Theta_1}{\partial \boldsymbol{u}}$ and $\frac{\partial \Theta_1}{\partial \dot{\boldsymbol{u}}}$ are $N \times N$ Jacobian matrices. Note also that $\varepsilon = 0$ is used when evaluating the group operator, so from Eq. (179) we have that $\boldsymbol{q} = \boldsymbol{u}$ and $\dot{\boldsymbol{q}} = \dot{\boldsymbol{u}}$ in the evaluation of G_{FO}. This means that $\Theta_1(\boldsymbol{q},\dot{\boldsymbol{q}},0) = \Theta_1(\boldsymbol{u},\dot{\boldsymbol{u}},0) = \dot{\boldsymbol{u}} - \mathscr{F}(\boldsymbol{u},0)$.

Evaluating the partial derivatives of Θ_1 for the group operator with $\varepsilon = 0$ gives

$$\frac{\partial \Theta_1}{\partial \boldsymbol{u}} = -\frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}, \quad \frac{\partial \Theta_1}{\partial \dot{\boldsymbol{u}}} = 1, \tag{184}$$

so then

$$G_{FO}\Theta_1 = -\frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}\boldsymbol{h} + \boldsymbol{h}^\dagger = -\Lambda \boldsymbol{b}\mathbf{u}^* + \boldsymbol{b}\Lambda^*\mathbf{u}^* = \mathscr{H}_1, \quad \text{where} \quad \mathscr{F}(\boldsymbol{u},0) = \Lambda \boldsymbol{u} \tag{185}$$

gives the Lie bracket part of the order-ε^1 homological equation. The complete homological equation can then be obtained by substituting Eq. (185) and Eq. (179) into Eq. (175). The resulting expressions can then be compared with Eq. (131) obtained from the previous normal form techniques.

[54] Note that throughout this Section we are assuming that $\boldsymbol{n}_2 = 0, \ldots$ etc. and so we use $\boldsymbol{n}_1 = \boldsymbol{n}$, and correspondingly $\boldsymbol{h}_1 = \boldsymbol{h}$, $\boldsymbol{b}_1 = \boldsymbol{b}$ and $\hat{\boldsymbol{n}}_{u1} = \hat{\boldsymbol{n}}_u$. Similarly for the analysis of second-order equations.

4.3.2 Second-order equations

Now let us consider equations of second-order written in the form of Eq. (145) so that

$$\frac{\mathrm{d}^2\mathbf{q}}{\mathrm{d}t^2} = \hat{\mathscr{F}}(\mathbf{q},\dot{\mathbf{q}},\varepsilon) \quad \text{which can be redefined as} \quad \ddot{\mathbf{q}} - \hat{\mathscr{F}}(\mathbf{q},\dot{\mathbf{q}},\varepsilon) = \Theta_2(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},\varepsilon) = 0. \tag{186}$$

The group can now be prolonged again using the same approach as above (but with $\mathbf{q}$ instead of $\boldsymbol{q}$ etc.). First the differential of $\dot{\mathbf{q}}$ is written as

$$\mathrm{d}\dot{\mathbf{q}} = \frac{\partial \mathbf{T}^\dagger}{\partial t}\mathrm{d}t + \frac{\partial \mathbf{T}^\dagger}{\partial \mathbf{u}}\mathrm{d}\mathbf{u} + \frac{\partial \mathbf{T}^\dagger}{\partial \dot{\mathbf{u}}}\mathrm{d}\dot{\mathbf{u}}. \tag{187}$$

The function $\mathbf{T}^\dagger$ comes from the previously prolongated term in Eq. (178) and because $\mathbf{T}^\dagger$ is not a function of t, then $\partial\mathbf{T}^\dagger/\partial t = 0$. Now setting $\partial\mathbf{T}^\dagger/\partial t = 0$ and dividing Eq. (187) by $\mathrm{d}t$ gives

$$\ddot{\mathbf{q}} = \frac{\partial \mathbf{T}^\dagger}{\partial \mathbf{u}}\dot{\mathbf{u}} + \frac{\partial \mathbf{T}^\dagger}{\partial \dot{\mathbf{u}}}\ddot{\mathbf{u}}. \tag{188}$$

Substituting $\mathbf{T}^\dagger = \dot{\mathbf{u}} + \varepsilon\mathbf{b}\Lambda^*\mathbf{u}^*$ (i.e. from Eq. (178)) leads to

$$\ddot{\mathbf{q}} = \varepsilon\mathbf{b}\Lambda^*\left(\frac{\partial \mathbf{u}^*}{\partial \mathbf{u}}\right)\dot{\mathbf{u}} + (I+0)\,\ddot{\mathbf{u}} \rightsquigarrow \ddot{\mathbf{q}} = \ddot{\mathbf{u}} + \varepsilon\mathbf{b}\Lambda^{*2}\mathbf{u}^*.$$

So finally the prolonged group is

$$\begin{aligned} t &= t, \\ \mathbf{q} &= \mathbf{T}(\mathbf{u},\varepsilon) = \mathbf{u} + \varepsilon\mathbf{b}\mathbf{u}^*, \\ \dot{\mathbf{q}} &= \mathbf{T}^\dagger(\mathbf{u},\dot{\mathbf{u}},\varepsilon) = \dot{\mathbf{u}} + \varepsilon\mathbf{b}\Lambda^*\mathbf{u}^*, \\ \ddot{\mathbf{q}} &= \mathbf{T}^\ddagger(\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}},\varepsilon) = \ddot{\mathbf{u}} + \varepsilon\mathbf{b}\Lambda^{*2}\mathbf{u}^*. \end{aligned} \tag{189}$$

Now define new augmented vectors as

$$\breve{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix}, \quad \breve{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \\ \ddot{\mathbf{u}} \end{bmatrix}, \quad \breve{\mathbf{h}} = \begin{bmatrix} \mathbf{h} \\ \mathbf{h}^\dagger \\ \mathbf{h}^\ddagger \end{bmatrix}, \quad \breve{\mathbf{T}} = \begin{bmatrix} \mathbf{T} \\ \mathbf{T}^\dagger \\ \mathbf{T}^\ddagger \end{bmatrix}, \tag{190}$$

such that the second, third and fourth equations in Eq. (189) can be combined to give a near-identity transformation in the new variables as

$$\breve{\mathbf{q}} = \hat{\mathbf{T}}(\breve{\mathbf{u}},\varepsilon) = \breve{\mathbf{u}} + \varepsilon\breve{\mathbf{h}}, \tag{191}$$

where from above we know that $\mathbf{h} = \mathbf{b}\mathbf{u}^*$, $\mathbf{h}^\dagger = \mathbf{b}\Lambda^*\mathbf{u}^*$, and from Eq. (153)

$$\mathbf{h}^\ddagger = \frac{\mathrm{d}\mathbf{h}^\dagger}{\mathrm{d}t} = \mathbf{b}\Lambda^*\frac{\mathrm{d}\mathbf{u}^*}{\mathrm{d}t} = (\Lambda^*)^2\mathbf{u}^*. \tag{192}$$

Next the augmented variables can be substituted into Eq. (174), so that the group operator becomes

$$G_{SO}\Theta_2 = \frac{\partial \Theta_2}{\partial \breve{\mathbf{u}}}\breve{\mathbf{h}} = \frac{\partial \Theta_2}{\partial \mathbf{u}}\mathbf{h} + \frac{\partial \Theta_2}{\partial \dot{\mathbf{u}}}\mathbf{h}^\dagger + \frac{\partial \Theta_2}{\partial \ddot{\mathbf{u}}}\mathbf{h}^\ddagger. \tag{193}$$

Evaluating the partial derivatives of Θ_2 gives

$$\frac{\partial \Theta_2}{\partial \mathbf{u}} = -\frac{\partial \hat{\mathscr{F}}}{\partial \mathbf{u}}, \quad \frac{\partial \Theta_2}{\partial \dot{\mathbf{u}}} = -\frac{\partial \hat{\mathscr{F}}}{\partial \dot{\mathbf{u}}}, \quad \frac{\partial \Theta_2}{\partial \ddot{\mathbf{u}}} = 1, \tag{194}$$

so then

$$G_{SO}\Theta_2 = -\frac{\partial \hat{\mathscr{F}}}{\partial \mathbf{u}}\mathbf{h} - \frac{\partial \hat{\mathscr{F}}}{\partial \dot{\mathbf{u}}}\mathbf{h}^{\dagger} + \mathbf{h}^{\ddagger} = \hat{\Lambda}\mathbf{b}\mathbf{u}^{*} + \mathbf{b}(\Lambda^{*})^{2}\mathbf{u}^{*} = -\hat{\mathscr{H}}_1, \tag{195}$$

because

$$\hat{\mathscr{F}}(\mathbf{q},\varepsilon) = \hat{\mathscr{F}}(\mathbf{T}(\mathbf{u},\varepsilon),\varepsilon)|_{\varepsilon=0} = \hat{\Lambda}\mathbf{u} \quad \rightsquigarrow \quad \left.\frac{\partial \hat{\mathscr{F}}}{\partial \mathbf{u}}\right|_{\varepsilon=0} = -\hat{\Lambda} \qquad \text{and} \qquad \left.\frac{\partial \hat{\mathscr{F}}}{\partial \dot{\mathbf{u}}}\right|_{\varepsilon=0} = 0. \tag{196}$$

Note that this analysis works providing all terms relating to $\dot{\mathbf{q}}$ and $\dot{\mathbf{u}}$ are included in the nonlinear vector of the equation of motion. i.e. from Eq. (145) $\ddot{\mathbf{q}} = \hat{\mathscr{F}}(\mathbf{q},\dot{\mathbf{q}},\varepsilon) = -\hat{\Lambda}\mathbf{q} - \varepsilon\hat{\mathbf{n}}(\mathbf{q},\dot{\mathbf{q}})$. If damping terms relating to $\dot{\mathbf{q}}$ were included at ε^0, then this would change the homological equation, because $\left.\frac{\partial \hat{\mathscr{F}}}{\partial \dot{\mathbf{u}}}\right|_{\varepsilon=0} \neq 0$. This is the reason that damping terms are recommend to be included in the nonlinear vector in Section 4.5.1 for second-order differential equations. However, this restriction does not apply to first order differential equations.

It's interesting to note that Eq. (195) gives $-\hat{\mathscr{H}}_1$ whereas in the first-order case we obtained $\mathscr{H}_1$ directly — see Eq. (185). The reason is that for the definition of Θ_2 given in Eq. (186), the term $(\Theta[\mathbf{q}] - \Theta[\mathbf{u}])_{\varepsilon^1} = \boldsymbol{n}_u - \boldsymbol{n}$ rather than $\boldsymbol{n} - \boldsymbol{n}_u$ as given in the homological equation for second order equations Eq. (156). If Θ_2 is redefined as $\Theta_2 = \hat{\mathscr{F}} - \ddot{\mathbf{q}} = 0$ then then Eq. (195) becomes equal to $+\hat{\mathscr{H}}_1$. In that case the homological equation given in Eq. (195) becomes the same as Eq. (156). Next we consider the Hamiltonian Normal form first proposed by Birkhoff [11].

4.4 Hamiltonian normal form

A standard result from classical mechanics (see for example [117, 54], and also [116, 120]) states that[55]: If the Hamiltonian, H, of a nonlinear oscillator system can be written in the form

$$H(\mathbf{q}) = H_0(\mathbf{q}) + \varepsilon H_1(\mathbf{q}) + \mathscr{O}(\varepsilon^2) \tag{197}$$

then it is possible to define a coordinate transformation from the original coordinate set $\mathbf{q}$ to a new set of coordinates $\mathbf{u}$ so that there is a new Hamiltonian function, called the Kamiltonian, given by

$$K(\mathbf{u}) = H_0(\mathbf{u}) + \varepsilon(H_1(\mathbf{u}) + \{H_0, G\}) + \mathscr{O}(\varepsilon^2), \tag{198}$$

using the near-identity transformation

$$\mathbf{q} = \mathbf{u} + \varepsilon\{\mathbf{u}, G\} + \mathscr{O}(\varepsilon^2), \tag{199}$$

where G is a generator function, ε is a series expansion parameter, and $\{f,g\}$ denotes the Poisson bracket for two arbitrary functions f and g

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \quad \text{where } q \text{ and } p \text{ are some generalised coordinates.} \tag{200}$$

In the non-resonant case, $\{H_0, G\} = 0$, which is often referred to as the Birkhoff normal form [11, 6]. In contrast, the solution to the resonant case when, $\{H_0, G\} \neq 0$, is attributed to Gustavson [57] and as a result is called the Birkhoff-Gustavson normal form [38]. The question of how to choose an appropriate generating function has a long history, and the interested reader can find full details of this in a number of papers, e.g. [74, 34, 117, 37, 24, 38, 95, 174] and the textbooks [116, 120].

4.4.1 Hamiltonian to Kamiltonian transformation

The approach taken here will be to define the Kamiltonian in terms of a residual function $R(\mathbf{u})$ such that

$$K(\mathbf{u}) = H_0(\mathbf{u}) + \varepsilon R(\mathbf{u}), \tag{201}$$

[55] A demonstration of this result for the Duffing example is given later in Section 4.4.2.

which at order ε^1 leads to the relationship

$$R(\mathbf{u}) = H_1(\mathbf{u}) + \{H_0, G\} \rightsquigarrow \{H_0, G\} = R(\mathbf{u}) - H_1(\mathbf{u}), \tag{202}$$

which is the order-ε^1 homological equation in Hamiltonian form. To see how this result arises, we start by substituting Eq. (199) into Eq. (197), so that

$$H(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) = H_0(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) + \varepsilon H_1(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) + \mathscr{O}(\varepsilon^2), \tag{203}$$

which when the right-hand-side is expanded gives to order ε^1

$$K(\mathbf{u}) = H(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) = H_0(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) + \varepsilon H_1(\mathbf{u}) + \mathscr{O}(\varepsilon^2). \tag{204}$$

Now consider the H_0 term on the right-hand-side. Birkhoff [11] showed that for a single degree-of-freedom system[56] $H_0(\mathbf{q}) = \rho q_p q_m$ where ρ is a constant related to the eigenvalues (a specific example is given in Eq. (212)) and $\mathbf{q} = [q_p \;\; q_m]^T$ are the generalised coordinates. As a result, using Eq. (199) with $\mathbf{u} = [u_p \;\; u_m]^T$ we have

$$H_0(\mathbf{q}) \implies H_0(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) = \rho q_p q_m = \rho \left(u_p + \varepsilon \frac{\partial G}{\partial u_m}\right)\left(u_m - \varepsilon \frac{\partial G}{\partial u_p}\right), \tag{205}$$

which gives to order ε^1

$$H_0(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) = \rho \left(u_p u_m + \varepsilon \left(u_m \frac{\partial G}{\partial u_m} - u_p \frac{\partial G}{\partial u_p}\right)\right) + \mathscr{O}(\varepsilon^2). \tag{206}$$

Then because $H_0(\mathbf{u}) = \rho u_p u_m$ we can write

$$H_0(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) = \rho \left(u_p u_m + \frac{\varepsilon}{\rho}\left(\frac{\partial H_0}{\partial u_p}\frac{\partial G}{\partial u_m} - \frac{\partial H_0}{\partial u_m}\frac{\partial G}{\partial u_p}\right)\right) + \mathscr{O}(\varepsilon^2), \tag{207}$$

which can be written as

$$H_0(\mathbf{u} + \varepsilon\{\mathbf{u}, G\}) = \rho u_p u_m + \varepsilon\{H_0, G\} + \mathscr{O}(\varepsilon^2), = H_0(\mathbf{u}) + \varepsilon\{H_0, G\}) + \mathscr{O}(\varepsilon^2) \tag{208}$$

which when substituted into Eq. (204) gives Eq. (198).

4.4.2 Example 13 Hamiltonian normal form for the conservative Duffing oscillator

Normally the coordinate displacement q, and momentum $p = m\dot{q}$ are used for the Hamiltonian, but to avoid confusion with coordinates q_p and q_m, here we use coordinates x for the displacement and $p = m\dot{x}$ for the momentum for the undamped, unforced (i.e. conservative) Duffing oscillator given by Eq. (132) (see Example 11 Section 4.2.2). Then we can rewrite the equations of motion in the Hamiltonian form as

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m}, \\ \dot{p} &= -\frac{\partial H}{\partial x} = -kx - k_3 x^3, \end{aligned} \tag{209}$$

with the Hamiltonian

$$H(\boldsymbol{p}) = \frac{p^2}{2m} + \frac{kx^2}{2} + \varepsilon\frac{k_3 x^4}{4} = H_0 + \varepsilon H_1, \tag{210}$$

where, m is mass, k linear stiffness, k_3 nonlinear stiffness, ε is a small parameter, $\alpha = k_3/m$, and ω_n^2 is the linear natural frequency and $\boldsymbol{p} = [x, p]^T$.

[56] This can be generalised to systems of N-degrees-of-freedom providing that the i^{th} $H_{i0}(\mathbf{q})$ function has the format $H_{i0}(\mathbf{q}) = \rho_i q_{ip} q_{im}$ for all $i \in N$, see [11]. However, and important detail to note is that for each degree-of-freedom, the result relies on $k = m\omega_n$ which only works in a straightforward way for the case without detuning (shown here). In the case when detuning is added, additional frequency terms will be generated at order ε^1. Examples of systems with more than one degree-of-freedom can be found in [38, 174].

Before we can use the result from Section 4.4.1, we need to apply a *diagonalisation transform* to get the equations into the form of Eq. (197). The usual choice for a diagonalisation transform is related directly to the modal transformation defined in Eq. (4)[57]. For consistency with the engineering approach we use the change of coordinates defined by Eq. (4) above. As a result the diagonalisation transformation of the Hamiltonian $H(\boldsymbol{p}) \to H(\mathbf{q})$, will be such that $x = q_p + q_m$ and $p = m\mathbf{i}\omega_n(q_p - q_m)$, which can be written using Eq. (4) as

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ m\lambda_1 & m\lambda_2 \end{bmatrix} \begin{bmatrix} q_p \\ q_m \end{bmatrix}. \tag{211}$$

Note that to use the Birkhoff form of H_0 we do not use detuning, and so $p = m\mathbf{i}\omega_n(q_p - q_m)$ is assumed.

Substituting these relationships into the Hamiltonian Eq. (210) gives

$$H(\mathbf{q}) = 2m\omega_n^2 q_p q_m + \varepsilon\frac{k_3}{4}(q_p + q_m)^4 = H_0(\mathbf{q}) + \varepsilon H_1(\mathbf{q}), \tag{212}$$

from which we note that $\rho = 2m\omega_n^2$ in this example. When the H_1 term is expanded we obtain

$$H_1(\mathbf{q}) = \frac{k_3}{4}\left(q_p^4 + 4q_p^3 q_m + 6q_p^2 q_m^2 + 4q_p q_m^3 + q_m^4\right) = \frac{k_3}{4}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \end{bmatrix}\begin{bmatrix} q_p^4 \\ q_p^3 q_m \\ q_p^2 q_m^2 \\ q_p q_m^3 \\ q_m^4 \end{bmatrix} = \mathbf{H}_1^* \mathbf{q}^\dagger, \tag{213}$$

where

$$\mathbf{H}_1^* = \frac{k_3}{4}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{q}^\dagger = \begin{bmatrix} q_p^4 \\ q_p^3 q_m \\ q_p^2 q_m^2 \\ q_p q_m^3 \\ q_m^4 \end{bmatrix}. \tag{214}$$

The next step is to make the coordinate transformation from $\mathbf{q} \to \mathbf{u}$ via Eq. (199).

Now we define G and R to mirror the structure of H_1 in Eq. (213), so that $G(\mathbf{u}) = \mathbf{G}^*\mathbf{u}^\dagger$, and $R(\mathbf{u}) = \mathbf{R}^*\mathbf{u}^\dagger$ where both are functions of $\mathbf{u}$ rather than $\mathbf{q}$. Notice that to order ε^1, $H_1(\mathbf{u}) = \mathbf{H}_1^*\mathbf{u}^\dagger$ where $\mathbf{u}^\dagger$ exactly mirrors $\mathbf{q}^\dagger$ but with $q_p = u_p$ and $q_m = u_m$. This means that the homological equation Eq. (202)) can be written as

$$\{H_0, \mathbf{G}^*\mathbf{u}^\dagger\} = \mathbf{R}^*\mathbf{u}^\dagger - \mathbf{H}_1^*\mathbf{u}^\dagger \rightsquigarrow \{H_0, \mathbf{G}^*\mathbf{u}^\dagger\} = (\mathbf{R}^* - \mathbf{H}_1^*)\mathbf{u}^\dagger. \tag{215}$$

Next we consider the Poisson bracket $\{H_0, \mathbf{G}^*\mathbf{u}^\dagger\}$ which, using the definition for the Poisson bracket with $\mathbf{u} = [u_p, u_m]^T$, gives

$$\{H_0, \mathbf{G}^*\mathbf{u}^\dagger\} = \frac{\partial H_0}{\partial u_p}\frac{\partial \mathbf{G}^*\mathbf{u}^\dagger}{\partial u_m} - \frac{\partial H_0}{\partial u_m}\frac{\partial \mathbf{G}^*\mathbf{u}^\dagger}{\partial u_p}. \tag{216}$$

This leads to

$$\{H_0, \mathbf{G}^*\mathbf{u}^\dagger\} = 2m\omega_n^2\left(u_m\mathbf{G}^*\frac{\partial \mathbf{u}^\dagger}{\partial u_m} - u_p\mathbf{G}^*\frac{\partial \mathbf{u}^\dagger}{\partial u_p}\right) = 2m\omega_n^2\left[G_1(-4)\ G_2(-2)\ G_3(0)\ G_4(2)\ G_5(4)\right]\mathbf{u}^\dagger, \tag{217}$$

such that the homological equation becomes

$$\left[G_1(-4)\ G_2(-2)\ G_3(0)\ G_4(2)\ G_5(4)\right]\mathbf{u}^\dagger = \frac{1}{2m\omega_n^2}([0\,0\,R_3\,0\,0] - \frac{k_3}{4}[1\,4\,6\,4\,1])\mathbf{u}^\dagger, \tag{218}$$

[57] However, most treatments are from a mathematical perspective, and the scaling $m = \omega_n = 1$ is applied (either directly, or through some form of non-dimensionalisation) such that the transformation to new coordinates $\hat{\boldsymbol{x}} = [\hat{x}, \hat{p}]^T$, is carried out such that $\hat{x} = x + \mathbf{i}p$ and $\hat{p} = x - \mathbf{i}p$ (see for example [195]). This choice of coordinates will lead to the desired form of the Hamiltonian as defined by Birkhoff [11].

where the R_i coefficients have been set to zero except where a zero appears in $\{H_0, G\}$ on the left hand side of Eq. (218). So Eq. (218) can be satisfied by setting

$$R_3 = \frac{3k_3}{2}, \quad \text{and} \quad \mathbf{G}^* = \frac{\alpha}{8\omega_n^2}\left[\frac{1}{4} \;\; 2 \;\; 0 \;\; -2 \;\; -\frac{1}{4}\right] \quad \text{where} \quad \alpha = \frac{k_3}{m}, \tag{219}$$

and $\mathbf{G}^*$ is multiplied by $\mathbf{u}^\dagger$ to give the generator function.

Next we consider finding an approximate solution for x using the Poisson bracket $\{\mathbf{u}, G\}$

$$\{\mathbf{u}, G\} = \frac{\partial \mathbf{u}}{\partial u_p}\frac{\partial G}{\partial u_m} - \frac{\partial \mathbf{u}}{\partial u_m}\frac{\partial G}{\partial u_p}. \tag{220}$$

Because $\mathbf{u} = [u_p, u_m]^T$ is a vector this gives

$$\{\mathbf{u}, \mathbf{G}^*\mathbf{u}^\dagger\} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{G}^* \frac{\partial \mathbf{u}^\dagger}{\partial u_m} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{G}^* \frac{\partial \mathbf{u}^\dagger}{\partial u_p} \equiv \frac{\alpha}{8\omega_n^2}\begin{bmatrix} 2 & 0 & -6 & -1 \\ -1 & -6 & 0 & 2 \end{bmatrix} \mathbf{u}^*, \tag{221}$$

where $\mathbf{u}^*$ for this example is

$$\mathbf{u}^* = \begin{bmatrix} u_p^3 \\ u_p^2 u_m \\ u_p u_m^2 \\ u_m^3 \end{bmatrix}. \tag{222}$$

Then to obtain the approximate expression for x, we use Eq. (221) with Eq. (199) and Eq. (211) to obtain

$$x = q_p + q_m = u_p + u_m + \varepsilon \frac{\alpha}{8\omega_n^2}(u_p^3 + u_m^3 - 6(u_p^2 u_m + u_p u_m^2)) + \mathscr{O}(\varepsilon^2). \tag{223}$$

Now that the normal form has been obtained, we can substitute an assumed solution, which we take as the two base solutions (without frequency detuning or phase lag) given as

$$u_p = \frac{U}{2}\mathrm{e}^{\mathrm{i}\omega_n t} \quad \text{and} \quad u_m = \frac{U}{2}\mathrm{e}^{-\mathrm{i}\omega_n t}, \tag{224}$$

which then leads to a final expression for approximate displacement

$$x = (U - \varepsilon \frac{3\alpha U^3}{16\omega_n^2})\cos(\omega_n t) + \varepsilon \frac{\alpha U^3}{32\omega_n^2}\cos(3\omega_n t) + \mathscr{O}(\varepsilon^2). \tag{225}$$

This can be compared to Eq. (143), which is the result from Example 12 (Section 4.2.2). □

As noted in Example 11, one characteristic of the approximation in Eq. (225) is that each order ε will update each of the harmonic components in the series. So as more ε terms are added, the coefficients of each harmonic components are modified. This is in contrast to the real normal form expression from Example 12, Eq. (168), where the higher-order corrections are orthogonal to the fundamental part of the solution. However, because there are free functions (see for example [85]) in the $\mathbf{G}^*$ matrix, Eq. (225) can also be made to have orthogonal higher-order correction terms. In this case the coefficient G_3 can be chosen to have any value with no change to the homological equation. In this example, if $G_3 = 3$ is selected then the series solution becomes

$$x = U\cos(\omega_n t) + \varepsilon \frac{\alpha U^3}{32\omega_n^2}\cos(3\omega_n t) + \mathscr{O}(\varepsilon^2). \tag{226}$$

This is the same result as that obtained in Example 12, Eq. (168), when $\alpha_2 = 0$ and $\alpha_1 = \alpha$ are substituted.

Notice that although we have obtained a solution for x we have not yet defined the Kamiltonian or the equations or motion for this example. To do this we first use Eq. (201) from which the Kamiltonian can be written as

$$K(\mathbf{u}) = 2m\omega_n^2 u_p u_m + \varepsilon \frac{3k_3}{2} u_p^2 u_m^2 + \mathscr{O}(\varepsilon^2), \tag{227}$$

which is the normal form of the Hamiltonian up to order ε, and can be compared with previous results, for example in [195], which has the same structure, but different scaling, as part of the chosen coefficients and diagonalisation transform. Substituting Eq. (224) into Eq. (227) gives

$$K(\mathbf{u}) = k\frac{U^2}{2} + \varepsilon\frac{3k_3U^4}{32} + \mathscr{O}(\varepsilon^2), \tag{228}$$

which implies that $K(\mathbf{u})$ is independent of mass m, and is dependent on just the stiffness parameters k and k_3 together with the amplitude value U. As such, it represents a nonlinear potential energy function for the Duffing oscillator.

One of the advantages of using the Hamiltonian normal form is that you can work with Hamiltonians rather than equations of motion. However, the transformations introduce a scaling factor, which is related to the nature of the diagonalising transformation. To see this we note that in the original coordinates, $\boldsymbol{p} = [x, p]^T$ the following standard relationship holds

$$\dot{\boldsymbol{p}} = \{\boldsymbol{p}, H\} \tag{229}$$

which when evaluated gives Eq. (209). We would now like to find the equivalent version of this for the transformed coordinates $\mathbf{q}$ and $\mathbf{u}$. First consider the coordinate change from $\boldsymbol{p} \to \mathbf{q}$, for which Eq. (229), via the chain rule, becomes

$$\begin{aligned}
\frac{\partial x}{\partial q_p}\dot{q}_p + \frac{\partial x}{\partial q_m}\dot{q}_m &= \frac{\partial H}{\partial q_p}\frac{\partial q_p}{\partial p} + \frac{\partial H}{\partial q_m}\frac{\partial q_m}{\partial p}, \\
\frac{\partial p}{\partial q_p}\dot{q}_p + \frac{\partial p}{\partial q_m}\dot{q}_m &= -\left(\frac{\partial H}{\partial q_p}\frac{\partial q_p}{\partial x} + \frac{\partial H}{\partial q_m}\frac{\partial q_m}{\partial x}\right).
\end{aligned} \tag{230}$$

To compute the partial derivatives on the left of Eq. (230) we use Eq. (211) to find that

$$\frac{\partial x}{\partial q_p} = \frac{\partial x}{\partial q_m} = 1, \qquad \frac{\partial p}{\partial q_p} = m\lambda_1, \qquad \frac{\partial p}{\partial q_m} = m\lambda_2. \tag{231}$$

To find the partial derivatives on the right of Eq. (230) it is useful to explicitly define the inverse relationship of the transformation using Eq. (211) giving

$$\begin{bmatrix} q_p \\ q_m \end{bmatrix} = \frac{1}{m(\lambda_2 - \lambda_1)} \begin{bmatrix} m\lambda_2 & -1 \\ -m\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}. \tag{232}$$

From this we obtain

$$\frac{\partial q_p}{\partial p} = \frac{-1}{m(\lambda_2 - \lambda_1)}, \qquad \frac{\partial q_m}{\partial p} = \frac{1}{m(\lambda_2 - \lambda_1)}, \qquad \frac{\partial q_p}{\partial x} = \frac{1}{2}, \qquad \frac{\partial q_m}{\partial x} = \frac{1}{2}. \tag{233}$$

Now, using these definitions, we can rewrite Eq. (230) as

$$\begin{aligned}
\dot{q}_p + \dot{q}_m &= \frac{\partial H}{\partial q_p}\left(\frac{-1}{m(\lambda_2 - \lambda_1)}\right) + \frac{\partial H}{\partial q_m}\left(\frac{1}{m(\lambda_2 - \lambda_1)}\right), \\
m\lambda_1\dot{q}_p + m\lambda_2\dot{q}_m &= -\frac{1}{2}\left(\frac{\partial H}{\partial q_p} + \frac{\partial H}{\partial q_m}\right),
\end{aligned} \tag{234}$$

which can be solved as a pair of simultaneous equations. For example, multiplying the first line in Eq. (234) by $-m\lambda_1$ and adding the two equations results in

$$\begin{aligned}
\dot{q}_m m(\lambda_2 - \lambda_1) &= \frac{\partial H}{\partial q_p}\left(-\frac{1}{2}\right) + \frac{\partial H}{\partial q_m}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{\partial H}{\partial q_p} + \frac{\partial H}{\partial q_m}\right), \rightsquigarrow \\
\dot{q}_m &= -\frac{1}{m(\lambda_2 - \lambda_1)}\frac{\partial H}{\partial q_p} = \frac{1}{2m\mathbf{i}\omega_n}\frac{\partial H}{\partial q_p} = -\mathbf{i}\omega_n q_m + \varepsilon\frac{3k_3}{2m\mathbf{i}\omega_n}(q_p + q_m)^3,
\end{aligned} \tag{235}$$

where $\frac{\partial H}{\partial q_p}$ has been obtained from Eq. (212). Similarly, multiplying the second line in Eq. (234) by $-m\lambda_2$ and adding the two equations results in

$$\dot{q}_p = \frac{1}{m(\lambda_2-\lambda_1)}\frac{\partial H}{\partial q_m} = \mathbf{i}\omega_n q_p - \varepsilon\frac{3k_3}{2m\mathbf{i}\omega_n}(q_p+q_m)^3, \tag{236}$$

where $\frac{\partial H}{\partial q_m}$ has been obtained from Eq. (212).

Note that if we define $\hat{\rho} = 1/(m(\lambda_2-\lambda_1))$, then we have the modified Hamiltonian relationships[58]

$$\dot{q}_p = \frac{1}{\hat{\rho}}\frac{\partial H}{\partial q_m}, \quad \dot{q}_m = -\frac{1}{\hat{\rho}}\frac{\partial H}{\partial q_p}, \quad \rightsquigarrow \quad \dot{\mathbf{q}} = \frac{1}{\hat{\rho}}\{\mathbf{q}, H\}. \tag{237}$$

Now the transformation from $\mathbf{q} \to \mathbf{u}$ leads to the relationship

$$\begin{aligned} \dot{u}_p &= \frac{1}{\hat{\rho}}\frac{\partial K}{\partial u_m} = \mathbf{i}\omega_n u_p + \mathbf{i}\varepsilon\frac{3\alpha}{2\omega_n}u_p^2 u_m, \\ \dot{u}_m &= -\frac{1}{\hat{\rho}}\frac{\partial K}{\partial u_p} = -\mathbf{i}\omega_n u_m - \mathbf{i}\varepsilon\frac{3\alpha}{2\mathbf{i}\omega_n}u_p u_m^2, \end{aligned} \tag{238}$$

where, $\alpha = k_3/m$, and so $\dot{\mathbf{u}} = \frac{1}{\hat{\rho}}\{\mathbf{u}, K\}$, which is exactly the same result we obtained using the complex normal form in Example 11, as given in Eq. (141). Note that the scaling factor, $\hat{\rho}$ relates to the term multiplying the inverse matrix in Eq. (232).

4.5 Damping in normal form transformations

For the linear systems in Section 2 and Section 3 dealing with damped oscillators is relatively straight forward (assuming viscous damping) and leads to well established results. However, in nonlinear normal form analysis, the inclusion of damping is more complicated. We motivate this discussion, with the following example.

4.5.1 Example 14: the damped, unforced escape equation

The damped, unforced escape equation can be written as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x + \beta x^2 = 0, \quad \rightsquigarrow \quad \dot{\boldsymbol{x}} = A\boldsymbol{x} + \tilde{\mathcal{N}}(\boldsymbol{x}): \quad A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad \tilde{\mathcal{N}}(\boldsymbol{x}) = \begin{Bmatrix} 0 \\ -\beta x_1^2 \end{Bmatrix}, \tag{239}$$

where the states are $\boldsymbol{x} = \{x_1 \;\; x_2\}^T = \{x \;\; \dot{x}\}^T$. Transforming to new coordinates $\boldsymbol{q} = \{q_p, q_m\}^T$ using the linear normal modes of the system, $\boldsymbol{x} = \Phi\boldsymbol{q}$ (see Section 3.2) gives modal matrices

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, \quad \Phi^{-1} = \frac{1}{\lambda_2-\lambda_1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix},$$

where λ_1 and λ_2 are the eigenvalues of A, given by $\lambda_{1,2} = -\zeta\omega_n \pm \mathbf{i}\omega_d$. The transformed equation of motion is given by

$$\dot{\boldsymbol{q}} = \Lambda\boldsymbol{q} + \varepsilon\boldsymbol{n}_1(\boldsymbol{q}) \quad \text{where} \quad \boldsymbol{n}_1(\boldsymbol{q}) = \Phi^{-1}\tilde{\mathcal{N}}(\Phi\boldsymbol{q}) = \frac{\beta}{\lambda_2-\lambda_1}\begin{bmatrix} (q_p+q_m)^2 \\ -(q_p+q_m)^2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{n}_2 = 0, \ldots. \tag{240}$$

We know from Example 10 (Section 4.1.1) that quadratic terms can be eliminated at order $k=2$ or ε^1 and so we assume that a nonlinear coordinate transformation from coordinate $\boldsymbol{q}$ to a new coordinate set $\boldsymbol{u}$, is possible obtain a linear system of the form

$$\dot{\boldsymbol{u}} = \Lambda\boldsymbol{u} + \mathscr{O}(\varepsilon^2) \qquad \text{or written in full} \qquad \begin{bmatrix} \dot{u}_p \\ \dot{u}_m \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} u_p \\ u_m \end{bmatrix} + \mathscr{O}(\varepsilon^2), \tag{241}$$

[58] We note that [6] also comments on modified Hamiltonian relationships for the case where $\hat{\rho} = \mathbf{i}$.

which means completely eliminating the nonlinear term $\boldsymbol{n}_1(\boldsymbol{q})$ in Eq. (240)[59]. To do this we make the transformation

$$\boldsymbol{q} = \boldsymbol{u} + \varepsilon \boldsymbol{h}(\boldsymbol{u}) = \begin{bmatrix} u_p \\ u_m \end{bmatrix} + \varepsilon \begin{bmatrix} h_1(u_p, u_m) \\ h_2(u_p, u_m) \end{bmatrix} = \begin{bmatrix} u_p \\ u_m \end{bmatrix} + \varepsilon \begin{bmatrix} b_1 u_p^2 + b_2 u_p u_m + b_3 u_m^2 \\ b_4 u_p^2 + b_5 u_p u_m + b_6 u_m^2 \end{bmatrix} + \mathscr{O}(\varepsilon^2), \tag{242}$$

where the dots are used to denote that the $h_j(u_p, u_m)$ series can have higher order terms, and we have included up to quadratic terms because the nonlinear term in Eq. (239) is quadratic. The b_j are coefficients that are yet to be identified. Substituting the first of Eq. (242) and Eq. (241) into Eq. (240) leads to order-ε^1 homological equation of the form

$$\varepsilon^1 : \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{h} - \Lambda \boldsymbol{h} = \boldsymbol{n}(\boldsymbol{u}) \quad \rightsquigarrow \quad \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}} \Lambda \boldsymbol{u} - \Lambda \boldsymbol{h} = \boldsymbol{n}(\boldsymbol{u}) \quad \rightsquigarrow \quad \boldsymbol{b}\Lambda^* \mathbf{u}^* - \Lambda \boldsymbol{b}\mathbf{u}^* = \boldsymbol{n}^* \mathbf{u}^*. \tag{243}$$

In order to compute the Lie derivative we use

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}} \Lambda \boldsymbol{u} = \begin{bmatrix} \frac{\partial h_1}{\partial u_p} & \frac{\partial h_1}{\partial u_m} \\ \frac{\partial h_2}{\partial u_p} & \frac{\partial h_2}{\partial u_m} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_p \\ u_m \end{bmatrix}. \tag{244}$$

As a result the order-ε^1 homological equation Eq. (243) can be written as

$$\begin{bmatrix} b_1(-\lambda_1) & b_2(-\lambda_2) & b_3(\lambda_1 - 2\lambda_2) \\ b_4(\lambda_2 - 2\lambda_1) & b_5(-\lambda_1) & -b_6(\lambda_2) \end{bmatrix} \mathbf{u}^* = -\frac{\beta}{\lambda_2 - \lambda_1} \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix} \mathbf{u}^*, \tag{245}$$

where $\mathbf{u}^* = [u_p^2 \quad u_p u_m \quad u_m^2]^2$. Eq. (245) can be solved for the b_j coefficients and substituted into Eq. (242) to give a transformation of

$$\boldsymbol{q} = \boldsymbol{u} + \boldsymbol{h}(\boldsymbol{u}) = \begin{bmatrix} u_p \\ u_m \end{bmatrix} + \varepsilon \begin{bmatrix} \frac{\beta}{(\lambda_2 - \lambda_1)\lambda_1} u_p^2 + \frac{2\beta}{(\lambda_2 - \lambda_1)\lambda_2} u_p u_m - \frac{\beta}{(\lambda_2 - \lambda_1)(\lambda_1 - 2\lambda_2)} u_m^2 \\ \frac{\beta}{(\lambda_2 - \lambda_1)(\lambda_2 - 2\lambda_1)} u_p^2 - \frac{2\beta}{(\lambda_2 - \lambda_1)\lambda_1} u_p u_m - \frac{\beta}{(\lambda_2 - \lambda_1)\lambda_2} u_m^2 \end{bmatrix} + \mathscr{O}(\varepsilon^2) \tag{246}$$

up to quadratic terms. □

This analysis relies on the denominator terms such as $(\lambda_2 - 2\lambda_1)$ that occur in Eq. (245) and Eq. (246) not becoming zero (otherwise the system would have resonances). In this case, the eigenvalues are complex, and therefore they are in the Poincaré domain (see Fig. 4), meaning non-resonant. However, what happens as the damping becomes small and $\zeta \to 0$ so that $\lambda_{1,2} \to \pm \mathbf{i}\omega_n$? In that case, we might want to consider the case when the eigenvalues are in the Siegel domain, and therefore the homological equation detects the resonant terms (see Section 4.1.2 for a more detailed discussion of resonance in this context). There are three methods that can do this, and these are:

Method 1: A solution to this problem proposed by Jezequel & Lamarque [80] was to reformulate the eigenvalues using the idea of a *detuning* parameter (using a k-order methodology). So for Example 14 where the eigenvalues are $\lambda_{1,2} = -\zeta\omega_n \pm \mathbf{i}\omega_d$ we write

$$\begin{aligned} \lambda_1 &= \mathbf{i}\omega_n + \delta \\ \lambda_2 &= -\mathbf{i}\omega_n + \bar{\delta} \end{aligned} \tag{247}$$

where δ is the detuning parameter such that, $\mathrm{Re}\delta = -\zeta\omega_n$, $\mathrm{Im}\delta = \omega_n(1 - \sqrt{1-\zeta^2})$ and $|\delta| \ll 1$ meaning that $\zeta \ll 1$. Here $\mathrm{Im}\delta$ is used to represent the detuning (meaning small difference) between ω_n and ω_d.

Applying this approach to Example 14, Eq. (244) becomes

$$\begin{aligned} \frac{\partial \mathbf{h}}{\partial \boldsymbol{u}} \Lambda \boldsymbol{u} &= \begin{bmatrix} \frac{\partial h_1}{\partial u_p} & \frac{\partial h_1}{\partial u_m} \\ \frac{\partial h_2}{\partial u_p} & \frac{\partial h_2}{\partial u_m} \end{bmatrix} \begin{bmatrix} \mathbf{i}\omega_n & 0 \\ 0 & -\mathbf{i}\omega_n \end{bmatrix} \begin{bmatrix} u_p \\ u_m \end{bmatrix} + \text{terms in } \delta, \\ \text{and} \quad \frac{1}{\lambda_2 - \lambda_1} &= \frac{1}{-2\mathbf{i}\omega_n} + \text{terms in } \delta \end{aligned} \tag{248}$$

[59] Note that because $\boldsymbol{n}_2 = 0, \ldots$ etc. we use $\boldsymbol{n}_1 = \boldsymbol{n}$, and correspondingly $\boldsymbol{b}_1 = \boldsymbol{b}$ and $\hat{\boldsymbol{n}}_{u1} = \hat{\boldsymbol{n}}_u$.

where the terms in δ are assumed to be relevant only at the next highest order than the one currently being computed (e.g in this example we are computing to quadratic, so δ terms are assumed to be included at cubic order). Now substituting these new expressions into Eq. (246) (with $\varepsilon = 1$) gives

$$\mathbf{q} = \boldsymbol{u} + \mathbf{h}(\boldsymbol{u}) = \begin{bmatrix} u_p \\ u_m \end{bmatrix} + \frac{\beta}{\omega_n^2} \begin{bmatrix} \frac{u_p^2}{2} & -u_p u_m & -\frac{u_m^2}{6} \\ -\frac{u_p^2}{6} & -u_p u_m & \frac{u_m^2}{2} \end{bmatrix}, \tag{249}$$

which once the base solutions are substituted, and using $x = x_1 = q_p + q_m$, gives the same solution for x as Eq. (105) (the undamped normal form). The normal form equations, Eq. (241), become

$$\begin{aligned} \dot{u}_p = &\quad +\mathbf{i}\omega_n u_p + \delta u_p, \\ \dot{u}_m = &\quad -\mathbf{i}\omega_n u_m + \bar{\delta} u_m, \end{aligned} \tag{250}$$

which is a linear damped system. We note at this order, damping appears in Eq. (250), but not in the series solution for x_1 (e.g. Eq. (105)) because the damping terms have been shifted up to the next order. To see the effect of damping in the solution for x_1 it would be necessary to compute the next order of the normal form.

The concept of detuning is a very important one for nonlinear normal forms and we will return to it in Section 4.6. However, to continue the discussion on including damping in normal form transformations of nonlinear systems we mention two further methods.

Method 2: This is to use the method of reduction of order presented in Section 2.3. Let's consider how this might work for the damped escape equation from Example 14 (Section 4.5.1). The damped, unforced escape equation (e.g. Eq. (239)) can be expressed as

$$\ddot{x} + \varepsilon\dot{x} + \omega_n^2 x + \varepsilon\hat{\beta}x^2 = 0, \tag{251}$$

where we have rescaled the β coefficient such that $\beta = \varepsilon\hat{\beta}$ and $\varepsilon = 2\zeta\omega_n$. This rescaling means that we are assuming that both the damping and nonlinear coefficient are of the order ε^1 small, which is a reasonable assumption for many weakly nonlinear oscillators, but will restrict the parameter values for which this method can be applied. Next make the reduction of order substitution $x = q(t)\mathrm{e}^{-\frac{\varepsilon}{2}t}$ so that the escape equation becomes

$$\ddot{q} + \omega_d^2 x + \varepsilon \mathrm{e}^{-\frac{\varepsilon}{2}t}\hat{\beta}q^2 = 0. \tag{252}$$

Now, assuming ε is a small quantity we can apply an approximation of the exponential function $\mathrm{e}^{-\frac{\varepsilon}{2}t} = 1 - \frac{\varepsilon}{2}t + \ldots\mathscr{O}(\varepsilon^2 t^2)$ we obtain

$$\ddot{q} + \omega_d^2 x + \varepsilon(1 - \frac{\varepsilon}{2}t)\hat{\beta}q^2 = 0 \rightsquigarrow \ddot{q} + \omega_d^2 x + \varepsilon\hat{\beta}q^2 + \ldots\mathscr{O}(\varepsilon^2) = 0, \tag{253}$$

which is valid when ε is small, and is equivalent to the case when damping is small, $\zeta \ll 1$. Now Eq. (253) has the same structure as the *undamped* escape equation, except now the damping is included in the damped natural frequency, ω_d. Applying the normal form transformation to Eq. (253) gives a normal form equation (e.g. Eq. (83) with $n_u^{(2)} = n_u^{(3)} = 0$ and ω_d replacing ω_n) of

$$\ddot{u} + \omega_d^2 u = 0. \tag{254}$$

and a solution for x (e.g. Eq. (105) with ω_d replacing ω_n) of

$$x = U\cos(\omega_d t) + \frac{\beta U^2}{6\omega_d^2}\cos(2\omega_d t) - \frac{\beta U^2}{2\omega_d^2}. \tag{255}$$

Notice that now the solution for x in Eq. (255) is in terms of ω_d, but there is no exponential decay, and just like the Jezequel & Lamarque [80] solution, higher order terms need to be computed to include these effects, which in this case are the $\mathscr{O}(\varepsilon^2)$ terms[60]. In fact computing the higher order terms in both the above methods is non-trivial, and so we present a third method which is used in preference to both previous methods.

[60]The interested reader might want to study the related method proposed by Burton [17] which avoids the expansion of the exponential term.

Method 3: In this method we treat the damping terms as part of the nonlinear vector such that the unforced version of Eq. (79) becomes

$$M\ddot{\mathbf{x}} + K\mathbf{x} + \mathscr{N}_d(\mathbf{x}, \dot{\mathbf{x}}) = 0, \tag{256}$$

where the $\mathscr{N}_d$ vector now also contains the damping terms. The particular logic for doing this for systems of equations written in second-order form, has been described in Section 4.3.2, and for such systems this is the most straightforward method to implement. Further details can be found in [183]. An example will be shown in Section 4.6.1

4.6 Frequency detuning

We now consider an important modelling technique that will allow us to distinguish between a response frequency, ω_r, that is different from the linear damped or undamped natural frequencies, ω_d and ω_n respectively. This will enable us to develop the idea of amplitude-dependent *backbone* curves in Section 4.8, although it should be noted that backbone curves (and other types of parameter variation) can be obtained using many other techniques — see for example [163, 112, 148, 149, 150, 90, 50, 53, 119, 173, 101, 119]. We consider the multi-frequency case (i.e. more than a single-degree-of-freedom) where

$$\omega_{ni}^2 = \omega_{ri}^2 + \varepsilon\delta_i + \mathscr{O}(\varepsilon^2) \quad \text{for} \quad i = 1, 2, 3...N. \tag{257}$$

Then we can form a matrix equation

$$\hat{\Lambda} = \Upsilon + \varepsilon\Delta + \mathscr{O}(\varepsilon^2) \quad \text{where} \quad \Upsilon = \begin{bmatrix} \omega_{r1}^2 & 0 & \dots & 0 & 0 \\ 0 & \omega_{r2}^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \omega_{rN}^2 \end{bmatrix}, \tag{258}$$

the matrix $\hat{\Lambda}$ is given by Eq. (70) and Δ is a diagonal matrix of the δ_i values.

Now consider the case when the homological equation is derived up to second order (Eq. (154) and Eq. (155) combined), which gives

$$-\varepsilon\hat{\mathbf{n}}_{\mathbf{u1}}(\mathbf{u}) - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{u2}}(\mathbf{u}) + \varepsilon\ddot{\mathbf{h}}_1 + \varepsilon^2\ddot{\mathbf{h}}_2 = -\hat{\Lambda}(\varepsilon\mathbf{h}_1 + \varepsilon^2\mathbf{h}_2) - \varepsilon\hat{\mathbf{n}}_{\mathbf{1}}(\mathbf{u}) - \varepsilon^2\frac{\partial\hat{\mathbf{n}}_{\mathbf{1}}(\mathbf{u})}{\partial\mathbf{u}}\mathbf{h}_1 - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{2}}(\mathbf{u}), \tag{259}$$

where $\frac{\partial\hat{\mathbf{n}}}{\partial\mathbf{u}}$ is the Jacobian matrix (or gradient vector) depending on the dimensions of $\mathbf{u}$, and an overdot represents $\frac{\mathrm{d}}{\mathrm{d}t}$. Now we introduce detuning using Eq. (258) to give

$$-\varepsilon\hat{\mathbf{n}}_{\mathbf{u1}}(\mathbf{u}) - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{u2}}(\mathbf{u}) + \varepsilon\ddot{\mathbf{h}}_1 + \varepsilon^2\ddot{\mathbf{h}}_2 = (\Upsilon + \varepsilon\Delta)(\varepsilon\mathbf{h}_1 + \varepsilon^2\mathbf{h}_2) - \varepsilon\hat{\mathbf{n}}_{\mathbf{1}}(\mathbf{u}) - \varepsilon^2\frac{\partial\hat{\mathbf{n}}_{\mathbf{1}}(\mathbf{u})}{\partial\mathbf{u}}\mathbf{h}_1 - \varepsilon^2\hat{\mathbf{n}}_{\mathbf{2}}(\mathbf{u}). \tag{260}$$

Now it can be seen from Eq. (260) that the terms at orders ε^1 and ε^2 can be equated separately such that

$$\begin{aligned} \varepsilon^1: &\quad \hat{\mathbf{n}}_{\mathbf{1}} - \hat{\mathbf{n}}_{\mathbf{u1}} = -\ddot{\mathbf{h}}_1 - \Upsilon\mathbf{h}_1 = -\mathbf{b}_1\left(\Lambda^*{}_d\right)^2\mathbf{u}^* - \Upsilon\mathbf{b}_1\mathbf{u}^*, \\ \varepsilon^2: &\quad \Delta\mathbf{h}_1 + \frac{\partial\hat{\mathbf{n}}_{\mathbf{1}}}{\partial\mathbf{u}}\mathbf{h}_1 + \hat{\mathbf{n}}_{\mathbf{2}} - \hat{\mathbf{n}}_{\mathbf{u2}} = -\ddot{\mathbf{h}}_2 - \Upsilon\mathbf{h}_2 = -\mathbf{b}_2\tilde{\Upsilon}^2\mathbf{u}^+ - \Upsilon\mathbf{b}_2\mathbf{u}^+. \end{aligned} \tag{261}$$

The first line in Eq. (261) is the detuned order-ε^1 homological equation (compare with in Eq. (156)), where $\Lambda^*{}_d$ represents the Lie derivative coefficient matrix obtained from Eq. (127) (and Eq. (153)) when the detuned system is being considered. This type of detuned formulation will be used for the rest of the examples in this paper, starting with Examples 15 and 16 (Section 4.6.1 and Section 4.6.2 respectively).

The second line in Eq. (261) gives the detuned order-ε^2 homological equation which needs to be solved if the normal form is to be computed to order ε^2 or higher. In this case the basis function is $\mathbf{u}^+$ such that $\mathbf{h}_2 = \mathbf{b}_2\mathbf{u}^+$. Notice also that evaluating the $\ddot{\mathbf{h}}_2$ term leads to a $\tilde{\Upsilon}^2$ term. This term is equivalent to $(\Lambda^*{}_d)^2$, and is computed as the Lie derivative coefficient matrix obtained from Eq. (127) (and Eq. (153)) when the detuned system is being considered. This will be demonstrated in Example 18, Section 4.8.1. Now we consider two detuned examples computed to order ε^1.

4.6.1 Example 15 the unforced Van der Pol oscillator

The unforced Van der Pol oscillator can be written as

$$\ddot{x}+\varepsilon(\mu x^2-\gamma)\dot{x}+\omega_n^2 x=0,\ \text{with}\ x=q \rightsquigarrow \quad \ddot{q}+\omega_n^2 q+\varepsilon(\mu q^2-\gamma)\dot{q}=0, \quad \rightsquigarrow \quad \ddot{q}+\omega_n^2 q+\varepsilon\hat{\mathbf{n}}_1(q_p,q_m)=0, \tag{262}$$

where by using $q=q_p+q_m$ and $\dot{q}=\mathbf{i}\omega_r(q_p-q_m)$ we can obtain

$$\hat{\mathbf{n}}_1(q_p,q_m)=\mathbf{i}\omega_r(\mu q_p^3+\mu q_p^2 q_m-\mu q_p q_m^2-\mu q_m^3-\gamma q_p+\gamma q_m)=\mathbf{i}\omega_r[\mu \ \ \mu \ -\mu \ -\mu \ -\gamma \ \ \gamma]\mathbf{q}^*, \tag{263}$$

where $\mathbf{q}^*=[q_p^3 \ \ q_p^2 q_m \ \ q_p q_m^2 \ \ q_m^3 \ \ q_p \ \ q_m]^T$ and $\mathbf{n}_2=0$[61]. Notice that, as described in Section 4.5.1, we are including the linear damping term in the nonlinear vector $\hat{\mathbf{n}}(q_p,q_m)$.

Now apply a near-identity nonlinear transform, Eq. (116), of the form

$$q=u+\varepsilon\mathbf{b}\mathbf{u}^* \qquad \text{where} \qquad \mathbf{b}=[b_1 \ \ b_2 \ \ b_3 \ \ b_4 \ \ b_5 \ \ b_6], \tag{264}$$

and $\mathbf{u}^*=[u_p^3 \ \ u_p^2 u_m \ \ u_p u_m^2 \ \ u_m^3 \ \ u_p \ \ u_m]^T$. The transformed dynamic equation, Eq. (146), is expressed in the new coordinates as

$$\ddot{u}+\omega_n^2 u+\varepsilon\hat{\mathbf{n}}_u(u_p,u_m)=0. \tag{265}$$

Substituting Eq. (264) and Eq. (265) into Eq. (262) (with detuning) leads to the ε^1 homological equation $\mathscr{H}_1(\mathbf{u})$ given by

$$\varepsilon^1: \quad -\mathbf{b}\frac{\mathrm{d}^2\mathbf{u}^*}{\mathrm{d}t^2}-\omega_r^2\mathbf{b}\mathbf{u}^*=\hat{\mathbf{n}}-\hat{\mathbf{n}}_{\mathbf{u}}=-\mathbf{b}\Lambda_d^{*2}\mathbf{u}^*-\Upsilon\mathbf{b}\mathbf{u}^*. \tag{266}$$

Then using $\hat{\mathbf{n}}_{\mathbf{u}}=[\hat{n}_{u1} \ \ \hat{n}_{u2} \ \ \hat{n}_{u3} \ \ \hat{n}_{u4} \ \ \hat{n}_{u5} \ \ \hat{n}_{u6}]\mathbf{u}^*$, the second Lie derivative Eq. (153), with $q_i\rightarrow u_i$ (to order ε^0) gives

$$\begin{aligned}&(\mathbf{i}\omega_r[\mu \ \ \mu \ -\mu \ -\mu \ -\gamma \ \ \gamma]-[\hat{n}_{u1} \ \ \hat{n}_{u2} \ \ \hat{n}_{u3} \ \ \hat{n}_{u4} \ \ \hat{n}_{u5} \ \ \hat{n}_{u6}])\,\mathbf{u}^*=\\ &\left(-(\mathbf{i}\omega_r)^2\begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6\end{bmatrix}\begin{bmatrix}3&0&0&0&0&0\\0&1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&-3&0&0\\0&0&0&0&1&0\\0&0&0&0&0&-1\end{bmatrix}^2-\omega_r^2\begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6\end{bmatrix}\right)\mathbf{u}^*.\end{aligned} \tag{267}$$

Evaluating the right hand side gives four zero values which allows the $\hat{\mathbf{n}}_{\mathbf{u}}$ coefficients to be identified such that

$$(\mathbf{i}\omega_r[\mu \ \ \mu \ -\mu \ -\mu \ -\gamma \ \ \gamma]-\mathbf{i}\omega_r[0 \ \ \mu \ \ -\mu \ \ 0 \ \ -\gamma \ \ \gamma])\,\mathbf{u}^*=[b_1(8\omega_r^2) \ \ b_2(0) \ \ b_3(0) \ \ b_4(8\omega_n^2) \ \ b_5(0) \ \ b_6(0)]\mathbf{u}^*. \tag{268}$$

From this, the transform coefficient values are $b_1=\mathbf{i}\mu/8\omega_r$, $b_4=-\mathbf{i}\mu/8\omega_r$, and $b_2=b_3=b_5=b_6=0$. Note that b_2, b_3, b_5 and b_6 are the free functions in this example, as any value can be selected without affecting the outcome of the homological equation. Other choices will be discussed in Section 4.7. Next, using the fact that

$$\varepsilon\hat{\mathbf{n}}_{\mathbf{u}}(u_p,u_m)=\varepsilon\mathbf{i}\omega_r[0 \ \ \mu \ \ -\mu \ \ 0 \ \ -\gamma \ \ \gamma]\mathbf{u}^*=\varepsilon\mathbf{i}\omega_r(\mu u_p^2 u_m-\mu u_p u_m^2-\gamma u_p+\gamma u_m)=\varepsilon\mathbf{i}\omega_r(\mu u_p u_m-\gamma)(u_p-u_m),$$

and noting that $\mathbf{i}\omega_r(u_p-u_m)=\dot{u}$ gives the ε^1 real normal form as

$$\ddot{u}+\omega_n^2 u+\varepsilon(\mu u_p u_m-\gamma)\dot{u}=0, \quad \rightsquigarrow \quad \ddot{u}+\omega_n^2 u+\varepsilon(\mu\frac{U^2}{4}-\gamma)\dot{u}=0, \tag{269}$$

when the base solutions are used in the $u_p u_m$ term. In fact, substituting the base solutions into Eq. (269) gives

$$(\mathbf{i}\omega_r)^2\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t}+\mathrm{e}^{-\mathbf{i}\omega_r t})+\omega_n^2\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t}+\mathrm{e}^{-\mathbf{i}\omega_r t})+\mathbf{i}\omega_r\varepsilon(\mu\frac{U^2}{4}-\gamma)\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t}-\mathrm{e}^{-\mathbf{i}\omega_r t})=0. \tag{270}$$

[61] As $\mathbf{n}_2=0,\ldots$ etc. we set $\mathbf{n}_1=\mathbf{n}$, and correspondingly, $\mathbf{h}_1=\mathbf{h}$, $\mathbf{b}_1=\mathbf{b}$ and $\hat{\mathbf{n}}_{\mathbf{u}1}=\hat{\mathbf{n}}_{\mathbf{u}}$ throughout this example.

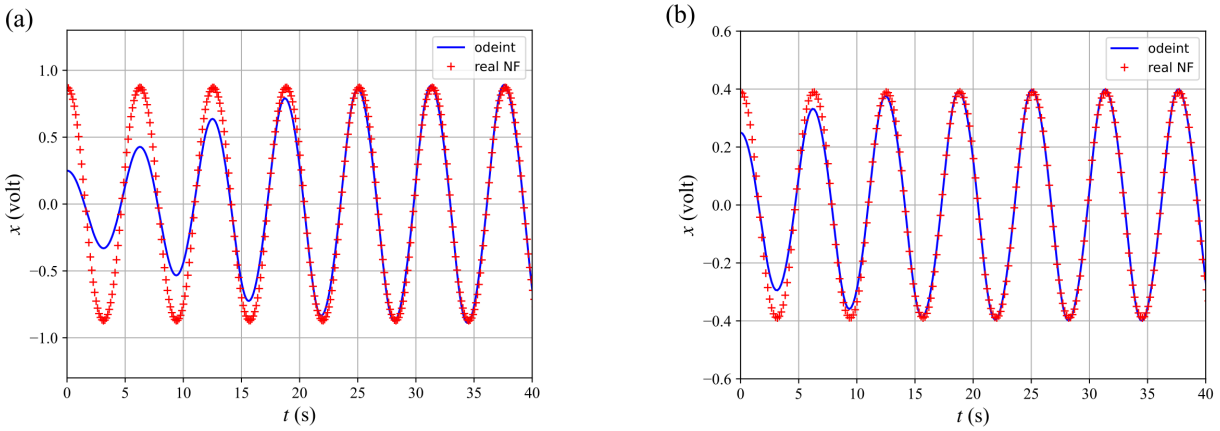

Fig. 6: Two simulations of the undamped Van der Pol oscillator showing the solution, x, from Example 15 (real normal form shown with red crosses). The normal form solution is compared with a reference solution generated from a fourth-order Runge-Kutta integration (the 'odeint' numerical integration routine from the Python numerical library — blue solid line). The real normal form solution is computed with Eq. (273) (red crosses). Parameter values are: $x(0) = 0.025$, $\gamma = 0.2$, and $\omega_n = 1$. In panel (a) the nonlinearity is relatively small $\mu = 1$, so that $U = 0.89$, and in panel (b) the nonlinearity is five times larger, $\mu = 5$, so that $U = 0.4$. The initial condition for the odeint simulation has been deliberately set to be less than the steady state amplitude of the limit cycle (i.e. periodic solution), in order to demonstrate how the transient solution converges to the limit cycle as time increases.

The real and imaginary parts of the $e^{\mathbf{i}\omega_r t}$ terms (or the $e^{-\mathbf{i}\omega_r t}$) can be balanced to give

$$\begin{aligned} &\text{Re}: \quad \omega_r^2 = \omega_n^2, \quad \rightsquigarrow \quad \omega_r = \omega_n, \\ &\text{Im}: \quad \omega_r \varepsilon(\mu \frac{U^2}{4} - \gamma) = 0, \quad \rightsquigarrow \quad U = \sqrt{4\frac{\gamma}{\mu}}, \end{aligned} \tag{271}$$

which corresponds to the classical solution to the unforced Van der Pol oscillator with a single limit cycle solution. Using $\omega_r = \omega_n$, the near-identity transform, to order ε^1 may now be written as

$$x = u + \varepsilon \mathbf{b}\mathbf{u}^* = \frac{U}{2}(e^{\mathbf{i}\omega_n t} + e^{-\mathbf{i}\omega_n t}) + \varepsilon \begin{bmatrix} \frac{\mathbf{i}\mu}{8\omega_n} & 0 & 0 & \frac{-\mathbf{i}\mu}{8\omega_n} & 0 & 0 \end{bmatrix} \mathbf{u}^*, \tag{272}$$

which gives the solution for x as

$$x = \sqrt{4\frac{\gamma}{\mu}}\cos(\omega_n t) - \varepsilon \frac{\mu}{32\omega_n}\left(\sqrt{4\frac{\gamma}{\mu}}\right)^3 \sin(3\omega_n t) + \mathscr{O}(\varepsilon^2). \tag{273}$$ □

A numerically computed example is shown in Fig. 6.

4.6.2 Example 16 the unforced Rayleigh oscillator

The unforced Rayleigh oscillator can be written as[62]

$$\ddot{x} + \varepsilon(\frac{\dot{x}^2}{3} - 1)\dot{x} + \omega_n^2 x = 0, \text{ with } x = q \rightsquigarrow \quad \ddot{q} + \omega_n^2 q + \varepsilon(\frac{\dot{q}^2}{3} - 1)\dot{q} = 0, \quad \rightsquigarrow \quad \ddot{q} + \omega_n^2 q + \varepsilon \hat{\mathbf{n}}(q_p, q_m) = 0, \tag{274}$$

where by using $q = q_p + q_m$ and $\dot{q} = \mathbf{i}\omega_r(q_p - q_m)$ we can obtain

$$\hat{\mathbf{n}}(q_p, q_m) = -\mathbf{i}\omega_r(\frac{\omega_r^2}{3}(q_p^3 - 3q_p^2 q_m + 3q_p q_m^2 - q_m^3) + q_p - q_m) = \mathbf{i}\omega_r[-\frac{\omega_r^2}{3} \quad \omega_r^2 \quad -\omega_r^2 \quad \frac{\omega_r^2}{3} \quad -1 \quad 1]\mathbf{q}^*, \tag{275}$$

where $\mathbf{q}^* = [q_p^3 \quad q_p^2 q_m \quad q_p q_m^2 \quad q_m^3 \quad q_p \quad q_m]^T$. Notice that, as described in Section 4.5.1, we are including the linear damping term in the nonlinear vector $\hat{\mathbf{n}}(q_p, q_m)$.

[62] As before $\mathbf{n}_2 = 0, \ldots$ etc. so we set $\mathbf{n}_1 = \mathbf{n}$, and correspondingly, $\mathbf{h}_1 = \mathbf{h}$, $\mathbf{b}_1 = \mathbf{b}$ and $\hat{\mathbf{n}}_{\mathbf{u}1} = \hat{\mathbf{n}}_{\mathbf{u}}$ throughout this example.

Now apply a near-identity nonlinear transform, Eq. (116), of the form

$$q = u + \varepsilon \mathbf{b}\mathbf{u}^* \qquad \text{where} \qquad \mathbf{b} = [b_1 \;\; b_2 \;\; b_3 \;\; b_4 \;\; b_5 \;\; b_6], \tag{276}$$

and $\mathbf{u}^* = [u_p^3 \;\; u_p^2 u_m \;\; u_p u_m^2 \;\; u_m^3 \;\; u_p \;\; u_m]^T$. The transformed dynamic equation, Eq. (146), is expressed in the new coordinates as

$$\ddot{u} + \omega_n^2 u + \varepsilon \hat{\mathbf{n}}_u(u_p, u_m) = 0. \tag{277}$$

Substituting Eq. (276) and Eq. (277) into Eq. (274) (with detuning) leads to the ε^1 homological equation $\hat{\mathscr{H}}_1(\mathbf{u})$ given by

$$\varepsilon^1 : \quad -\mathbf{b}\frac{\mathrm{d}^2\mathbf{u}^*}{\mathrm{d}t^2} - \omega_r^2 \mathbf{b}\mathbf{u}^* = \hat{\mathbf{n}} - \hat{\mathbf{n}}_{\mathbf{u}} = -\mathbf{b}\Lambda^{*2}_d \mathbf{u}^* - \Upsilon \mathbf{b}\mathbf{u}^*. \tag{278}$$

Then using $\hat{\mathbf{n}}_{\mathbf{u}} = [\hat{n}_{u1} \;\; \hat{n}_{u2} \;\; \hat{n}_{u3} \;\; \hat{n}_{u4} \;\; \hat{n}_{u5} \;\; \hat{n}_{u6}]\mathbf{u}^*$, the second Lie derivative Eq. (153), with $q_i \to u_i$ (to order ε^0) gives

$$\begin{aligned}
&\left(\mathbf{i}\omega_r[-\frac{\omega_r^2}{3} \;\; \omega_r^2 \;\; -\omega_r^2 \;\; \frac{\omega_r^2}{3} \;\; -1 \;\; 1] - [\hat{n}_{u1} \;\; \hat{n}_{u2} \;\; \hat{n}_{u3} \;\; \hat{n}_{u4} \;\; \hat{n}_{u5} \;\; \hat{n}_{u6}]\right)\mathbf{u}^* = \\
&\left(-(\mathbf{i}\omega_r)^2 \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}^2 - \omega_r^2 \begin{bmatrix} b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}\right)\mathbf{u}^*.
\end{aligned} \tag{279}$$

Evaluating the right hand side gives four zero values which allows the $\hat{\mathbf{n}}_{\mathbf{u}}$ coefficients to be identified such that

$$\left(\mathbf{i}\omega_r[-\frac{\omega_r^2}{3} \;\; \omega_r^2 \;\; -\omega_r^2 \;\; \frac{\omega_r^2}{3} \;\; -1 \;\; 1] - \mathbf{i}\omega_r[0 \;\; \omega_r^2 \;\; -\omega_r^2 \;\; 0 \;\; -1 \;\; 1]\right)\mathbf{u}^* = [b_1(8\omega_r^2) \;\; b_2(0) \;\; b_3(0) \;\; b_4(8\omega_n^2) \;\; b_5(0) \;\; b_6(0)]\mathbf{u}^*. \tag{280}$$

From this the transform coefficient values are $b_1 = \mathbf{i}\omega_r/24$, $b_4 = -\mathbf{i}\omega_r/24$, and $b_2 = b_3 = b_5 = b_6 = 0$. Next, using the fact that

$$\varepsilon\hat{\mathbf{n}}_{\mathbf{u}}(u_p, u_m) = \varepsilon\mathbf{i}\omega_r[0 \;\; \omega_r^2 \;\; -\omega_r^2 \;\; 0 \;\; -1 \;\; 1]\mathbf{u}^* = \varepsilon\mathbf{i}\omega_r(\omega_r^2 u_p^2 u_m - \omega_r^2 u_p u_m^2 - u_p + u_m) = \varepsilon\mathbf{i}\omega_r(\omega_r^2 u_p u_m - 1)(u_p - u_m),$$

and noting that $\mathbf{i}\omega_r(u_p - u_m) = \dot{u}$ gives the ε^1 real normal form as

$$\ddot{u} + \omega_n^2 u + \varepsilon(\omega_r^2 u_p u_m - 1)\dot{u} = 0, \tag{281}$$

Substituting the base solutions into Eq. (281) gives

$$(\mathbf{i}\omega_r)^2 \frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t} + \mathrm{e}^{-\mathbf{i}\omega_r t}) + \omega_n^2 \frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t} + \mathrm{e}^{-\mathbf{i}\omega_r t}) + \mathbf{i}\omega_r \varepsilon(\omega_r^2 \frac{U^2}{4} - 1)\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t} - \mathrm{e}^{-\mathbf{i}\omega_r t}) = 0. \tag{282}$$

The real and imaginary parts of the $\mathrm{e}^{\mathbf{i}\omega_r t}$ terms (or the $\mathrm{e}^{-\mathbf{i}\omega_r t}$) can be balanced to give

$$\begin{aligned}
&\mathrm{Re}: \quad \omega_r^2 = \omega_n^2, \quad \rightsquigarrow \quad \omega_r = \omega_n, \\
&\mathrm{Im}: \quad \omega_r \varepsilon(\omega_r^2 \frac{U^2}{4} - 1) = 0, \quad \rightsquigarrow \quad U = \sqrt{\frac{4}{\omega_r^2}},
\end{aligned} \tag{283}$$

which is the same as the unforced Van der Pol oscillator with parameters $\mu = \omega_r^2$ and $\gamma = 1$. Using $\omega_r = \omega_n$, the near-identity transform, to order ε^1 may now be written as

$$x = u + \varepsilon\mathbf{b}\mathbf{u}^* = \frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_n t} + \mathrm{e}^{-\mathbf{i}\omega_n t}) + \varepsilon \begin{bmatrix} \frac{\mathbf{i}\omega_n}{24} & 0 & 0 & \frac{-\mathbf{i}\omega_n}{24} & 0 & 0 \end{bmatrix}\mathbf{u}^*, \tag{284}$$

which gives the solution for x as

$$x = \sqrt{\frac{4}{\omega_n^2}}\cos(\omega_n t) - \varepsilon\frac{\omega_n}{96}\left(\sqrt{\frac{4}{\omega_n^2}}\right)^3 \sin(3\omega_n t) + \mathscr{O}(\varepsilon^2). \qquad \square \text{ (285)}$$

Using this type of real normal form approach, the Rayleigh and Van der Pol oscillators therefore have the same normal form structure, and only differ in terms of the constants in the nonlinear term. Next we consider how the choice of the free functions can be used.

4.7 Choice of free functions

As was previously noted, if zeros occur in the homological equation, then the b_i coefficient that corresponds to the zero can be chosen to have any value without changing the homological equation. We follow [85] in calling the b_i coefficients the 'free functions'. The key point to note is that although the free functions do not change the homological equation they do change the solution for x. In most of the examples shown so far, $b_i = 0$ is chosen whenever there was a free function choice, except in Example 13, Section 4.4.2, where we showed how a specific choice of the free function could be used to alter the solution for x (see Eq. (226) and the related discussion).

Another concept, called the *minimal normal form*, uses the choice of free functions to try an eliminate terms at higher order in the solution for x. For example, in the case where we are truncating at order ε^1, then the idea is to try and eliminate terms at order ε^2 and above. In order to carry out a minimal normal form, we need to compute the homological equation truncated at second order (or higher), which is taken as the second line in Eq. (261) .

Now consider eliminating the ε^2 terms (and higher if required) in the solution for x. In this context it means setting $\mathbf{h}_2 = 0$ (i.e setting all the b_i coefficients to zero for $\mathbf{h}_2$). It can be seen that if this is done for the second line in Eq. (261), the result is

$$\Delta\mathbf{h}_1 + \frac{\partial\hat{\mathbf{n}}_1}{\partial\mathbf{u}}\mathbf{h}_1 + \hat{\mathbf{n}}_2 - \hat{\mathbf{n}}_{\mathbf{u2}} = 0. \tag{286}$$

So, to obtain the minimal normal form to order ε^2 step one is to solve the ε^1 homological equation (first line in Eq. (261)) to find $\mathbf{h}_1$, then use Eq. (286) to solve for $\hat{\mathbf{n}}_{\mathbf{u2}}$. An example will be considered next.

4.7.1 Example 17: The minimal normal form for the undamped cubic-quintic oscillator

We consider the cubic-quintic oscillator from Example 12 (Section 4.2.4) written as

$$\begin{aligned}\ddot{x} + \omega_n^2 x + \varepsilon\alpha_1 x^3 + \varepsilon\alpha_2 x^5 = 0 \quad \text{with } x = q \rightsquigarrow \quad & \ddot{q} + \omega_n^2 q + \varepsilon\alpha_1 q^3 + \varepsilon\alpha_2 q^5 = 0, \quad \rightsquigarrow \\ & \ddot{q} = -\omega_n^2 q - \varepsilon\hat{\mathbf{n}}_{\mathbf{1}}(q_p, q_m) = 0,\end{aligned} \tag{287}$$

from which the ε^1 homological equation gives (see Eq. (164))

$$\begin{aligned}([\alpha_1\ \ 3\alpha_1\ \ 3\alpha_1\ \ \alpha_1\ \ \alpha_2\ \ 5\alpha_2\ \ 10\alpha_2\ \ 10\alpha_2\ \ 5\alpha_2\ \ \alpha_2] - [0\ \ 3\alpha_1\ \ 3\alpha_1\ \ 0\ \ 0\ \ 0\ \ 10\alpha_2\ \ 10\alpha_2\ \ 0\ \ 0])\mathbf{u}^* = \\ [b_1(8\omega_r^2)\ \ b_2(0)\ \ b_3(0)\ \ b_4(8\omega_r^2)\ \ b_5(24\omega_r^2)\ \ b_6(8\omega_r^2)\ \ b_7(0)\ \ b_8(0)\ \ b_9(8\omega_r^2)\ \ b_{10}(24\omega_r^2)]\mathbf{u}^*,\end{aligned} \tag{288}$$

although here we apply detuning using the analysis in Section 4.6 so that the expression has ω_r rather than ω_n as in Eq. (164). In this case the coefficient values are $b_1 = b_4 = \alpha_1/8\omega_r^2$, $b_6 = b_9 = 5\alpha_2/8\omega_r^2$, $b_5 = b_{10} = \alpha_2/24\omega_r^2$, and $b_2 = b_3 = b_7 = b_8 = 0$ which means that the ε^1 term in Eq. (116) will be given by

$$\mathbf{h}_1 = \mathbf{b}_1\mathbf{u}^* = \frac{\alpha_1}{8\omega_r^2}u_p^3 + \frac{\alpha_1}{8\omega_r^2}u_m^3 + \frac{\alpha_2}{24\omega_r^2}u_p^5 + \frac{5\alpha_2}{8\omega_r^2}u_p^4 u_m + \frac{5\alpha_2}{8\omega_r^2}u_p u_m^4 + \frac{\alpha_2}{24\omega_r^2}u_m^5. \tag{289}$$

We can use this to find the minimal normal form for the cubic-quintic oscillator using Eq. (286). To do this note that for the single-degree-of-freedom oscillator such as this $\mathbf{u} = u$, so from Eq. (287),

$$\begin{aligned}\hat{\mathbf{n}}_{\mathbf{1}}(q) = \alpha_1 q^3 + \alpha_3 q^5 \ \rightsquigarrow\ \alpha_1(u+\varepsilon\mathbf{h}_1)^3 + \alpha(u+\varepsilon\mathbf{h}_1)^5 \ \rightsquigarrow\ (\alpha_1(u)^3 + \alpha_2(u)^5)|_{\varepsilon=0} \ \rightsquigarrow \\ \frac{\partial\hat{\mathbf{n}}_{\mathbf{1}}(u)}{\partial u} = 3\alpha_1(u)^2 + 5\alpha_2(u)^4 = 3\alpha_1(u_p+u_m)^2 + 5\alpha_2(u_p+u_m)^4\end{aligned} \tag{290}$$

where $\varepsilon = 0$ has been used in the partial derivative, so that when the resulting expression are substituted into the second line of Eq. (261) then only terms up to order-ε^2 are retained.

Now we can write Eq. (286) for the cubic-quintic oscillator (noting that $\hat{\mathbf{n}}_\mathbf{2} = 0$ from Eq. (287)) as

$$\begin{aligned}(\Delta + 3\alpha_1(u_p+u_m)^2 + 5\alpha_2(u_p+u_m)^4)\left(\frac{\alpha_1}{8\omega_r^2}u_p^3 + \frac{\alpha_1}{8\omega_r^2}u_m^3 + \frac{\alpha_2}{24\omega_r^2}u_p^5 + \frac{5\alpha_2}{8\omega_r^2}u_p^4u_m + \frac{5\alpha_2}{8\omega_r^2}u_pu_m^4 + \frac{\alpha_2}{24\omega_r^2}u_m^5\right) &= \hat{\mathbf{n}}_\mathbf{u2}\\ \rightsquigarrow \hat{\mathbf{n}}_\mathbf{u2} = A_1u_p^3 + A_2u_m^3 + A_3u_p^5 + A_4u_p^4u_m + A_5u_p^3u_m^2 + A_6u_p^2u_m^3 + A_7u_pu_m^4 + A_8u_m^5 + \mathscr{O}(k=7)&\end{aligned} \tag{291}$$

where $\Delta = \omega_n^2 - \omega_r^2$ and

$$A_1 = A_2 = \frac{\Delta\alpha_1}{8\omega_r^2}, \quad A_3 = A_8 = \frac{9\alpha_1^2 + \Delta\alpha_2}{24\omega_r^2}, \quad A_4 = A_7 = \frac{6\alpha_1^2 + 5\Delta\alpha_2}{8\omega_r^2}, \quad A_5 = A_6 = \frac{3\alpha_1^2}{8\omega_r^2}, \tag{292}$$

and we have included terms up to $k = 5$ order in Eq. (291). Then by adding the ε^2 (e.g up to $k = 5$) terms to the ε normal form, Eq. (165), (e.g. to obtain Eq. (146)), the minimal normal form becomes

$$\begin{aligned}&\ddot{u} + \omega_n^2u + \varepsilon 3\alpha(u_p^2u_m + u_pu_m^2) + \varepsilon 10\alpha_2(u_p^3u_m^2 + u_p^2u_m^3)\\ &\quad + \varepsilon^2(A_1u_p^3 + A_2u_m^3 + A_3u_p^5 + A_4u_p^4u_m + A_5u_p^3u_m^2 + A_6u_p^2u_m^3 + A_7u_pu_m^4 + A_8u_m^5) + \mathscr{O}(\varepsilon^3, k=7) = 0.\end{aligned} \tag{293}$$

Then the assumed base solutions (from Eq. (22)), are substituted into Eq. (116) (lower) to give the solution for x as

$$x = U\cos(\omega_r t) + \varepsilon\left(\frac{\alpha_1U^3}{32\omega_r^2} + \frac{5\alpha_2U^5}{128\omega_r^2}\right)\cos(3\omega_r t) + \varepsilon\frac{\alpha_2U^5}{384\omega_r^2}\cos(5\omega_r t) + \mathscr{O}(\varepsilon^3). \qquad \square \tag{294}$$

Note that due to the fact that this is a single-degree-of-freedom oscillator and $\mathbf{h}_2 = 0$, Eq. (294) is given by $x = q = u + \varepsilon\mathbf{h}_1 + \varepsilon^2(0)$. When Eq. (294) is compared with Eq. (168) from Example 12, it can be seen that the expressions are almost the same except Eq. (294) has the detuned frequency, ω_r, and is computed to order $\mathscr{O}(\varepsilon^3)$ rather than $\mathscr{O}(\varepsilon^2)$. As can be seen from this example, this type of manipulation is just forcing all the terms that would appear in the solution for x, Eq. (294) into the normal form Eq. (293). The complexity of Eq. (293) is significantly increased, which may be a disadvantage for some applications. If instead of this we set $\mathbf{h}_2 \neq 0$ in the ε^2 homological equation (second line in Eq. (261)) then we can obtain the real normal form to order ε^2. This will be demonstrated in Example 18 (Section 4.8.1).

4.8 Backbone curves

An important application that can be achieved using normal form is the computation of backbone curves. The idea of a backbone curve was introduced in Section 1.1.3, and an example was shown in Fig. 2. To compute backbone curves using the types of normal form transformations described in this paper, detuning needs to be included. This is because, the backbone curve represents (an approximation) of the nonlinear amplitude-frequency variation which is a fundamental characteristic of many nonlinear oscillators. For example, the backbone curve relationship shown in Fig. 2, can be derived from the detuned version of Example 12 (Section 4.2.4). From Example 12, substituting the (detuned) base solutions (Eq. (22)) into Eq. (165),

$$(\mathbf{i}\omega_r)^2\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t} + \mathrm{e}^{-\mathbf{i}\omega_r t}) + \omega_n^2\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_r t} + \mathrm{e}^{-\mathbf{i}\omega_r t}) + \varepsilon 3\alpha\frac{U^3}{8}(\mathrm{e}^{\mathbf{i}\omega_r t} + \mathrm{e}^{-\mathbf{i}\omega_r t}) = 0. \tag{295}$$

The coefficients of the $\mathrm{e}^{\mathbf{i}\omega_r t}$ terms (or the $\mathrm{e}^{-\mathbf{i}\omega_r t}$) can be balanced to give

$$\omega_r^2 = \omega_n^2 + \varepsilon\frac{3\alpha U^2}{4} + \mathscr{O}(\varepsilon^2), \tag{296}$$

from which, when $\varepsilon = 1$, we can obtain

$$\omega_r = \sqrt{\omega_n^2 + \alpha\frac{3U^2}{4} + \mathscr{O}(2)}. \tag{297}$$

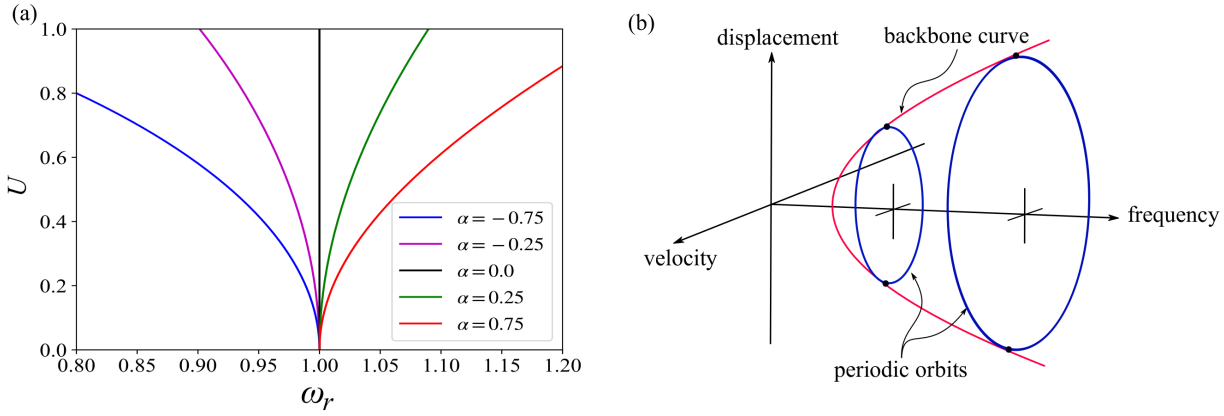

Fig. 7: Conservative backbone curves, showing (a) examples of conservative backbone curves for the Duffing oscillator computed using Eq. (297) for different values of α when $\omega_n = 1$, and (b) a schematic diagram of the conservative backbone concept. Here the conservative backbone curve is defined as the curve that joins all the points of maximum displacement for a family of periodic orbits generated by the nonlinear oscillator under investigation. For clarity, just two periodic orbits are shown here (blue curves). The maximum (and minimum) points of displacement amplitude are shown as black dots, and both the positive and negative backbone curves are shown (red curves). Typically just the upper half of this diagram is shown (as in panel (a)), and symmetry is assumed for the lower half. Note that units of frequency are in radians per second, and units of displacement amplitude, U are in mm or m.

It should be noted that the example shown in Fig. 2 had negative α and so substituting $-\alpha$ into Eq. (296) gives the same relationship as Eq. (26) from Example 2. Backbone curves computed for undamped, unforced systems are called *conservative* backbone curves. A schematic diagram of the concept and example curves for Eq. (297) are shown in Fig. 7. We now consider computing the backbone curve to order ε^2 for the undamped cubic-quintic oscillator

4.8.1 Example 18: the ε^2 real normal form of the undamped cubic-quintic oscillator

The ε^2 real normal form of the undamped cubic-quintic oscillator can be found by extending the analysis from the order-ε^1 approach shown in Example 12, Section 4.2.4 (noting that $\hat{\mathbf{n}}_2 = 0$ from Eq. (287), and that the quintic nonlinear terms are included in $\hat{\mathbf{n}}_1$ in these examples). To do this, we first express Eq. (291) as

$$\Delta\mathbf{h}_1 + \frac{\partial\hat{\mathbf{n}}}{\partial\mathbf{u}}\mathbf{h}_1 = A_1u_p^3 + A_2u_m^3 + A_3u_p^5 + A_4u_p^4u_m + A_5u_p^3u_m^2 + A_6u_p^2u_m^3 + A_7u_pu_m^4 + A_8u_m^5 = \hat{\mathbf{n}}^+\mathbf{u}^+, \tag{298}$$

where

$$\hat{\mathbf{n}}^+ = [A_1 \;\; A_2 \;\; A_3 \;\; A_4 \;\; A_5 \;\; A_6 \;\; A_7 \;\; A_8] \quad \text{and} \quad \mathbf{u}^+ = \begin{bmatrix} u_p^3 \\ u_m^3 \\ u_p^5 \\ u_p^4u_m \\ u_p^3u_m^2 \\ u_p^2u_m^3 \\ u_pu_m^4 \\ u_m^5 \end{bmatrix}, \tag{299}$$

such that if we use $\mathbf{u}^+$ as the basis for all the terms in the ε^2 homological equation, and the A_j coefficients are given in Eq. (292). Then (the second line in) Eq. (261) becomes

$$\hat{\mathbf{n}}^+\mathbf{u}^+ - \hat{\mathbf{n}}^+_{\mathbf{u2}}\mathbf{u}^+ = -\mathbf{b}_2\tilde{\Upsilon}^2\mathbf{u}^+ - \Upsilon\mathbf{b}_2\mathbf{u}^+, \tag{300}$$

where we have used the second Lie derivative (Eq. (153)) in order to evaluate $\ddot{\mathbf{h}}_2$ and thus find $\tilde{\Upsilon}^2$. In addition, $\hat{\mathbf{n}}^+_{\mathbf{u2}}$ and $\mathbf{b}_2$ are coefficient matrices of the same dimension as $\hat{\mathbf{n}}^+$.

Computing the ε^2 homological equation for the undamped cubic-quintic oscillator gives

$$\begin{aligned}&([A_1\ \ A_2\ \ A_3\ \ A_4\ \ A_5\ \ A_6\ \ A_7\ \ A_8]-[\hat{n}_{u21}\ \hat{n}_{u22}\ \hat{n}_{u23}\ \hat{n}_{u24}\ \hat{n}_{u25}\ \hat{n}_{u26}\ \hat{n}_{u27}\ \hat{n}_{u28}])\mathbf{u}^+=\\&\left(-(\mathbf{i}\omega_r)^2[b_1\ \ b_2\ \ b_3\ \ b_4\ \ b_5\ \ b_6\ \ b_7\ \ b_8]\begin{bmatrix}3&0&0&0&0&0&0&0\\0&-3&0&0&0&0&0&0\\0&0&5&0&0&0&0&0\\0&0&0&3&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&-1&0&0\\0&0&0&0&0&0&-3&0\\0&0&0&0&0&0&0&-5\end{bmatrix}^2-\omega_r^2[b_1\ \ b_2\ \ b_3\ \ b_4\ \ b_5\ \ b_6\ \ b_7\ \ b_8]\right)\mathbf{u}^+.\end{aligned}\tag{301}$$

Evaluating the right hand side gives two zero values which allows the $\hat{\mathbf{n}}^+_{\mathbf{u2}}$ coefficients to be identified such that

$$\begin{aligned}&([A_1\ \ A_2\ \ A_3\ \ A_4\ \ A_5\ \ A_6\ \ A_7\ \ A_8]-[0\ 0\ 0\ 0\ A_5\ A_6\ 0\ 0])\mathbf{u}^+=\\&\qquad[b_1(8\omega_r^2)\ \ b_2(8\omega_r^2)\ \ b_3(24\omega_r^2)\ \ b_4(8\omega_n^2)\ \ b_5(0)\ \ b_6(0)\ \ b_7(8\omega_r^2)\ \ b_8(24\omega_r^2)]\mathbf{u}^+.\end{aligned}\tag{302}$$

Next we substitute for A_5 and A_6 from Eq. (292) and then use the fact that

$$\varepsilon^2\hat{\mathbf{n}}_{\mathbf{u2}}(u_p,u_m)=\varepsilon^2[0\ \ 0\ \ 0\ \ 0\ \ \frac{3\alpha_1^2}{8\omega_r^2}\ \ \frac{3\alpha_1^2}{8\omega_r^2}\ \ 0\ \ 0]\mathbf{u}^+=\varepsilon^2\frac{3\alpha_1^2}{8\omega_r^2}(u_p^3u_m^2+u_p^2u_m^3),$$

which is up to $k=5$, as for Example 17. Then by adding the ε^2 (e.g up to $k=5$) terms to the ε normal form, Eq. (165), (e.g. to obtain Eq. (146)), the ε^2 normal form becomes

$$\ddot{u}+\omega_n^2u+\varepsilon 3\alpha_1(u_p^2u_m+u_pu_m^2)+\left(\varepsilon 10\alpha_2+\varepsilon^2\frac{3\alpha_1^2}{8\omega_r^2}\right)(u_p^3u_m^2+u_p^2u_m^3)=0.\tag{303}$$

Using the values for $\mathbf{b}_1$ from Eq. (289), and $\mathbf{b}_2$ from solving Eq. (302), the near-identity transform, to order ε^2 may now be written as

$$\begin{aligned}x=u+\varepsilon\mathbf{b}_1\mathbf{u}^*+\varepsilon^2\mathbf{b}_2\mathbf{u}^+=\frac{U}{2}(\mathrm{e}^{\mathbf{i}(\omega_rt)}+\mathrm{e}^{-\mathbf{i}(\omega_rt)})+\varepsilon\left[\frac{\alpha_1}{8\omega_r^2}\ \ 0\ \ 0\ \ \frac{\alpha_1}{8\omega_r^2}\ \ \frac{\alpha_2}{24\omega_r^2}\ \ \frac{5\alpha_2}{8\omega_r^2}\ \ 0\ \ 0\ \ \frac{5\alpha_2}{8\omega_r^2}\ \ \frac{\alpha_2}{24\omega_r^2}\right]\mathbf{u}^*+\\\varepsilon^2[\frac{\Delta\alpha_1}{64\omega_r^4}\ \ \frac{\Delta\alpha_1}{64\omega_r^4}\ \ \frac{9\alpha_1^2+\Delta\alpha_2}{576\omega_r^4}\ \ \frac{6\alpha_1^2+5\Delta\alpha_2}{64\omega_r^4}\ \ 0\ \ 0\ \ \frac{6\alpha_1^2+5\Delta\alpha_2}{64\omega_r^4}\ \ \frac{9\alpha_1^2+\Delta\alpha_2}{576\omega_r^4}]\mathbf{u}^+,\end{aligned}\tag{304}$$

where $\mathbf{u}^*$ is given by Eq. (162). Then the assumed base solutions (from Eq. (22)), are substituted into Eq. (304) to give the solution for x as $x=q=u+\varepsilon\mathbf{h}_1+\varepsilon^2\mathbf{h}_2$ which leads to

$$\begin{aligned}x=U\cos(\omega_rt)+\left[\varepsilon\frac{\alpha_1U^3}{32\omega_r^2}+\varepsilon\frac{5\alpha_2U^5}{128\omega_r^2}+\varepsilon^2\frac{\Delta\alpha_1U^3}{256\omega_r^4}+\varepsilon^2\frac{3\alpha_1^2U^5}{512\omega_r^4}+\varepsilon^2\frac{\Delta\alpha_2U^5}{1024\omega_r^4}\right]\cos(3\omega_rt)\\+\left[\varepsilon\frac{\alpha_2U^5}{384\omega_r^2}+\varepsilon^2\frac{\alpha_1^2U^5}{1024\omega_r^4}+\varepsilon^2\frac{\Delta\alpha_2U^5}{9216\omega_r^4}\right]\cos(5\omega_rt)+\mathscr{O}(\varepsilon^3,k=7).\end{aligned}\tag{305}$$

To find an expression for ω_r^2 the (detuned) base solutions (Eq. (22)) are taken to be the assumed solution, and are substituted into Eq. (303) to give

$$(\mathbf{i}\omega_r)^2\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_rt}+\mathrm{e}^{-\mathbf{i}\omega_rt})+\omega_n^2\frac{U}{2}(\mathrm{e}^{\mathbf{i}\omega_rt}+\mathrm{e}^{-\mathbf{i}\omega_rt})+\varepsilon 3\alpha_1\frac{U^3}{8}(\mathrm{e}^{\mathbf{i}\omega_rt}+\mathrm{e}^{-\mathbf{i}\omega_rt})+\left(\varepsilon 10\alpha_2+\varepsilon^2\frac{3\alpha_1^2}{8\omega_r^2}\right)\frac{U^5}{32}(\mathrm{e}^{\mathbf{i}\omega_rt}+\mathrm{e}^{-\mathbf{i}\omega_rt})=0.$$

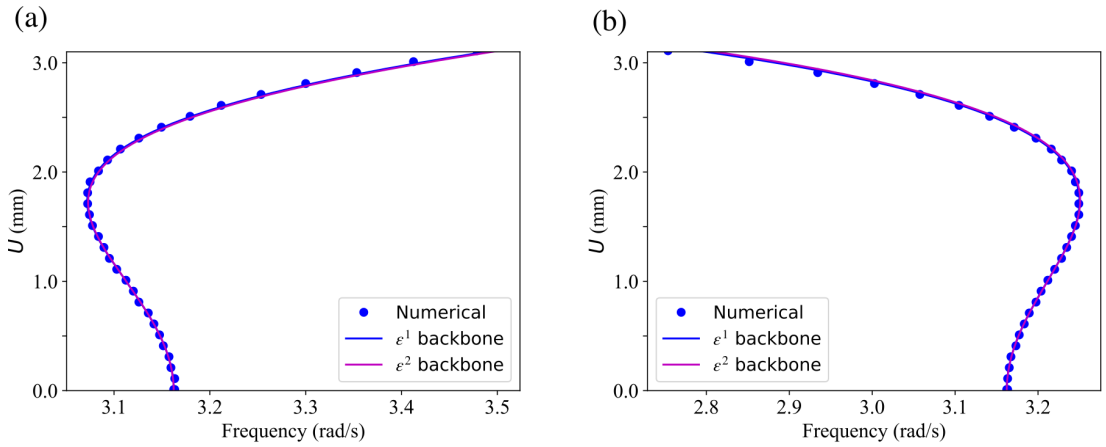

Fig. 8: Numerically computed conservative backbone curves for the oscillator in Example 18 with $\omega_n = \sqrt{10}$ rad/s, and $\varepsilon = 1$. Showing (a) the case for $\alpha_1 = -0.5$ & $\alpha_2 = 0.1$, and (b) the case when $\alpha_1 = 0.5$ & $\alpha_2 = -0.1$. The blue dots are numerically computed maximum values of the displacement amplitude, and are taken as the reference solution. The blue line is the ε^1 backbone curve solution computed by substituting the assumed solution into Eq. (165), and the purple line is the ε^2 backbone curve solution computed from Eq. (307). It can be noted from both plots that as U increases, there is a combination of both softening and hardening frequency behaviour. In (a) the backbone curve first softens and then hardens as U increases, whereas in (b) (where the coefficients α_1 & α_2 have the opposite signs) the backbone curve first hardens and then softens as U increases. From a perturbation viewpoint, it could be expected that the solutions will give better agreement with the reference solution for "small" U (and α_1, α_2) values. In this interpretation, for values $U > 1$ we should be cautious about how accurate the predictions from these types of approximate solutions might be.

(306)

The coefficients of the $e^{i\omega_r t}$ terms (or the $e^{-i\omega_r t}$) can be balanced to give

$$\omega_r^2 = \omega_n^2 + \varepsilon\frac{3\alpha_1 U^2}{4} + \varepsilon\frac{5\alpha_2 U^4}{8} + \varepsilon^2\frac{3\alpha_1^2 U^4}{128\omega_r^2} + \mathcal{O}(\varepsilon^3, k=7). \qquad \square \quad (307)$$

A numerically computed example is shown in Fig. 8 for the case when $\alpha_2 = 0$. Further details of this type of analysis is given in Nasir et al. [122].

4.9 Normal form transformations of bifurcation phenomena

An important use of normal form transformation in the field of nonlinear dynamics is to simplify models of important dynamic phenomena, such as bifurcations. From a structural dynamics perspective, we can start with model containing multiple terms and coefficients, and use the normal form process to eliminate as many of them as possible at a given order, resulting in a simpler system with fewer coefficients [100, 78, 61]. In this use of normal form transformations, it is not necessarily expected that the process will provide an approximate solution of the equations. Instead, a systematic reduction of terms (and equations) is carried out, until there is no further reduction possible. Ultimately, this can result in just a single equation and parameter remaining that determines the behaviour of the bifurcation. In other words, a reduction to the simplest non-trivial case is sometimes possible. We consider the example of the Hopf bifurcation.

4.9.1 Example 19 a normal form analysis for a Hopf bifurcation

We revisit the unforced Van der Pol oscillator from Example 15, and start by writing it in a slightly modified form as

$$\ddot{x} + (\mu x^2 - 2\zeta\omega_n)\dot{x} + \omega_n^2 x = 0, \quad \rightsquigarrow \quad \dot{\boldsymbol{x}} = A\boldsymbol{x} + \tilde{\mathcal{N}}(\boldsymbol{x}): \quad A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 2\zeta\omega_n \end{bmatrix}, \quad \tilde{\mathcal{N}}(\boldsymbol{x}) = \begin{Bmatrix} 0 \\ -\mu x_1^2 x_2 \end{Bmatrix}, \quad (308)$$

where the states are $\boldsymbol{x} = \{x_1 \;\; x_2\}^T = \{x \;\; \dot{x}\}^T$. Next we will use the Jordan normal form technique from Section 2.2, specifically Example 6, to transform the systems so that the linear matrix A has the real form for complex eigenvalues.

To do this we first note that the eigenvalues of A in this example are $\lambda_{1,2} = \zeta\omega_n \pm \mathbf{i}\omega_d$, where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency. Therefore, for the linear part of the problem we have a similar situation to that in Example 6, and we can make the coordinate transformation $\boldsymbol{x} = P\boldsymbol{q}$ where in this example

$$P = \begin{bmatrix} 0 & 1 \\ \omega_d & \zeta\omega_n \end{bmatrix} \quad , \quad P^{-1} = \begin{bmatrix} \frac{-\zeta\omega_n}{\omega_d} & \frac{1}{\omega_d} \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\mathcal{N}}(P\boldsymbol{q}) = \begin{Bmatrix} 0 \\ -\mu q_m^2(\omega_d q_p + \zeta\omega_n q_m) \end{Bmatrix}, \tag{309}$$

where $\boldsymbol{q} = \{q_p \;\; q_m\}^T$. Next we can write down the transformed system as

$$\dot{\boldsymbol{q}} = P^{-1}AP\boldsymbol{q} + P^{-1}\tilde{\mathcal{N}}(P\boldsymbol{q}) \quad \rightsquigarrow \quad \begin{bmatrix} \dot{q}_p \\ \dot{q}_m \end{bmatrix} = \begin{bmatrix} \zeta\omega_n & -\omega_d \\ \omega_d & \zeta\omega_n \end{bmatrix} \begin{bmatrix} q_p \\ q_m \end{bmatrix} + \begin{Bmatrix} -\mu q_m^2 q_p - \mu\gamma q_m^3 \\ 0 \end{Bmatrix}. \tag{310}$$

where $\gamma = \frac{\zeta\omega_n}{\omega_d}$.

Next we use a process of *complexification* to obtain new coordinates of the form $z = q_p + \mathbf{i}q_m$ and $\bar{z} = q_p - \mathbf{i}q_m$, so that $q_p = \frac{z+\bar{z}}{2}$, $q_m = \frac{z-\bar{z}}{2\mathbf{i}}$, $\dot{z} = \dot{q}_p + \mathbf{i}\dot{q}_m$ and $\dot{\bar{z}} = \dot{q}_p - \mathbf{i}\dot{q}_m$. Using these relationships with Eq. (310) we then obtain

$$\dot{z} = \lambda z - \mu\left(\frac{z-\bar{z}}{2\mathbf{i}}\right)^2\left(\frac{z+\bar{z}}{2}\right) - \mu\gamma\left(\frac{z-\bar{z}}{2\mathbf{i}}\right)^3 \quad \text{and} \quad \dot{\bar{z}} = \bar{\lambda}\bar{z} - \mu\left(\frac{z-\bar{z}}{2\mathbf{i}}\right)^2\left(\frac{z+\bar{z}}{2}\right) - \mu\gamma\left(\frac{z-\bar{z}}{2\mathbf{i}}\right)^3, \tag{311}$$

where $\lambda = \lambda_1 = \zeta\omega_n + \mathbf{i}\omega_d$ and $\bar{\lambda} = \lambda_2 = \zeta\omega_n - \mathbf{i}\omega_d$.

Now expanding the nonlinear terms and projecting them onto the nonlinear basis vector $\mathbf{z}^* = [z^3 \;\; z^2\bar{z} \;\; z\bar{z}^2 \;\; \bar{z}^3]^T$ we obtain the system

$$\begin{bmatrix} \dot{z} \\ \dot{\bar{z}} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix} + \frac{\mu}{8} \begin{bmatrix} 1-\mathbf{i}\gamma & -1+\mathbf{i}3\gamma & -1-\mathbf{i}3\gamma & 1+\mathbf{i}\gamma \\ 1-\mathbf{i}\gamma & -1+\mathbf{i}3\gamma & -1-\mathbf{i}3\gamma & 1+\mathbf{i}\gamma \end{bmatrix} \mathbf{z}^* \quad \text{or} \quad \dot{\boldsymbol{z}} = \Lambda\boldsymbol{z} + N_z\mathbf{z}^*, \tag{312}$$

$$\text{where} \quad \boldsymbol{z} = [z, \bar{z}]^T, \quad \Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \quad \text{and} \quad N_z = \frac{\mu}{8} \begin{bmatrix} 1-\mathbf{i}\gamma & -1+\mathbf{i}3\gamma & -1-\mathbf{i}3\gamma & 1+\mathbf{i}\gamma \\ 1-\mathbf{i}\gamma & -1+\mathbf{i}3\gamma & -1-\mathbf{i}3\gamma & 1+\mathbf{i}\gamma \end{bmatrix}. \tag{313}$$

We now seek a near-identity normal form transformation from $\boldsymbol{z}$ to a new coordinate set $\boldsymbol{w} = [w, \bar{w}]^T$, such that $\boldsymbol{z} = \boldsymbol{w} + \boldsymbol{b}\mathbf{w}^*$ where $\boldsymbol{b}$ is a coefficient matrix, $\mathbf{w}^* = [w^3 \;\; w^2\bar{w} \;\; w\bar{w}^2 \;\; \bar{w}^3]^T$, and the transformed system is governed by

$$\dot{\boldsymbol{w}} = \Lambda\boldsymbol{w} + N_w\mathbf{w}^*, \tag{314}$$

where N_w is the new (simplified) coefficient matrix. Substituting for $\boldsymbol{z}$ in Eq. (312) and using Eq. (314) leads to a homological equation of the form

$$N_z - N_w = \boldsymbol{b}\frac{\partial\mathbf{w}^*}{\partial\boldsymbol{w}}\Lambda\boldsymbol{w} - \Lambda\boldsymbol{b}\mathbf{w}^*, \tag{315}$$

which can be written as

$$\begin{aligned} &\left(\frac{\mu}{8}\begin{bmatrix} 1-\mathbf{i}\gamma & -1+\mathbf{i}3\gamma & -1-\mathbf{i}3\gamma & 1+\mathbf{i}\gamma \\ 1-\mathbf{i}\gamma & -1+\mathbf{i}3\gamma & -1-\mathbf{i}3\gamma & 1+\mathbf{i}\gamma \end{bmatrix} - \begin{bmatrix} n_{w1}^* & n_{w2}^* & n_{w3}^* & n_{w4}^* \\ n_{w5}^* & n_{w6}^* & n_{w7}^* & n_{w8}^* \end{bmatrix}\right)\mathbf{w}^* = \\ &\left(\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{bmatrix}\begin{bmatrix} 3\lambda & 0 & 0 & 0 \\ 0 & 2\lambda+\bar{\lambda} & 0 & 0 \\ 0 & 0 & \lambda+2\bar{\lambda} & 0 \\ 0 & 0 & 0 & 3\bar{\lambda} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{bmatrix}\right)\mathbf{w}^* = \\ &\begin{bmatrix} b_1(2\lambda) & b_2(\lambda+\bar{\lambda}) & b_3(2\bar{\lambda}) & b_4(3\bar{\lambda}-\lambda) \\ b_5(3\lambda-\bar{\lambda}) & b_6(2\lambda) & b_7(\lambda+\bar{\lambda}) & b_8(2\bar{\lambda}) \end{bmatrix}\mathbf{w}^*. \end{aligned} \tag{316}$$

Now we wish to consider what happens very close to the Hopf bifurcation point. To do this we need to consider what happens when the damping ratio, ζ, is allowed to vary. The Hopf bifurcation will occur when $\zeta = 0$, and to understand the process, three cases are defined in Table 3. Case iii) is the initial case for this example, with the additional condition that $\zeta > 0$. Case ii) is the bifurcation point itself, and Case i) can be thought of as the case before the bifurcation happens. It should also be understood that Cases i) and iii) are very close to the bifurcation point[63]

Case i) $\zeta < 0$	Case ii) $\zeta = 0$	Case iii) $\zeta > 0$
$\lambda = (-\zeta)\omega_n + \mathbf{i}\omega_d$	$\lambda = +\mathbf{i}\omega_n$	$\lambda = \zeta\omega_n + \mathbf{i}\omega_d$
$\bar{\lambda} = (-\zeta)\omega_n - \mathbf{i}\omega_d$	$\bar{\lambda} = -\mathbf{i}\omega_n$	$\bar{\lambda} = \zeta\omega_n - \mathbf{i}\omega_d$
$\mathrm{Re}(\lambda, \bar{\lambda}) < 0$	$\mathrm{Re}(\lambda, \bar{\lambda}) = 0$	$\mathrm{Re}(\lambda, \bar{\lambda}) > 0$
$\lambda + \bar{\lambda} = -2\zeta\omega_n$	$\lambda + \bar{\lambda} = 0$	$\lambda + \bar{\lambda} = 2\zeta\omega_n$
$\lambda - \bar{\lambda} = 2\mathbf{i}\omega_d$	$\lambda - \bar{\lambda} = 2\mathbf{i}\omega_n$	$\lambda - \bar{\lambda} = 2\mathbf{i}\omega_d$
$\gamma = \frac{(-\zeta)\omega_n}{\omega_d}$	$\gamma = 1$	$\gamma = \frac{\zeta\omega_n}{\omega_d}$

Table 3: Three cases in the neighbourhood of the Hopf bifurcation for the Van der Pol oscillator.

It can be seen from Table 3 that $\lambda + \bar{\lambda} = 0$ in Case ii) when $\zeta = 0$, and so we treat these terms as resonant and using Eq. (316), set all n^*_{wi} terms to zero, except for $n^*_{w2} = \frac{\mu}{8}(-1 + \mathbf{i}3)$ and $n^*_{w2} = \frac{\mu}{8}(-1 - \mathbf{i}3)$ such that using Eq. (314) the normal form can be written

$$\begin{bmatrix} \dot{w} \\ \dot{\bar{w}} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \begin{bmatrix} w \\ \bar{w} \end{bmatrix} + \frac{\mu}{8} \begin{bmatrix} 0 & -1+\mathbf{i}3\gamma & 0 & 0 \\ 0 & 0 & -1-\mathbf{i}3\gamma & 0 \end{bmatrix} \mathbf{w}^* \quad \text{or} \quad \begin{aligned} \dot{w} &= \lambda w + n^*_{w2} w^2 \bar{w}, \\ \dot{\bar{w}} &= \bar{\lambda}\bar{w} + n^*_{w7} w \bar{w}^2. \end{aligned} \tag{317}$$

Note that the eight coefficients in N_z from Eq. (313) has now been reduced to just two. Even so, further reduction and insight can be gained by changing to polar coordinates such that $w = r(t)\mathrm{e}^{\mathbf{i}\theta(t)}$ and $\bar{w} = r(t)\mathrm{e}^{-\mathbf{i}\theta(t)}$. Substituting these relationships into the right-hand expression of Eq. (317) gives

$$\begin{aligned} \dot{r}\mathrm{e}^{\mathbf{i}\theta} + r\mathbf{i}\dot{\theta}\mathrm{e}^{\mathbf{i}\theta} &= \lambda r\mathrm{e}^{\mathbf{i}\theta} + n^*_{w2} r^3 \mathrm{e}^{\mathbf{i}\theta}, \\ \dot{r}\mathrm{e}^{-\mathbf{i}\theta} - r\mathbf{i}\dot{\theta}\mathrm{e}^{-\mathbf{i}\theta} &= \bar{\lambda} r\mathrm{e}^{-\mathbf{i}\theta} + n^*_{w7} r^3 \mathrm{e}^{-\mathbf{i}\theta}, \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \dot{r} + r\mathbf{i}\dot{\theta} &= \lambda r + n^*_{w2} r^3, \\ \dot{r} - r\mathbf{i}\dot{\theta} &= \bar{\lambda} r + \bar{n}^*_{w2} r^3, \end{aligned} \tag{318}$$

where we have used the fact that $n^*_{w7} = \bar{n}^*_{w2}$ in the right-hand expressions. Now treating the right-hand expression of Eq. (318) as a pair of simultaneous equations, and adding, then subtracting them leads to

$$\begin{aligned} \dot{r} &= \frac{\lambda + \bar{\lambda}}{2} r + \frac{n^*_{w2} + \bar{n}^*_{w2}}{2} r^3, \\ \dot{\theta} &= \frac{\lambda - \bar{\lambda}}{2\mathbf{i}} + \frac{n^*_{w2} - \bar{n}^*_{w2}}{2\mathbf{i}} r^2, \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \dot{r} &= \mathrm{Re}(\lambda) r + \mathrm{Re}(n^*_{w2}) r^3, \\ \dot{\theta} &= \mathrm{Im}(\lambda) + \mathrm{Im}(n^*_{w2}) r^2, \end{aligned} \tag{319}$$

where $\mathrm{Re}(n^*_{w2}) = -\frac{\mu}{8}$, $\mathrm{Im}(n^*_{w2}) = \frac{3\mu\gamma}{8}$ and $\mathrm{Re}(\lambda)$ & $\mathrm{Im}(\lambda)$ can be obtained from Table 3.

Now consider the equation for $\dot{r}$ on the right-hand side of Eq. (319). The righthand side of this equation is zero when

$$\mathrm{Re}(\lambda) r + \mathrm{Re}(n^*_{w2}) r^3 = 0 \quad \rightsquigarrow \quad \text{three possible solutions} \quad r = 0 \quad \text{and} \quad r = \pm\sqrt{-\frac{\mathrm{Re}(\lambda)}{\mathrm{Re}(n^*_{w2})}}. \tag{320}$$

The solution $r = 0$ exists for all three cases listed in Table 3, but the solutions $r = \pm\sqrt{-\mathrm{Re}(\lambda)/(\mathrm{Re}(n^*_{w2}))}$ only exist for Case iii) when $\zeta > 0$, because $\mathrm{Re}(n^*_{w2}) = -\frac{\mu}{8} < 0$ (assuming $\mu > 0$) and so $-\mathrm{Re}(\lambda)/(\mathrm{Re}(n^*_{w2})) > 0$. □

[63] A more rigorous mathematical approach is normally applied by defining the size of this region more specifically — see for example [77, 52]. Note also that the phrase 'in the neighbourhood of' is taken to mean 'sufficiently close to'.

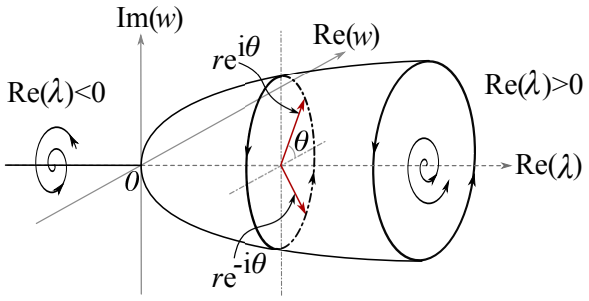

Fig. 9: A schematic representation of the limit cycle (i.e. periodic orbit) behaviour obtained from the normal form of a (supercritical) Hopf bifurcation, as derived in Example 20 (Section 4.9.1). For $\mathrm{Re}(\lambda) < 0$ there is just one solution, $r = 0$, and the dynamics consists of stable spirals converging to the $r = 0$ equilibrium point. At $\mathrm{Re}(\lambda) = 0$ the Hopf bifurcation occurs and a limit cycle is created. When $\mathrm{Re}(\lambda) > 0$ there are now two further solutions that are shown as red vectors (in polar coordinates) in the figure. These solutions relate to the limit cycle (stable in this example) where the radius of the limit cycle corresponds to r and the angle, θ, denotes the rotation (and $-\theta$ the counter-rotation) of the two solution vectors. Furthermore when $\mathrm{Re}(\lambda) > 0$ the solution at $r = 0$ becomes unstable, and therefore there are now unstable spirals emanating from the equilibrium point, see discussions in [77, 55, 52, 162].

Of course, in the case when $\mu < 0$, then the sign of $\mathrm{Re}(n^*_{w2})$ changes, meaning that the additional solutions will occur instead for Case i), when $\zeta < 0$[64]. So it can be seen that these successive coordinate changes, i.e. normal form transformations, have reduced the system to the point where the sign of just a single coefficient can be used to determine the type of Hopf bifurcation (supercritical or subcritical). A schematic representation of the Hopf bifurcation normal form is shown in Fig. 9.

It is important to note that this is a relatively simple example. However, even if we had all possible quadratic and cubic terms in the nonlinear vector on the righthand side of Eq. (310), then these could still be projected onto the $\mathbf{z}^*$ basis in the form of $N_z\mathbf{z}^*$ (with different entries in N_z from those shown above). In this way, *all* the terms in such as system can be reduced to a normal form controlled by just a single parameter.

4.10 Summary

In this Section, the normal form transformations of nonlinear systems have been considered. Several important topics have been introduced, including the use of near-identity transformations, derivation of the homological equations using Lie series, Hamiltonian normal form, frequency detuning, choice of free functions, backbone curves and bifurcations. Although the analysis has been developed for multi-degree-of-freedom systems, all the examples in this Section have been single-degree-of-freedom. The complexity of applying normal form techniques to nonlinear oscillators with more than one degree-of-freedom is significant, as will be seen in the next Section. In particular it is the topic of backbone curves that will be of most interest as we consider coupled nonlinear oscillators next.

5 Normal form transformations for coupled nonlinear oscillators

In this final Section we will consider how normal form transformations can be applied to systems of coupled nonlinear oscillators. All the analysis developed in Section 4 can be applied to systems with more than a single-degree-of-freedom. However, one important addition is needed. In the base solutions for Example 1, (see Eq. (13)) we included a phase lag term, which we have (deliberately) neglected throughout the discussion of the nonlinear oscillators so far. Although this might be acceptable in many cases for single-degree-of-freedom oscillators, it needs to be included in the case of coupled (e.g. multi-degree-of-freedom) oscillators. Therefore, in this Section we define the assumed base solutions for the i^{th} degree-of-freedom as

$$u_{pi} = \frac{U}{2}\mathrm{e}^{\mathbf{i}(\omega_{ri}t-\phi_i)}, \quad \text{and} \quad u_{mi} = \frac{U}{2}\mathrm{e}^{-\mathbf{i}(\omega_{ri}t-\phi_i)}. \tag{321}$$

where ω_{ri} and ϕ_i are the response frequency and phase lag for the ith degree-of-freedom respectively.

[64] We do not go into further detail here, but this will correspond to a subcritical rather than supercritical type of Hopf bifurcation. Much more in depth treatments can be found in many texts on nonlinear dynamics, see for example the discussions in [77, 55, 52, 162].

It should be noted that the assumptions regarding these solution types will capture only periodic solutions with a frequency ω_{ri}. Coupled nonlinear systems typically exhibit far more complex behaviours, such as quasi-periodic and chaotic motions — see for example [2, 59, 40, 53] and references therein. The type of analysis presented here will only capture the most basic underlying dynamical behaviour. We now consider an example.

5.0.1 Example 20: Two degree-of-freedom undamped system with cubic spring nonlinearities

We consider the two-degree-of-freedom lumped mass system[65], as shown in Fig. 10 (a), in which the spring forces contain linear and cubic terms with coefficients k, k_2 and κ, κ_2 respectively for masses, m. The equations of motion may be written as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \kappa x_1^3 - \kappa_2 (x_2 - x_1)^3 \\ \kappa x_2^3 + \kappa_2 (x_2 - x_1)^3 \end{bmatrix} = 0, \quad \rightsquigarrow \quad M\ddot{\mathbf{x}} + K\mathbf{x} + \mathcal{N}(\mathbf{x}) = 0, \tag{322}$$

where $\mathbf{x} = [x_1 \;\; x_2]^T$ in this case. The linearised version of this system, where $\kappa = \kappa_2 = 0$, results in the modal transform

$$\mathbf{x} = \Phi\mathbf{q} \quad \text{where} \quad \Phi = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \rightsquigarrow \quad \ddot{\mathbf{q}} + \begin{bmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{bmatrix} \mathbf{q} + \frac{\kappa}{m} \begin{bmatrix} q_1^3 + 3q_1 q_2^2 \\ \gamma q_2^3 + 3q_1^2 q_2 \end{bmatrix} = 0, \tag{323}$$

where $\gamma = 1 + 8\kappa_2/\kappa$, $\omega_{n1}^2 = \frac{k}{m}$ and $\omega_{n2}^2 = \frac{k+2k_2}{m}$ are the linear natural frequencies. Now we define[66]

$$\Lambda = \begin{bmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{bmatrix}, \quad \text{and} \quad \varepsilon\hat{\mathbf{n}}(\mathbf{q}) = \varepsilon\frac{\hat{\kappa}}{m} \begin{bmatrix} q_1^3 + 3q_1 q_2^2 \\ \gamma q_2^3 + 3q_1^2 q_2 \end{bmatrix} \tag{324}$$

where $\kappa = \varepsilon\hat{\kappa}$ has been used[67]. Now we are transforming from q_i to u_i, using T_{SO} from Eq. (116). Note that $u_i = u_{ip} + u_{im}$ which can also be expressed as the vector relationship $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_m$. Therefore we obtain

$$\begin{aligned} \varepsilon\hat{\mathbf{n}}(\mathbf{q}) = \varepsilon\hat{\mathbf{n}}(\mathbf{u}_p, \mathbf{u}_m) = \varepsilon\frac{\hat{\kappa}}{m} \begin{bmatrix} (u_{1p} + u_{1m})^3 + 3(u_{1p} + u_{1m})(u_{2p} + u_{2m})^2 \\ \gamma(u_{2p} + u_{2m})^3 + 3(u_{1p} + u_{1m})^2(u_{2p} + u_{2m}) \end{bmatrix} = \\ \frac{\hat{\kappa}}{m} \begin{bmatrix} 1 & 3 & 3 & 1 & 3 & 6 & 3 & 3 & 6 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 6 & 6 & 3 & 3 & \gamma & 3\gamma & 3\gamma & \gamma \end{bmatrix} \mathbf{u}^* = \varepsilon\frac{\hat{\kappa}}{m}\mathbf{n}^*\mathbf{u}^*, \end{aligned} \tag{325}$$

[65] This is example is taken from Chapter 5 of [128] and [68], and further details can be found there. See also the nonlinear normal mode analysis given by Vakakis et al. [175].

[66] As before $\mathbf{n}_2 = 0, \ldots$ etc. so we set $\mathbf{n}_1 = \mathbf{n}$, and correspondingly, $\mathbf{h}_1 = \mathbf{h}$, $\mathbf{b}_1 = \mathbf{b}$ and $\hat{\mathbf{n}}_{\mathbf{u}1} = \hat{\mathbf{n}}_{\mathbf{u}}$ throughout this example.

[67] Note that the specific structure in the nonlinear vector, $\hat{\mathbf{n}}(\mathbf{q})$, of Eq. (324) is of a relatively simple form because of the structure of the original nonlinear vector $\mathcal{N}(\mathbf{x})$ in this example. In general for problems of this type each row would have terms of q_1^3, $q_1 q_2^2$, $q_1^2 q_2$ and q_2^3 such that there would be eight coefficients in total in $\hat{\mathbf{n}}(\mathbf{q})$.

where $\mathbf{n}^*$ is the coefficient matrix for the nonlinear terms in the untransformed equation and

$$\mathbf{u}^* = \begin{bmatrix} u_{p1}^3 \\ u_{p1}^2 u_{m1} \\ u_{p1} u_{m1}^2 \\ u_{m1}^3 \\ u_{p1} u_{p2}^2 \\ u_{p1} u_{p2} u_{m2} \\ u_{p1} u_{m2}^2 \\ u_{m1} u_{p2}^2 \\ u_{m1} u_{p2} u_{m2} \\ u_{m1} u_{m2}^2 \\ u_{p1}^2 u_{p2} \\ u_{p1}^2 u_{m2} \\ u_{p1} u_{m1} u_{p2} \\ u_{p1} u_{m1} u_{m2} \\ u_{m1}^2 u_{p2} \\ u_{m1}^2 u_{m2} \\ u_{p2}^3 \\ u_{p2}^2 u_{m2} \\ u_{p2} u_{m2}^2 \\ u_{m2}^3 \end{bmatrix} \quad \text{and} \quad \Lambda_d \boldsymbol{u} = \begin{bmatrix} \mathbf{i}\omega_{r1} & 0 & 0 & 0 \\ 0 & -\mathbf{i}\omega_{r1} & 0 & 0 \\ 0 & 0 & \mathbf{i}\omega_{r2} & 0 \\ 0 & 0 & 0 & -\mathbf{i}\omega_{r2} \end{bmatrix} \begin{bmatrix} u_{p1} \\ u_{m1} \\ u_{p2} \\ u_{m2} \end{bmatrix} = \mathbf{i}\omega_{r1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & -r \end{bmatrix} \begin{bmatrix} u_{p1} \\ u_{m1} \\ u_{p2} \\ u_{m2} \end{bmatrix}, \tag{326}$$

where $r = \omega_{r2}/\omega_{r1}$, and Λ_d is the detuned frequency matrix obtained from $\dot{\boldsymbol{u}} = \Lambda_d \boldsymbol{u}$ using Eq. (321). Next the detuned Lie derivative is computed by substituting $\Lambda = \Lambda_d$ into Eq. (127) to give the coefficient matrix $\Lambda^*{}_d$. This

is repeated in Eq. (153) to give the following

$$\Lambda^*{}_d = \mathbf{i}\omega_{r1}\mathbf{I}_{20}\begin{bmatrix} 3 \\ 1 \\ -1 \\ -3 \\ 1+2r \\ 1 \\ 1-2r \\ -1+2r \\ -1 \\ -1-2r \\ 2+r \\ 2-r \\ r \\ -r \\ -2+r \\ -2-r \\ 3r \\ r \\ -r \\ -3r \end{bmatrix}, \quad \rightsquigarrow \quad (\Lambda^*{}_d)^2 = -\omega_{r1}^2\mathbf{I}_{20}\begin{bmatrix} 9 \\ 1 \\ 1 \\ 9 \\ (1+2r)^2 \\ 1 \\ (1-2r)^2 \\ (-1+2r)^2 \\ 1 \\ (-1-2r)^2 \\ (2+r)^2 \\ (2-r)^2 \\ r^2 \\ r^2 \\ (-2+r)^2 \\ (-2-r)^2 \\ 9r^2 \\ r^2 \\ r^2 \\ 9r^2 \end{bmatrix}, \tag{327}$$

where $\mathbf{I}_{20}$ is a 20×20 identity matrix. These relationships are substituted into the detuned homological equation, Eq. (261), to give

$$-\mathbf{b}\Lambda^{*2}_{d}\mathbf{u}^{*}-\Upsilon\mathbf{b}\mathbf{u}^{*}=\omega_{r1}^{2}\begin{bmatrix} b_1(8) & b_{21}(9-r^2)\\ b_2(0) & b_{22}(1-r^2)\\ b_3(0) & b_{23}(1-r^2)\\ b_4(8) & b_{24}(9-r^2)\\ b_5(4(r^2+r)) & b_{25}(1-4r+3r^2)\\ b_6(0) & b_{26}(1-r^2)\\ b_7(4(r^2-r)) & b_{27}(1-4r+3r^2)\\ b_8(4(r^2-r)) & b_{28}(1-4r+3r^2)\\ b_9(0) & b_{29}(1-r^2)\\ b_{10}(4(r^2+r)) & b_{30}(1+4r+3r^2)\\ b_{11}(3+4r+r^2) & b_{31}(4(1+r))\\ b_{12}(3-4r+r^2) & b_{32}(4(1-r))\\ b_{13}(r^2-1) & b_{33}(0)\\ b_{14}(r^2-1) & b_{34}(0)\\ b_{15}(r^2-2r) & b_{35}(4(1-r))\\ b_{16}(3-4r+r^2) & b_{36}(4(1+r))\\ b_{17}(r^2-1) & b_{37}(8r^2)\\ b_{18}(r^2-1) & b_{38}(0)\\ b_{19}(r^2-1) & b_{39}(0)\\ b_{20}(r^2-1) & b_{40}(8r^2) \end{bmatrix}^{T}=\mathbf{n}^{*}\mathbf{u}^{*}-\mathbf{n}^{*}_{\mathbf{u}}\mathbf{u}^{*}, \tag{328}$$

where $\mathbf{b}$ and $\mathbf{n}^{*}_{\mathbf{u}}$ are coefficient matrices of the same dimension as $\mathbf{n}^{*}$ defined in Eq. (325). As with previous examples there is now a choice of free functions for b_i and $i=1,2,3...20$. Notice that some terms have a zero associated with the b_i (as we saw previously), and these are the *unconditional resonances* (as defined in Section 4.1.2). In addition, now that we have multiple degrees-of-freedom, and therefore more than one response frequency, there are other b_i values which could have a zero term associated with them depending on the frequency ratio value, r. These are the *conditional resonances*, and as we discussed in Section 4.1.2 are also called *internal resonances*.

Notice also that the simplest way to satisfy Eq. (328) is to set $b_i=n^{*}_{ui}=0$ whenever $n^{*}_{i}=0$. From Eq. (325) it can be seen that this applies to $i=11,12,13...28,29,30$. After setting the associated $b_i=0$ for $i=11,12,13...28,29,30$ we are left with unconditionally resonance terms for $b_2,b_3,b_6,b_9,b_{33},b_{34},b_{38},b_{39}$ and conditionally resonant terms for b_7,b_8,b_{32},b_{35} based on the condition $r=1$. This is called a one-to-one resonant interaction, because it occurs when $\omega_{r1}=\omega_{r2}$.

Therefore to satisfy Eq. (328) for both the conditional and unconditional resonances we set the coefficients of $\mathbf{n}^{*}_{\mathbf{u}}$ to be

$$\mathbf{n}^{*}_{\mathbf{u}}=\frac{\hat{\kappa}}{m}\begin{bmatrix} 0 & 3 & 3 & 0 & 0 & 6 & 3\delta_{1r} & 3\delta_{1r} & 6 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 3\delta_{1r} & 6 & 6 & 3\delta_{1r} & 0 & 0 & 3\gamma & 3\gamma & 0 \end{bmatrix}\mathbf{u}^{*} \tag{329}$$

where δ_{1r} is the Kronecker delta function defined such that $\delta_{1r}=1$ (i.e. when $r=1$) and $\delta_{1r}=0$ for all other

values (i.e. when $r \neq 1$). Substituting Eq. (329) into Eq. (146) gives the ε^1 normal form as

$$\ddot{\mathbf{u}} + \begin{bmatrix} \omega_{n1}^2 & 0 \\ 0 & \omega_{n2}^2 \end{bmatrix} \mathbf{u} + \frac{3\kappa}{m} \begin{bmatrix} u_{p1}^2 u_{m1} + u_{p1} u_{m1}^2 + 2u_{p1}u_{p2}u_{m2} + 2u_{m1}u_{p2}u_{m2} + \delta_{1r}(u_{p1}u_{m2}^2 + u_{m1}u_{p2}^2) \\ 2u_{p2}u_{p1}u_{m1} + 2u_{m2}u_{p1}u_{m1} + \gamma u_{p2}^2 u_{m2} + \gamma u_{p2} u_{m2}^2 + \delta_{1r}(u_{p1}^2 u_{m2} + u_{m1}^2 u_{p2}) \end{bmatrix} = 0, \quad (330)$$

where $\mathbf{u} = [u_1 \;\; u_2]^T$. Furthermore, using Eq. (116), we find that the modal displacements are given by

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \frac{\kappa}{m\omega_{r1}^2} \begin{bmatrix} \frac{1}{8}(u_{p1}^3 + u_{m1}^3) + \frac{3}{4(r^2+r)}(u_{p1}u_{p2}^2 + u_{m1}u_{m2}^2) + \frac{3}{4(r^2-r)}(u_{p1}u_{m2}^2 + u_{m1}u_{p2}^2) \\ \frac{\gamma}{8r^2}(u_{m2}^3 + u_{p2}^3) + \frac{3}{4(1+r)}(u_{m2}u_{m1}^2 + u_{p2}u_{p1}^2) + \frac{3}{4(1-r)}(u_{p1}^2 u_{m2} + u_{m1}^2 u_{p2}) \end{bmatrix}, \quad (331)$$

when $r \neq 1$, and

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \frac{\kappa}{m\omega_{r1}^2} \begin{bmatrix} \frac{1}{8}(u_{p1}^3 + u_{m1}^3) + \frac{3}{4(r^2+r)}(u_{p1}u_{p2}^2 + u_{m1}u_{m2}^2) \\ \frac{\gamma}{8r^2}(u_{m2}^3 + u_{p2}^3) + \frac{3}{4(1+r)}(u_{m2}u_{m1}^2 + u_{p2}u_{p1}^2) \end{bmatrix}, \quad (332)$$

for the case when $r = 1$. □

Expressions for the backbone curves can be found by taking the base solutions from Eq. (321) as the assumed solutions and substituting them into Eq. (330). For example, carrying this out for the first line of Eq. (330) gives

$$\begin{aligned} (\omega_{n1}^2 - \omega_{r1}^2) U_1 \left(e^{\mathbf{i}(\omega_{r1}t-\phi_1)} + e^{-\mathbf{i}(\omega_{r1}t-\phi_1)}\right) + \frac{3\kappa}{4m} U_1 \Big[(U_1^2 + 2U_2^2) \left(e^{\mathbf{i}(\omega_{r1}t-\phi_1)} + e^{-\mathbf{i}(\omega_{r1}t-\phi_1)}\right) \\ + \delta_{1r} U_2^2 \left(e^{-\mathbf{i}(\omega_{r1}t-\phi_1)} e^{-\mathbf{i}2(\phi_1-\phi_2)} + e^{\mathbf{i}(\omega_{r1}t-\phi_1)} e^{\mathbf{i}2(\phi_1-\phi_2)}\right) \Big] = 0. \end{aligned} \quad (333)$$

Using the same approach for the second line of Eq. (330) and then balancing the $e^{\mathbf{i}(\omega_{r1}t-\phi_1)}$ terms gives[68]

$$\begin{aligned} &\left\{ \omega_{n1}^2 - \omega_{r1}^2 + \frac{3\kappa}{4m} \left[U_1^2 + U_2^2 \left(2 + \delta_{1r} e^{\mathbf{i}2(\phi_1-\phi_2)}\right) \right] \right\} U_1 = 0, \\ &\left\{ \omega_{n2}^2 - \omega_{r2}^2 + \frac{3\kappa}{4m} \left[\gamma U_2^2 + U_1^2 \left(2 + \delta_{1r} e^{-\mathbf{i}2(\phi_1-\phi_2)}\right) \right] \right\} U_2 = 0. \end{aligned} \quad (334)$$

There are several cases of backbone curves, depending on the parameters in Eq. (334). The first case is for *single mode backbone curves* where we set either $U_1 = 0$ or $U_2 = 0$ to give

$$\begin{aligned} &\text{S1}: \;\; \text{when} \quad U_2 = 0, \quad \omega_{r1}^2 = \omega_{n1}^2 + \frac{3\kappa}{4m} U_1^2, \\ &\text{S2}: \;\; \text{when} \quad U_1 = 0, \quad \omega_{r2}^2 = \omega_{n2}^2 + \frac{3\kappa\gamma}{4m} U_2^2. \end{aligned} \quad (335)$$

These expression relate to the non-resonant case, $r \neq 1$ (therefore $\delta_{1r} = 0$), when there is no *phase coupling* between the modal coordinates u_1 and u_2 (and then by definition q_1 and q_2) via the $e^{-\mathbf{i}2(\phi_1-\phi_2)}$ term in Eq. (334).

To consider the resonant case, when $r = 1$ (and therefore $\delta_{1r} = 1$), we first introduce a general frequency response parameter Ω and then rewrite Eq. (334) such that we consider the response of both equations at a specific Ω value which gives

$$\Omega^2 = \omega_{n1}^2 + \frac{3\kappa}{4m} \left[U_1^2 + U_2^2 (2+p) \right] = \omega_{n2}^2 + \frac{3\kappa}{4m} \left[\gamma U_2^2 + U_1^2 (2+p) \right], \quad \text{where} \quad p = e^{\mathbf{i}2(\phi_1-\phi_2)}. \quad (336)$$

In order to maintain real solutions we require that $p = \pm 1$ such that[69]

> $p = +1$ corresponds to $|\phi_1 - \phi_2| = 0, \pm n\pi$, for $n = 1, 2, 3...$ This corresponds to the *in-unison case*, with either in-phase (0) or out-of-phase ($\pm n\pi$) cases,

[68] As in previous examples, balancing the $e^{-\mathbf{i}(\omega_{r1}t-\phi_1)}$ terms gives equivalent expressions.

[69] This analysis was first developed in [114], where these two solutions were referred to as the *normal mode* and *elliptic mode* due to their shapes in the configuration space.

$p=-1$ corresponds to $|\phi_1-\phi_2|=\pm n\pi/2$, for $n=1,2,3...$ This is the *out-of-unison case*, with $\pm n\pi/2$ phase differences in the response behaviour.

Here we consider just the $p=+1$ case. Details of the out-of-unison case can be found for example in [114, 68].

Setting $p=+1$ and $n=1$ yields two backbone curves, labelled $D12_i^+$ and $D12_i^-$, with the phase differences

$$D12_i^+: \quad |\phi_1-\phi_2|=0, \qquad D12_i^-: \quad |\phi_1-\phi_2|=\pi. \tag{337}$$

Note that the notation is D for double (i.e. two mode) interaction, followed by the modes that interact (1 and 2 in this case), the subscript i means in-unison, and plus (in-phase) or minus (out-of-phase)[70] Substituting $p=+1$ in Eq. (336) leads to

$$D12_i^\pm: \qquad U_1^2=\left(1-4\frac{\kappa_2}{\kappa}\right)U_2^2-\frac{2m}{3\kappa}\left(\omega_{n2}^2-\omega_{n1}^2\right) \qquad \text{and} \qquad \Omega^2=\frac{3\omega_{n1}^2-\omega_{n2}^2}{2}+\frac{3(\kappa-\kappa_2)}{m}U_2^2. \tag{338}$$

From the first expression of Eq. (338) it can be seen that to ensure real solutions the following conditions can be imposed

$$U_2^2>\frac{2m}{3(\kappa-4\kappa_2)}(\omega_{n2}^2-\omega_{n1}^2), \quad \text{and} \quad \kappa\geq 4\kappa_2\,. \tag{339}$$

Numerically computed results are shown in Fig. 10 (b), (c) and (d). The numerical results are for the case where the central spring connecting the two masses is much less stiff than the springs connecting the masses to the rigid walls at each end, e.g. $k_2\ll k$ and $\kappa_2\ll\kappa$. We call this type of system *weakly coupled*, and for very small amplitudes, the two masses both oscillate at a frequency very close to one, on the $S1$ and $S2$ curves. Then as the U_i or X_i amplitudes increase, it can be seen that the $S2$ backbone curve has a bifurcation at which point the stable $D12_i^\pm$ curves coalesce with $S2$. The bifurcation is caused by the underlying periodic orbit undergoing a secondary-Hopf type bifurcation. This manifests itself in terms of the backbone curve, as a supercritical pitchfork bifurcation event. After the bifurcation point, the $S2$ becomes unstable, and the stable solutions are given by the $D12_i^\pm$ backbone curves. The type of modal coordinate motion that occurs for each backbone curve is shown in the inset diagrams in Fig. 10 (b). Further details and discussion of this example is given in [128].

Note also that these type of backbone curves can be interpreted as the underlying “skeleton" which govern some of the key characteristics of the forced-damped responses. For example the curvature of the forced-damped resonance curve, as was shown in Fig. 2 (c). In this example, the structure of the backbone curves is considerably more complex than those shown in Fig. 2 (c), but the same principle holds true. A detailed discussion of the forced-damped responses relevant to this example is given in [67] — see also [21, 68, 73].

5.1 Summary

The example considered in this Section demonstrates how normal form methods can be used to help understand the behaviour of coupled nonlinear oscillators. We have limited the examples in this Section to just one, because (i) we are now going beyond the introductory remit of this paper, and (ii) there are multiple published examples for the interested reader to find in the literature. For example, normal form analysis of structural elements such as beams, plates, shells and cables can be found in references [124, 194, 171, 68, 128, 44, 33, 50], to highlight just a few. Perhaps the most important features highlighted in this Section is the role of the conditional resonances, and how they can be interpreted using backbone curves.

To demonstrate the approach with as much simplicity as possible, this example has not included either damping or forcing, which are typically needed for real applications. That said, forced-damped cases have also been treated using normal form methods — see for example [128, 15] and references therein. An alternative is to capture the forced-damped behaviour using numerical continuation packages, such as COCO [153], MATCONT [35], MANLAB [56], and AUTO [36], and the conservative backbone curves are found using normal form techniques — for example as shown in Fig. 2[71]. Of course backbone curves are independent of normal forms, and can be found using multiple other methods as well, see for example [128, 41, 42] and references therein. Further potential developments will be discussed in Section 6.3.

[70] In [128] and [68] these curves are referred to as $S3^\pm$ for the $p=1$ case and $S4^\pm$ for the $p=-1$ case.

[71] Interesting recent studies on the stability of the forced-damped curve and the relationship to the backbone curve have been undertaken by Cenedese and Haller [27, 28].

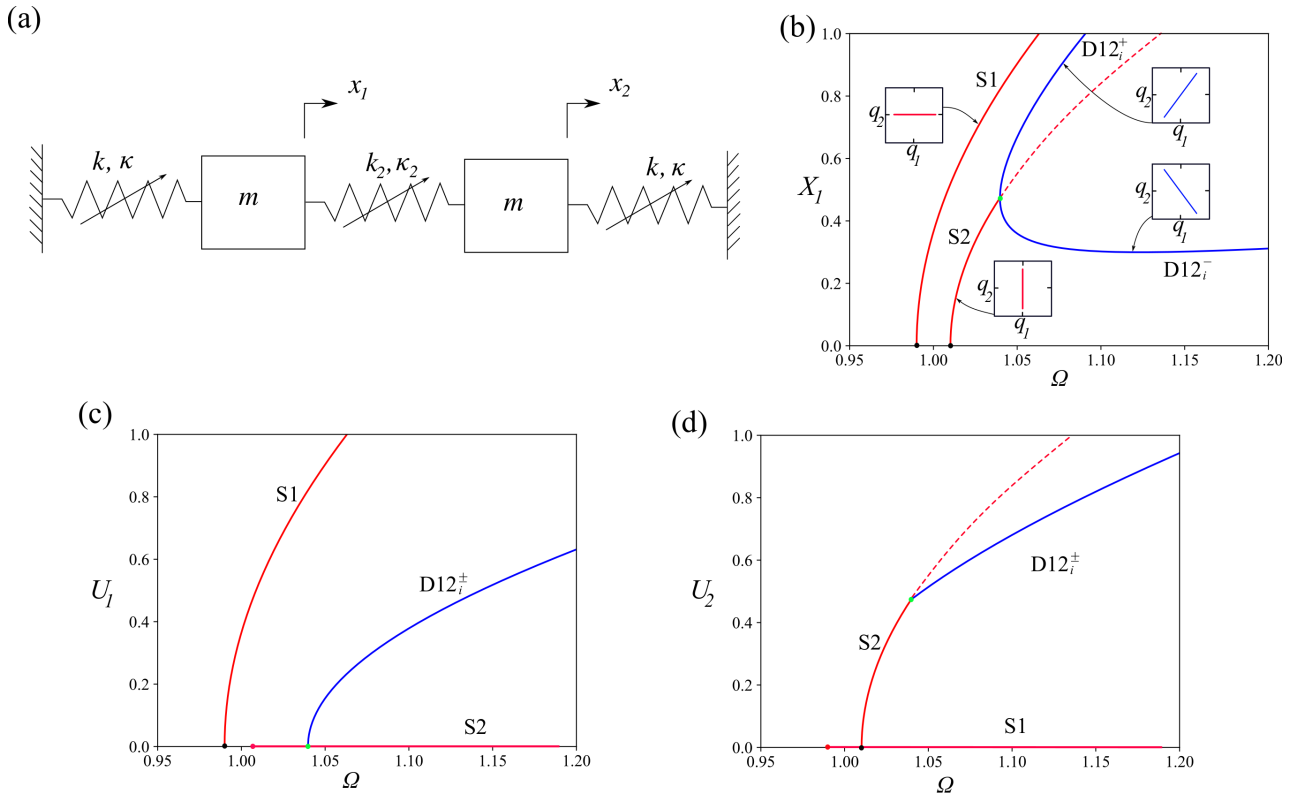

Fig. 10: The two-degree-of-freedom nonlinear oscillator considered in Example 20, Section 5.0.1. Showing (a) the schematic diagram of the two-degree-of-freedom nonlinear oscillator, (b) plot showing the backbone curves in the projection of the displacement amplitude X_1 against Ω, where ($X_i = ||x_i||$ for $i = 1, 2$). This shows the in-phase backbone, $S1$, the out-of-phase backbone, $S2$, and the conditionally resonant double mode (also called "mixed mode") backbones $D12_i^\pm$. The inset diagrams show the behaviour in modal coordinates q_1, q_2 (noting that $q_i = u_i$ to order ε^0). Green dots represent the points where the double backbone curves $D12_i^\pm$ bifurcate from the single mode curve $S2$. Black dots represent the birth of a nonlinear normal mode family of periodic solutions via a Hamiltonian Hopf bifurcation. (c) The backbone curves in the projection of U_1 against Ω, where red dots represent the start of a non-oscillatory curve e.g. when either $U_1 = 0$ or $U_2 = 0$. (d) Projection of U_2 against Ω, where the dash red line represents an unstable part of the backbone curve. Parameter values are $m = 1$ kg, $k = 0.98$ N/m, $k_2 = 0.0202$ N/m, $\kappa = 0.2$ N/m^3 and $\kappa_2 = 0.02$ N/m^3. Note that $\kappa > 4\kappa_2$ in this example, and that parameters are chosen to show the phenomena rather than relate to a physical example.

6 Conclusions and future directions for research

6.1 Summary and conclusions

In this paper an introduction to using normal form transformations for linear and nonlinear structural dynamics has been given. After some motivating examples, the paper investigated linear single-degree-of-freedom systems, which is a topic that is fundamental to structural dynamics. In that discussion, we outlined the importance of exponential base solutions for both damped and undamped oscillators. However, it was also highlighted how simplifying the matrix structure of the oscillator model could significantly simplify the solution procedure. Coordinate transformations were considered via Jordan normal form and the method of reduction of order.

Simplifying matrix structures leads naturally to coordinate transformations, and structural dynamists are generally very familiar with this concept via the idea of modal analysis. This was the starting point for the discussion of multi-degree-of-freedom linear systems. Modal transformations were demonstrated both for differential equations written in state-space form and classical coupled ordinary differential equations in second-order form. This type of modal coordinate transformation was then used as a first step when analysing nonlinear oscillators.

We began the discussion of nonlinear systems, by highlighting some of the key issues. Most importantly the concept of nonlinear resonance. Following that, the idea of near-identity transformations was described in detail

for both first-order and second-order sets of equations. This led to expressions for the homological equation, the solution of which is the central operation of a normal form method. Another important detail considered at this point was to show the equivalence of the real and complex normal forms using a Duffing oscillator example.

Normal form transformations can be expressed in terms of Lie series, and to give an insight into this connection we described a derivation of the homological equations using Lie series. This required prolongation of the near-identity transformation, and gives additional insight into the structures that underly the complex and real normal form transformations. This was followed by a description of the Hamiltonian normal form. This type of normal form is not used very often in structural dynamics, and is more commonly used in maths and physics. However, for completeness, it has been included to show the relationship with both the complex and real normal form variants described previously.

In order to describe amplitude-frequency nonlinear effects, the concept of frequency detuning was introduced. Following this, the choice of free functions in the homological equation was discussed, and an example of a minimal normal form included. Next the topic of backbone curves was described, although as pointed out at the time, this is an independent concept that can be used for nonlinear oscillators without normal form methods.

Finally, we considered the case of a coupled nonlinear oscillator system. This final example showed the concept of conditional resonances, a phenomena that is very naturally modelled via the homological equation. It also demonstrated how the real normal form method could be used to find approximate analytical expressions for backbone curves. The conditional (nonlinear) resonance was represented as a bifurcation of one of the backbone curves into a double backbone curve solution.

In Section 1.1.4 we outlined five useful properties of normal form transformations for structural dynamicists. We now revisit each of these in turn:

I. *Obtaining approximate solutions for* x: This was demonstrated for most of the examples throughout the paper. For nonlinear systems certain conditions are required for this to be possible; specifically the eigenvalues of the underlying linear problem need to be semisimple — see [120]) for details. For multi-degree-of-freedom nonlinear systems, the algebraic complexity of trying to obtain an approximate solutions for $\mathbf{x}$ quickly becomes prohibitive.

II. *Obtain information on the nonlinear resonances:* This is one of the most important aspects, particularly for nonlinear multi-degree-of-freedom oscillators, as we saw in Example 20. It will also be discussed in the next Section.

III. *Obtain frequency amplitude relationships known as backbone curves:* Again this is particularly relevant for nonlinear multi-degree-of-freedom oscillators (e.g. Example 20) and will also be discussed in the next Section.

IV. *Simplify models of important dynamic phenomena, such as bifurcations.* An example was shown for the Hopf bifurcation in Section 4.9. This is one of the most important dynamic phenomena for vibration problems, and using normal form transformations it can be reduced to a model with just a single parameter that determines the key behaviour.

V. Normal form transformations used to achieve model-order reduction or system identification. In fact, (and related to point IV above) normal forms can be used not just to reduce the number of states, but also the number of parameters. This comes from the ethos of normal forms, which is to systematically produce the simplest model that captures the important dynamics, both in terms of the number of states and the number of parameters. Model-order reduction or system identification were not discussed in detail in this paper, but they are mentioned in Section 6.3 where we will discuss future research directions.

Other concluding points regarding the normal form techniques introduced in this paper are:

a) *Coordinate transformations can simplify, but not without work!* Both linear and nonlinear systems can be simplified using coordinate transformations, as demonstrated in this paper. However, this is not without work. The algebraic intensity, and work required in some of the methods presented here is a significant limitation of both their application and popularity.

b) *Formats and styles are important.* As has been shown in this paper, normal form transformations are not unique, and are reliant on many choices. This gives a lot of flexibility in how they can be tailored to suit a particular application. However, it can also be confusing when there are so many different variants, and small variations in the outcomes. That said, the final results should be equivalent, regardless of which style was used, as was shown for example in Section 4.2.5 with the equivalence between the complex and real normal form. For linear systems, there is less to be chosen, but the scaling of eigenvectors is one example where choices made in the method can affect the outcome.

c) *The homological equation is central to the method.* For nonlinear systems, this offers the key advantage of the normal form technique over other methods. By developing a homological equation, that is linear to solve, and also identifies the conditional and unconditional resonances, the normal form approach can give insight into the likely behaviour of the nonlinear system. This insight is explicit, in terms of structures like backbone curves and their bifurcations.

d) *Normal forms transformations can be used as part of a reduced order modelling process.* In general, a normal form transformation will result in a system that has the same number of degrees-of-freedom as the original system. The examples in this paper have all been either single, two or three degree-of-freedom. As the number of degrees-of-freedom increases beyond this, it makes sense to combine the normal form method with a reduction in the order of the system, in order to capture just the essential dynamics. There have been many investigations into combining normal form and order reduction methods (mentioned below), and this is a rapidly expanding area of future research[72].

6.2 Comparison with other methods; some advantages and disadvantages

In the context of structural dynamics, the method of normal forms has been compared to other techniques that are used to carry out similar tasks, for example to compute nonlinear normal modes or backbone curves [66, 41, 42, 180]. As was described in Section 4.1.3 there are subtle but important distinctions between normal form methods and perturbation techniques, and so we will outline some of the potential advantages of using normal form techniques compared to these and other methods. Choosing a particular technique to use will depend on multiple factors, but the typical advantages of normal form methods are:

1. There is no explicit requirement of “smallness" of either the nonlinear terms or amplitude of response[73] in the same way that a perturbation method inherently has — see for example the study carried out by [104]. Observations by [66, 41, 42] also show examples of how normal form approximations to backbone curves can be extended to relatively large amplitudes without significant loss if accuracy, in comparison to harmonic balance and the method of multiple scales.
2. During the normal form procedure, no secular terms are generated, and as a result there are no “solvability conditions" (to use the terminology of [84]) that occur, for example, in the method of multiple scales[74]. Advocates of other methods point out that secular terms can be used to infer that resonances are present in the system being considered. However, in a normal form approach, the resonance’s are revealed explicitly in the homological equation. Furthermore, the homological equation also gives information on more subtle features of the resonance, such as whether they are conditional or unconditional, without needing to carry out additional work.
3. The resonant terms can be dealt with in a systematic way via the homological equation, whereas in perturbation methods this process is ad-hoc and (depending on the exact details) can require additional terms to “fix" the solutions depending on the exact resonant conditions, as mentioned above.
4. For reduced-order models of high-dimensional systems, normal form techniques provide a very strong modelling framework, particularly in terms of *intrusive* methods — see for example the comparison made by [180]. That said, normal forms is not the only approach to this problem. For example, harmonic balance methods have also been developed in combination with reduction of order techniques to create reduced-order models — see for example [176].

In terms of disadvantages of the normal form technique, Kodama and Mikhailov [92] pointed out some issues that occur when normal form transformations are applied to certain partial differential equations. This subject

[72] It should be noted that care needs to be taken with these types of reduction methods, as terms from neglected modes can affect the normal form of the reduced model.

[73] The exception is where it is deliberately introduced, as discussed in Section 4.1.3.

[74] These issues have been known about for some time. For example, they are described in [141], but first became apparent in a structural dynamics context in the work of [147], where the authors tried to apply the method of multiple scales to systems with internal resonances. In their conclusions the authors stated: *“This indicates a yet unclear mathematical drawback in the specific application of the method of multiple scales to this two time-scale-dependent problem."* The underlying reason for these issues are explained in detail in [84] (which references the theoretical work of [91]). In single-degree-of-freedom systems, the solvability conditions are (typically) benign, and do not restrict choices too much (see [98] for an example of how this restriction can be effectively removed by adding an additional time scale). However, as the system becomes more complex, particularly for systems with more degrees-of-freedom, solvability conditions start to become restrictive, and to get the method of multiple scales to work (without adding additional time scales) the solutions are “fixed" by adjusting both free functions and initial conditions — see the two-degree-of-freedom examples shown in [125] (Chapter 10 page 267) and [41] where it is observed that the initial conditions cannot be set to zero, and instead they need to be used to satisfy the solvability conditions.

has been subsequently further explored by other authors [71, 93, 191]. In addition, normal forms techniques are often regarded as being algebraically intense, and some authors have developed symbolic computation methods (particularly using Maple) in order to try and mitigate this effect [10, 190, 193, 122].

6.3 Future directions for research

In terms of future research, it seems likely that normal form methods will continue to be an important class of methods for studying the behaviour of nonlinear oscillator systems. From the authors perspective, there are several key areas that would be of interest for future research development:

1. *Investigation of nonlinear resonances.* This longstanding topic continues to be of significant interest, and there are still several aspects that can be investigated. From a structural dynamics perspective, the understanding of the phenomena has been progressed in several different ways. Firstly from a geometric perspective, and with particular application to the geometric definition of nonlinear normal modes (NNMs) as invariant manifolds — first put forward by Shaw and Pierre [155] — and formalised using normal forms by Touzé [168, 169] (see also the overview given by Volvert and Kerschen [182]). From a normal form perspective this study of resonance in coupled oscillators is mainly being advanced via the analysis of backbone curves — see for example [21, 68, 15, 50, 53]. However, other methods are also relevant in this area. Particularly following the work of Haller and Ponsioen [59] in defining spectral submanifolds, which are now able to capture some of the more subtle and interesting [139] phenomena in this domain. In particular, the spectral submanifold technique can be applied in a normal form style (or as an alternative in a graph style), and an interesting comparison of the spectral submanifold method and other normal form techniques has been carried out by Breunung and Haller [15]. Also related to this topic is the relationship between NNMs (or backbone curves) and the forced-damped responses of coupled oscillator systems [170, 169, 67, 70, 15]. Recent developments have included analysis of rotor stator contact [154], analyses using a Melnikov function approach [28, 27], parametric antiresonance [86], and vibration of nancomposite shells [118] to name just a few. This will continue to be a key area of future research.
2. *Identification methods from experimental data. This is also a longstanding topic of interest, that has not been approached using normal forms until relatively recently — see [50, 33]. In contrast there are a selection of other approaches that are typically used for this techniques, see for example [135, 39, 69, 148, 33, 150] and references therein. Other recent developments include analysis of wheel-set dynamics [187] and power grids [94]. It is likely that this research topic will grow as an area of interest.* In particular, there is considerable potential cross-over between using normal form transformations to create reduced-order models for engineering structures, and identification methods. Specifically the reduced-order models of structural elements, such as beams, cables plates and shells, lead to much simpler models with a number of key parameters that can potentially be identified. See for example [169, 179, 173, 133].
3. *Nonlinear model reduction.* Reducing large scale systems (for example from finite element models) down to smaller models that capture the key dynamical behaviour is another area of great importance in the structural dynamics field — see for example [172, 19, 131, 20, 62, 138, 177]. As mentioned above, normal form methods have been used extensively in combination with methods to reduce the order of the system — see for example [110, 180, 156, 173], and it is likely that this topic will grow considerably in the future. In particular, the connection of invariant manifold computation techniques (described in point **1.**) has recently been developed to work for structures with complex geometries, such as those represented in finite element models [180, 181, 79]. A comparison has also been recently made between reduced-order normal form methods, and other similar methods, such as modal derivatives [180]. This is an important development for the application of these methods to more complex engineering problems.
4. *Hamiltonian normal form.* This is mostly used in physics, astronomy and celestial mechanics, as described for example by [63, 49] and references therein. That said, there are also some applications in maths and physics that relate to structural dynamics, such as resonances in chains of oscillators [178], triangular connected spring-mass systems [87], resonance-assisted tunnelling [46], and this is an open area for structural dynamists in the future.

7 Author's Contributions

The author wrote the paper, including generating the simulations for all figures.

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