Anticipatory 1D and 2D Linear Systems

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Abstract. Notions of anticipatory systems for discrete-time and continuous-time 1D linear systems and 2D discrete linear systems are introduced. A discrete-time system is called anticipatory if its state vector and output vector depend on the future values of inputs. A continuous-time system is called anticipatory if its state vector and output vector depend on the derivatives of inputs. Necessary and sufficient conditions for the anticipation of singular discrete-time and continuous-time 1-D linear systems are established. It is shown that the discrete-time system obtained by discretization from continuous-time one is anticipatory for any value of the discretization step if and only if the continuous-time system is anticipatory. Necessary and sufficient conditions for the anticipation of the singular 2D Fornasini – Marchesini model and the singular 2D Roesser model are established.

Key Words. 1D and 2D anticipatory systems, Roesser model, discretization.

INTRODUCTION

In recent years a dynamic development of the theory of anticipatory systems especially the theory of anticipatory discrete-time linear systems has been observed [26,4,5]. The definitions of anticipatory systems are different and usually not very precise [26]. Dubois in [4,5] has introduced the concepts of incursion and hyperincursion for dynamical systems. In this paper precise definitions of anticipatory continuous-time and discrete-time linear systems will be proposed. A discrete-time system will be called anticipatory if its state vector and output vector depend on the future values of inputs. A continuous-time system will be called anticipatory if its state vector and output vector depend on the derivatives of inputs [3]. In [8-11] it has been shown that in singular discrete-time systems the state vectors may depend on the future values of inputs and in singular continuous-time systems the state vectors may depend on the derivatives of inputs. The electrical circuits are examples of singular systems [9]. Therefore, the following question arises. Can an electrical circuit be an anticipatory system? Let a singular continuous-time linear system be an anticipatory system. By discretization of this singular continuous-time system we obtain a suitable singular discrete-time system. Will be the obtained discrete-time system also anticipatory?
The main purpose of this paper is just to give answers to these questions. Necessary and sufficient conditions for the anticipation of singular discrete-time and continuous-time linear systems will be established. It will be shown that:

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1) the singular electrical circuits are not anticipatory systems,
2) the discrete-time system obtained by discretization from continuous-time one is anticipatory for any value of the discretization step if and only if the continuous-time system is also anticipatory.

Necessary and sufficient conditions for the anticipation of the singular 2D Fornasini-Marchesini model and the singular 2D Roesser model will be established.

DISCRETE-TIME SYSTEMS

Let $R^{p\times n}$ be the set of real $p\times n$ matrices and $R^p := R^{p\times 1}$. Consider the discrete-time linear system

$$Ex_{i+1} = Fx_i + Gu_i \tag{1a}$$

$$y_i = Cx_i + Du_i\ ,\ i\in Z_+ := \{0,1,2,...\} \tag{1b}$$

where $x_i \in R^n$, $u_i \in R^m$, $y_i \in R^p$ are the state vector, input vector and output vector at the point i, respectively and $E,F \in R^{n\times n}$, $G\in R^{n\times m}$, $C\in R^{p\times n}$, $D\in R^{p\times m}$.

If $\det E \neq 0$ then the system (1) is called standard and if $\det E = 0$ then the system is called singular.

It is assumed that the pencil (E,F) is regular, i.e.

$$\det[Ez-F]\neq 0 \text{ for some } z\in \mathbf{C} \tag{2}$$

where $\mathbf{C}$ is the field of complex numbers

If the condition (2) is satisfied then

$$[Ez-F]^{-1} = \sum_{i=-\mu}^{\infty}\Phi_i z^{-(i+1)} \tag{3}$$

where μ is the nilpotence index and Φ_i is the fundamental matrix defined by

$$E\Phi_i - F\Phi_{i-1} = \Phi_i E - \Phi_{i-1}F = \begin{cases} I_n & for\quad i=0 \\ 0 & for\quad i\neq 0\end{cases} \tag{4}$$

where I_n is the $n\times n$ identity matrix

The solution of (1a) has the form [12,11,22-24]

$$x_i = \Phi_i Ex_0 + \sum_{k=0}^{i+\mu-1}\Phi_{i-k-1}Gu_k \tag{5}$$

From (5) it follows that if $\mu > 1$ then the solution x_i depends on the future values of inputs u_k for $k>i$.

Definition 1. The system (1) is called anticipatory if the state vector x_i and output vector y_i at the point i depends on the future values of u_k for $k>i$.

Theorem 1. The standard system (1) is not anticipatory.

Proof. If $\det E \neq 0$ then there exists E^{-1} and

$$[Ez-F]^{-1} = [E(I_n z - E^{-1}F)]^{-1} = (I_n z - E^{-1}F)^{-1}E^{-1} = \sum_{i=0}^{\infty} (E^{-1}F)^i E^{-1} z^{-(i+1)} \tag{6}$$

since

$$(I_n z - E^{-1}F)^{-1} = \sum_{i=0}^{\infty} (E^{-1}F)^i z^{-(i+1)}$$

From (6) we have $\Phi_i = \begin{cases} (E^{-1}F)^i E^{-1} & for \quad i \geq 0 \\ 0 & for \quad i < 0 \end{cases}$ and μ=0. From (5) it follows that in this case x_i (and also y_i) does not depend on the future values of inputs. □

Theorem 2. The singular system (1) is anticipatory if and only if

$$\text{rank}E > \text{deg.det}[Ez-F] \tag{7}$$

where deg.det[Ez-F] denotes the degree of the polynomial det[Ez-F].

Proof. Using the Weierstrass decomposition of the regular pencil (E,F) [12] we shall show that the nilpotence index μ>1 if and only if (7) holds. If the condition (2) is satisfied then there exists nonsingular matrix $P, Q \in R^{n \times n}$ such that [12]

$$P[Ez-F]Q = \begin{bmatrix} I_{n_1}z - A_1 & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix} \tag{8}$$

where $n_1 = \text{deg.det}[Ez-F]$, $n_2 = n - n_1$. $A_1 \in R^{n_1 \times n_1}$ and $N \in R^{n_2 \times n_2}$ is the nilpotent matrix with index μ, $N^{\mu-1} \neq 0$, $N^{\mu} = 0$. The index μ is equal to maximal dimension of the Jordan block corresponding to the zero eigenvalue of the pairs (E,F) [12]. From (8) it follows that $\text{rank}\, E = n_1$ if and only if N=0 and μ=1. The condition (7) is satisfied if and only if μ>1. From (5) it follows that in this case x_i depends on u_k for $k>i$. □

CONTINUOUS-TIME SYSTEMS

Consider the continuous-time linear system

$$E\dot{x} = Ax + Bu, \qquad x(0) = x_0 \tag{9a}$$

$$y = Cx + Du \tag{9b}$$

where $\dot{x} = \frac{dx}{dt}$, $x = x(t) \in R^n$, $u = u(t) \in R^m$, $y = y(t) \in R^p$ are the state vector, input vector and output vector, respectively and $E, A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.
If $\det E \neq 0$ the system (9) is called standard and if $\det E = 0$ the system is called singular.
It is assumed that the pencil (E,A) is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C} \tag{10}$$

If the condition (10) is satisfied then

$$[Es - A]^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i s^{-(i+1)} \tag{11}$$

where μ is the nilpotence index and Φ_i is the fundamental matrix defined by [17,23,24]

$$E\Phi_i - A\Phi_{i-1} = \Phi_i E - \Phi_{i-1} A = \begin{cases} I_n & for \quad i = 0 \\ 0 & for \quad i \neq 0 \end{cases} \tag{12}$$

The solution $x(t)$ of the equation (9a) has the form [17,9]

$$x(t) = e^{\Phi_0 At}\Phi_0 Ex_0 + \int_0^t e^{\Phi_0 A(t-\tau)}\Phi_0 Bu(\tau)d\tau + \sum_{j=1}^{\mu} \Phi_{-j}\left(Bu^{(j-1)} + Ex_0\delta^{(j-1)}\right) \tag{13}$$

where $u^{(j)} = \dfrac{d^j u}{dt^j}$, $\delta^{(j)}$ denotes the derivative of the j-th order of the Dirac impulse $\delta(t)$. From (13) it follows that if $\mu>1$ then the solution $x(t)$ depends on the derivatives of $u(t)$.

Definition 2. The system (9) is called anticipatory if the state vector x and the output vector y depend on the derivatives of u.

Theorem 3. The standard system (9) is not anticipatory.

Proof. If $\det E \neq 0$ then in a similar way as for the system (1) it can be shown that

$$\Phi_i = \begin{cases} (E^{-1}A)^i E^{-1} & for \quad i \geq 0 \\ 0 & for \quad i < 0 \end{cases} \tag{14}$$

and μ=0.

In this case from (13) it follows that x does not depend on the derivatives of u. □

Theorem 4. The singular system (9) is anticipatory if and only if

$$\text{rank}E > \text{deg.det}[Es-A] \tag{15}$$

Proof. In a similar way as for the system (1) it can be shown that the condition (15) is satisfied if and only if the nilpotence index μ>1. From (13) it follows that in this case x depends on the derivatives of u. □

ELECTRICAL CIRCUITS

It is well-known [17,9] that electrical circuits are examples of singular continuous-time linear systems. The following question arises. Are the electrical circuits also examples of anticipatory systems? To answer the question let us consider an electrical circuit with n meshes consisting of resistors, inductances $L_1, L_2, \ldots, L_r$ and m voltage sources. Let $i_1, i_2, \ldots, i_n$ be the mesh currents. Using the mesh method we may write the equation (9a) in which

$x = [i_1 \quad i_2 \quad \ldots \quad i_n]^T$, $u = [e_1 \quad e_2 \quad \ldots \quad e_m]^T$ (T – denotes the transpose)

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in R^{n \times n}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in R^{n \times m}, A_1 := \begin{bmatrix} -\frac{R_{11}}{L_1} & \frac{R_{12}}{L_1} & \cdots & \frac{R_{1r}}{L_1} \\ \frac{R_{21}}{L_2} & -\frac{R_{22}}{L_2} & \cdots & \frac{R_{2r}}{L_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{R_{r1}}{L_r} & \frac{R_{r2}}{L_r} & \cdots & -\frac{R_{rr}}{L_r} \end{bmatrix},$$

$$A_2 := \begin{bmatrix} \frac{R_{1,r+1}}{L_1} & \cdots & \frac{R_{1n}}{L_1} \\ \frac{R_{2,r+1}}{L_2} & \cdots & \frac{R_{2n}}{L_2} \\ \vdots & \ddots & \vdots \\ \frac{R_{r,r+1}}{L_r} & \cdots & \frac{R_{rn}}{L_r} \end{bmatrix}, A_3 := \begin{bmatrix} R_{r+1,1} & \cdots & R_{r+1,r} \\ R_{r+2,1} & \cdots & R_{r+2,r} \\ \vdots & \ddots & \vdots \\ R_{n1} & \cdots & R_{nr} \end{bmatrix}, A_4 := \begin{bmatrix} -R_{r+1,r+1} & R_{r+1,r+2} & \cdots & R_{r+1,n} \\ R_{r+2,r+1} & -R_{r+2,r+2} & \cdots & R_{r+2,n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n,r+1} & R_{n,r+2} & \cdots & -R_{nn} \end{bmatrix} \tag{16}$$

and the resistance R_{ij} satisfy the conditions

$$R_{ij} = R_{ji} \begin{cases} > 0 & for \quad i = j \\ \geq 0 & for \quad i \neq j \end{cases} \text{ and } R_{ii} \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} R_{ij}, \; i=1,\ldots,n \tag{17}$$

It can be easily shown [17,9] that the matrix A_4 is nonsingular, $\det A_4 \neq 0$. We shall show that for this electrical circuit the matrix N in Weierstrass decomposition of the pencil (E,A) is zero matrix and the nilpotence index μ=1. In this case we choose

$$P=\begin{bmatrix} I_r & -A_2A_4^{-1} \\ 0 & I_{n-r} \end{bmatrix},\ Q=\begin{bmatrix} I_r & 0 \\ -A_4^{-1}A_3 & A_4^{-1} \end{bmatrix}$$

and we obtain

$$P[Es-A]Q=\begin{bmatrix} I_r & -A_2A_4^{-1} \\ 0 & I_{n-r} \end{bmatrix}^{-1}\begin{bmatrix} I_rs-A_1 & -A_2 \\ -A_3 & -A_4 \end{bmatrix}\times\begin{bmatrix} I_r & 0 \\ -A_4^{-1}A_3 & A_4^{-1} \end{bmatrix}=\begin{bmatrix} I_rs-\overline{A}_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix} \quad (18)$$

where $\overline{A}_1 = A_1 - A_2A_4^{-1}A_3$. From (18) it follows that $N=0$, $n_1=r$, $n_2=n-r$ and rank$E$=deg.det[Es-A]. Dual results can be obtained for the electrical circuits consisting of resistors, capacitors and

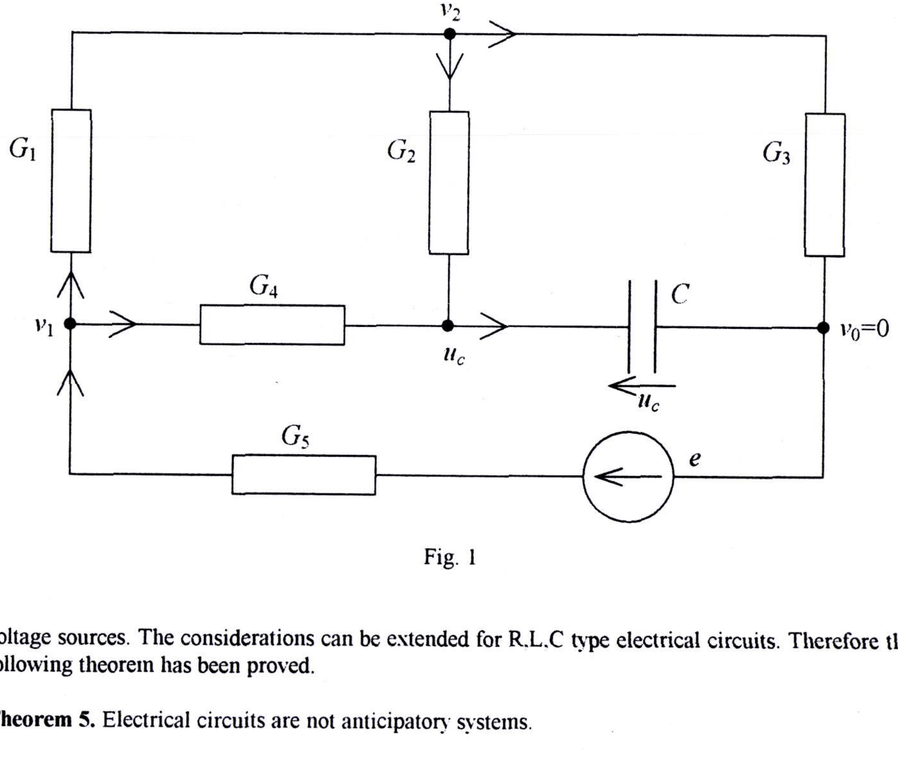

Fig. 1

voltage sources. The considerations can be extended for R,L,C type electrical circuits. Therefore the following theorem has been proved.

Theorem 5. Electrical circuits are not anticipatory systems.

Example 1. Consider the electrical circuit shown in Fig. 1 with given conductances G_k, k=1,...,5, capacity C and source voltage e.
Using the Kirchoff's node low for this circuit we may write the equations

$$
\begin{aligned}
C\dot{u}_C &= G_4(v_1 - u_C) + G_2(v_2 - u_C) \\
G_5(e - v_1) &= G_1(v_1 - v_2) + G_4(v_1 - u_C) \\
G_1(v_1 - v_2) &= G_2(v_2 - u_C) + G_3 v_2
\end{aligned} \tag{19}
$$

Choosing as the state variables $x_1 = u_C$, $x_2 = v_1$, $x_3 = v_2$ we obtain the equation (9a) in which

$$
E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ x = \begin{bmatrix} u_C \\ v_1 \\ v_2 \end{bmatrix},\ A = \begin{bmatrix} -\frac{G_{11}}{C} & \frac{G_{12}}{C} & \frac{G_{13}}{C} \\ G_{21} & -G_{22} & G_{23} \\ G_{31} & G_{32} & -G_{33} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ G_5 \\ 0 \end{bmatrix},\ u=e \tag{20}
$$

$G_{11} = G_2 + G_4$, $G_{12} = G_{21} = G_4$, $G_{13} = G_{31} = G_2$, $G_{22} = G_1 + G_4 + G_5$, $G_{23} = G_{32} = G_1$, $G_{33} = G_1 + G_2 + G_3$.

In this case $r=1$, $n=3$, $m=1$ and

$$
A_1 = \left[-\tfrac{G_{11}}{C}\right],\quad A_2 = \left[\tfrac{G_{12}}{C} \quad \tfrac{G_{13}}{C}\right],\quad A_3 = \begin{bmatrix} G_{21} \\ G_{31} \end{bmatrix},\quad A_4 = \begin{bmatrix} -G_{22} & G_{23} \\ G_{32} & -G_{33} \end{bmatrix},\quad B_1 = [0].
$$

$$
B_2 = \begin{bmatrix} G_5 \\ 0 \end{bmatrix}
$$

The matrix A_4 is nonsingular since $G_{22}G_{33} > G_{23}G_{32}$, and the inverse matrix

$$
A_4^{-1} = \frac{1}{G_{23}G_{32} - G_{22}G_{33}} \begin{bmatrix} G_{33} & G_{23} \\ G_{32} & G_{22} \end{bmatrix}
$$

has negative entries.
From (20) it follows that

$$
\dot{u}_C = \overline{A}_1 u_C + \overline{B}_1 e \tag{21}
$$

where

$$
\overline{A}_1 = A_1 - A_2 A_4^{-1} A_3 = \left[-\frac{G_{11}}{C} - \frac{1}{C[G_{23}G_{32} - G_{22}G_{33}]} \left((G_{12}G_{33} + G_{13}G_{32})G_{21} + (G_{12}G_{23} + G_{13}G_{22})G_{31} \right) \right]
$$

$$
\overline{B}_1 = B_1 - A_2 A_4^{-1} B_2 = \left[\frac{G_5(G_{12}G_{33} + G_{13}G_{32})}{G_{22}G_{33} - G_{23}G_{32}} \right]
$$

The solution $u_C(t)$ of (21) has the form

$$u_C(t) = e^{\overline{A}_1 t} u_C(0) + \int_0^t e^{\overline{A}_1(t-\tau)} \overline{B}_1 e(\tau) d\tau$$

and it does not depend on the derivatives of e.

Knowing $u_C(t)$ we may find $v_1(t)$ and $v_2(t)$ from the equation

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = -A_4^{-1} A_3 u_C(t) - A_4^{-1} B_2 e(t)$$

Note that $v_1(t)$ and $v_2(t)$ also do not depend on the derivatives of e. Therefore, the electrical circuit is a singular system but it is not an anticipatory system.

INFLUENCE OF THE VALUE OF THE STEP DISCRETISATION ON THE ANTICIPATION

Substituting the derivative $\dot{x}$ in (9a) by $\dfrac{x_{i+1} - x_i}{\Delta t}$ we obtain the equation (1a) in which

$$F = E + \Delta t A, \; G = \Delta t B \tag{22}$$

Let the continuous-time system (9) be anticipatory. The following question arises. Can the discrete-time system (1) obtained by the discretization from the continuous-time system (9) be anticipatory system for any value of the discretization step Δt?

We shall show that the discrete-time system (1) is anticipatory for any $\Delta t > 0$ if and only if the continuous-time system (9) is anticipatory. The proof of this hypothesis is based on the following

Lemma. Let s_i, i=1,2,...,n be the eigenvalues of the pair $(E, A) \in R^{n \times n}$, i.e. the roots of the equation $\det[Es - A] = 0$. Then $z_i = 1 + \Delta t s_i$, i=1,2,...,n are the eigenvalues of the pair (E,F).

Proof. Using (22) we may write

$$\det[Ez - F] = \det[Ez - E - \Delta t A] = \det[E(z-1) - \Delta t A] \det\left[\Delta t\left(E\frac{z-1}{\Delta t} - A\right)\right] = (\Delta t)^n \det\left[E\frac{z-1}{\Delta t} - A\right] = \\ = (\Delta t)^n \det[Es - A] \tag{23}$$

From (23) it follows that $s_i = \dfrac{z_i - 1}{\Delta t}$ or $z_i = 1 + \Delta t s_i$ for i=1,2,...,n. □

Theorem 6. The discrete-time system (1) obtained by discretization from the continuous-time system (9) is anticipatory for any value of discretization step $\Delta t > 0$ if and only if the continuous-time system is anticipatory.

Proof. By theorems 2 and 4 it is enough to shown that

$$\deg.\det[Ez - F] = \deg.\det[Es - A] \tag{24}$$

From Lemma it follows that the number of eigenvalues of both pairs (E,A) and (E,F) is the same and the equality (24) holds. □

2D LINEAR SYSTEMS

Singular Fornasini-Marchesini model.

Consider the 2D linear system described by the equations

$$Ex_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + Bu_{ij} \qquad i,j \in Z_+ \tag{25a}$$

$$y_{ij} = Cx_{ij} + Du_{ij} \tag{25b}$$

where $x_{ij} \in R^n$ is the state vector at the point (i,j), $u_{ij} \in R^m$ is the input vector, $y_{ij} \in R^p$ is the output vector and $E \in R^{n \times n}, A_k \in R^{n \times n}$, $k = 0,1,2$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$.
The system (model) (25) is called the singular first Fornasini-Marchesini model if $\det E = 0$ and standard if $\det E \neq 0$. It is assumed that

$$\det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} d_{ij} z_1^i z_2^j \tag{26}$$

and $d_{n_1n_2} \neq 0$ for some positive integers n_1, n_2 $(n_1 \leq n,\ n_2 \leq n)$.
If the assumption is satisfied then [12]

$$[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2]^{-1} = \sum_{i=-\mu_1}^{\infty}\sum_{j=-\mu_2}^{\infty} T_{ij} z_1^{-(i+1)} z_2^{-(j+1)} \tag{27}$$

where the pair (μ_1, μ_2) is the nilpotence index of (25) and the transition matrices T_{ij} are defined by

$$ET_{ij} = \begin{cases} A_0T_{-1,-1} + A_1T_{0,-1} + A_2T_{-1,0} + I_n & for\ i = j = 0 \\ A_0T_{i-1,j-1} + A_1T_{i,j-1} + A_2T_{i-1,j} & for\ i \neq 0 \\ & or/and\ \ j \neq 0 \end{cases} \tag{28}$$

and $T_{ij} = 0$ for $i < -\mu_1$ or/and $j < -\mu_2$.

The solution x_{ij} of (25) with the boundary conditions

$$x_{i0} \text{ for } i \in Z_+ \text{ and } x_{0j} \text{ for } j \in Z_+ \tag{29}$$

and input sequence u_{ij} is given by [12]

$$x_{ij} = \sum_{k=1}^{i+\mu_1}\sum_{l=1}^{j+\mu_2} T_{i-k-1,j-l-1}Bu_{kl} + \sum_{k=1}^{i+\mu_1}\left(T_{i-k-1,j-l-1}[A_0, B]\begin{bmatrix} x_{k0} \\ u_{k0}\end{bmatrix} + T_{i-k,j-1}A_1 x_{k0} \right) + \tag{30}$$
$$+ \sum_{l=1}^{j+\mu_2}\left(T_{i-1,j-l-1}[A_0, B]\begin{bmatrix} x_{0l} \\ u_{0l}\end{bmatrix} + T_{i-1,j-l}A_2 x_{0l} \right) + T_{i-1,j-1}[A_0, B]\begin{bmatrix} x_{00} \\ u_{00}\end{bmatrix} \text{ for } i,j \geq 0$$

From (30) it follows that if $\mu_1 \geq 1$ *or/and* $\mu_2 \geq 1$ then the solution x_{ij} depends on the future values of inputs u_{kl} for $k > i, l > j$.

Definition 3. The system (model) (25) is called anticipatory if the state vector x_{ij} and output vector y_{ij} depend on the future values of inputs. u_{kl} for $k > i, l > j$.

Theorem 7. The singular 2D system (25) is anticipatory if and only if

$$\deg_{z_i} adj[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] \geq \deg_{z_i} \det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] \tag{31}$$

for at least one of $i = 1,2$

where $\deg_{z_i}$ denotes the degree of the polynomial matrix (or polynomial) with respect to z_i. $i = 1,2$ and $adj\, A(z_1, z_2)$ stands for the adjoin matrix of $A(z_1, z_2)$.

Proof. Let $\deg_{z_i} adj[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] = q_i$ and $\deg_{z_i} \det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] = n_i$ for $i = 1,2$.

Then using the procedure of the division of polynomials from the formula

$$[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2]^{-1} = \frac{adj[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2]}{\det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2]} \tag{32}$$

we obtain (27) where $\mu_i = q_i - n_i + 1$ for $i = 1,2$. Therefore, if (31) holds then $\mu_i \geq 1$ and from (30) it follows that the solution x_{ij} depends on the future values of inputs. In this case from (25b) it follows that y_{ij} depends also on the future values of inputs. By definition 3 the system (25) is anticipatory if and only if (31) holds. □

Remark: Let $rank\, E = rank[E, A_1, A_2]$. Then (31) is satisfied only if

$rank\, E \geq \deg_{z_i} \det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2]$ for $i = 1,2$.

Example 2. Consider the model (25a) with

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad (33)$$

In this case $n = 3, m = 1$

$$\det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] = \begin{vmatrix} -1 & z_1z_2 - z_2 & 0 \\ -z_2 & -z_1 & z_1z_2 \\ 0 & -1 & 0 \end{vmatrix} = -z_1z_2$$

and $d_{11} = -1, n_1 = n_2 = 1$.
Using (32) we obtain

$$[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2]^{-1} = \frac{1}{z_1z_2} \begin{bmatrix} -z_1z_2 & 0 & z_1z_2^2 - z_1^2z_2^2 \\ 0 & 0 & -z_1z_2 \\ -z_2 & 1 & -z_1z_2^2 + z_2^2 - z_1 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 & 0 & z_2 - z_1z_2 \\ 0 & 0 & -1 \\ -z_1^{-1} & z_1^{-1}z_2^{-1} & -z_2 + z_1^{-1}z_2 - z_2^{-1} \end{bmatrix} = \begin{matrix} T_{-2,-2}z_1z_2 + T_{-1,-2}z_2 + T_{0,-2}z_1^{-1}z_2 + T_{-1,-1} + \\ + T_{-1,0}z_2^{-1} + T_{0-1}z_1^{-1} + T_{00}z_1^{-1}z_2^{-1} \end{matrix}$$

where

$$T_{-2,-2} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, T_{-1,-2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, T_{0,-2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T_{-1,-1} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$T_{-1,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, T_{0,-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, T_{00} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Hence $q_1 = q_2 = 2$ and $\mu_i = q_i - n_i + 1 = 2$ for $i = 1,2$.
Using (30) we obtain

$$x_{ij} = \sum_{k=1}^{i+2}\sum_{l=1}^{j+2} T_{i-k-1,j-l-1}Bu_{kl} + \sum_{k=1}^{i+2}\left(T_{i-k-1,j-1}[A_0, B]\begin{bmatrix} x_{k0} \\ u_{k0} \end{bmatrix} + T_{i-k,j-1}A_1x_{k0} \right) +$$

$$+\sum_{l=1}^{j+2}\left(T_{i-1,j-l-1}[A_0,B]\begin{bmatrix}x_{0l}\\u_{0l}\end{bmatrix}+T_{i-1,j-l}A_2x_{0l}\right)+T_{i-1,j-1}[A_0,B]\begin{bmatrix}x_{00}\\u_{00}\end{bmatrix} \tag{34}$$

and

$$x_{1,1}=\begin{bmatrix}-u_{22}+u_{12}-u_{11}\\-u_{11}\\-u_{12}+u_{02}-u_{01}-u_{10}\end{bmatrix}$$

$$x_{1,j}=\begin{bmatrix}-u_{2,j+1}+u_{1,j+1}-u_{1,j}+u_{0,j+1}\\-u_{1,j}\\-u_{1,j+1}+u_{0,j+1}-u_{0,j}-u_{1,j-1}-u_{0,j}\end{bmatrix} \text{ for } j>1$$

$$x_{i,1}=\begin{bmatrix}-u_{i+1,2}+u_{i,2}-u_{i,1}\\-u_{i,1}\\-u_{i,2}+u_{i-1,2}-u_{i-1,1}-u_{i,0}-u_{i+1,0}+x_{i,0}-x_{i+1,0}\end{bmatrix} \text{ for } i>1$$

$$x_{i,j}=\begin{bmatrix}-u_{i+1,j+1}+u_{i,j+1}-u_{i,j}\\-u_{i,j}\\-u_{i,j+1}+u_{i-1,j+1}-u_{i-1,j}-u_{i,j-1}\end{bmatrix} \text{ for } i>1 \text{ and } j>1$$

Singular Roesser model

Consider the 2D linear system described by the equations

$$E\begin{bmatrix}x^h_{i+1,j}\\x^v_{i,j+1}\end{bmatrix}=A\begin{bmatrix}x^h_{ij}\\x^v_{ij}\end{bmatrix}+Bu_{ij} \tag{35a}$$

$$y_{ij}=C\begin{bmatrix}x^h_{ij}\\x^v_{ij}\end{bmatrix}+Du_{ij} \tag{35b}$$

where $x^h_{ij}\in R^{n_1}$ is the horizontal state vector, $x^v_{ij}\in R^{n_2}$ is the vertical state vector, $u_{ij}\in R^m$ is the input vector, $y_{ij}\in R^p$ is the output vector and

$$E=\begin{bmatrix}E_{11}&E_{12}\\E_{21}&E_{22}\end{bmatrix},\ E_{11}\in R^{n_1\times n_1},\ E_{22}\in R^{n_2\times n_2},\ A=\begin{bmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{bmatrix},\ A_{11}\in R^{n_1\times n_1},\ A_{22}\in R^{n_2\times n_2},\ B=\begin{bmatrix}B_1\\B_2\end{bmatrix},\ B_1\in R^{n_1\times m},\ B_2\in R^{n_2\times m},\ C\in R^{p\times(n_1+n_2)},\ D\in R^{p\times m}$$

The system (model) (35) is called the singular Roesser model if $\det E = 0$ and standard Roesser model if $\det E \neq 0$.

It is assumed that

$$d(z_1, z_2) = \det\begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix} = \sum_{i=0}^{r_1}\sum_{j=0}^{r_2} d_{ij} z_1^i z_2^j \tag{36}$$

and $d_{r_1 r_2} \neq 0$ for some positive integers $r_1, r_2 (r_1 \leq n_1, r_2 \leq n_2)$.

If the assumption is satisfied then [12]

$$\begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix}^{-1} = \sum_{i=-\mu_1}^{\infty}\sum_{j=-\mu_2}^{\infty} T_{ij} z_1^{-(i+1)} z_2^{-(j+1)} \tag{37}$$

where the pair (μ_1, μ_2) is the nilpotence index of (35) and the transition matrices T_{ij} are defined by [12]

$$[E_1, 0]T_{i,j-1} + [0, E_2]T_{i-1,j} - AT_{i-1,j-1} = \begin{cases} I_n & for \;\; i = j = 0 \\ 0 & for \;\; i \neq 0 \;\; or/and \;\; j \neq 0 \end{cases} \tag{38}$$

and $T_{ij} = 0$ for $i < -\mu_1$ or/and $j < -\mu_2$

$$E_1 = \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix}, E_2 = \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix}$$

The solution x_{ij} of (35a) with the boundary conditions

$$x_{0j}^h, j \in Z_+, \; x_{i0}^v, i \in Z_+ \tag{39}$$

is given by [12]

$$x_{ij} = \sum_{k=0}^{i+\mu_1-1}\sum_{l=0}^{j+\mu_2-1} T_{i-k-1,j-l-1}Bu_{kl} + \sum_{l=0}^{j+\mu_2-1} T_{i,j-l-1}E_1 x_{0l}^h + \sum_{k=0}^{i+\mu_1-1} T_{i-k-1,j}E_2 x_{k0}^v \tag{40}$$

From (40) it follows that if $\mu_1 > 1$ or/and $\mu_2 > 1$ then the solution x_{ij} depends on the future values of inputs u_{kl} for $k > i, l > j$.

Definition 4. The system (model) (35) is called anticipatory if the state vector x_{ij} and output vector y_{ij} depend on the future values of inputs, u_{kl} for $k > i, l > j$.

Theorem 8. The singular system (35) is anticipatory if and only if

$$rank\, E_i > \deg_{z_i} d(z_1, z_2) \quad \text{for} \quad i = 1,2 \tag{41}$$

where $d(z_1, z_2)$ is defined by (36).

Proof. It is easy to show that

$$\deg_{z_i} adj \begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix} = rank\, E_i \qquad \text{for } i = 1,2 \tag{42}$$

Using the well-known procedure of the division of polynomials from the formula

$$\begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix}^{-1} = \frac{1}{d(z_1, z_2)} adj \begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix} \tag{43}$$

we obtain $\mu_i = rank\, E_i - r_i + 1$ for $i = 1,2$. Therefore, if (41) holds then $\mu_i > 1$ and from (40) it follows that the solution x_{ij} depends on the future values of inputs. □

Example 3. Consider the model (35a) with

$$E = \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \; A = \left[\begin{array}{cc:cc} 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ \hdashline -1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right], B = \left[\begin{array}{c} 1 \\ 0 \\ \hdashline -1 \\ 1 \end{array}\right] \tag{44}$$

In this case $n_1 = n_2 = 2$, $m = 1$

$$d(z_1, z_2) = \det \begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix} = \begin{vmatrix} z_1 & 1 & 0 & 1 \\ 0 & z_1 - 1 & 0 & -1 \\ 1 & 2 & z_2 & -1 \\ 0 & -1 & 0 & 0 \end{vmatrix} = -z_1 z_2$$

and $r_1 = r_2 = 1$.

Using (43) we obtain

$$\begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} z_1 & 1 & 0 & 1 \\ 0 & z_1 - 1 & 0 & -1 \\ 1 & 2 & z_2 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}^{-1} =$$

$$= \frac{1}{-z_1 z_2} \begin{bmatrix} z_2 & z_2 & 0 & z_1 z_2 \\ 0 & 0 & 0 & -z_1 z_2 \\ -1 & -z_1 - 1 & z_1 & -z_1^2 + 2z_1 \\ 0 & -z_1 z_2 & 0 & -z_1^2 z_2 + z_1 z_2 \end{bmatrix} = \begin{bmatrix} z_1^{-1} & z_1^{-1} & 0 & 1 \\ 0 & 0 & 0 & -1 \\ -z_1^{-1}z_2^{-1} & -z_2^{-1} - z_1^{-1}z_2^{-1} & z_2^{-1} & -z_1 z_2^{-1} + 2z_2^{-1} \\ 0 & -1 & 0 & -z_1 + 1 \end{bmatrix} =$$

$$= T_{-2-1}z_1 + T_{-20}z_1 z_2^{-1} + T_{-1-1} + T_{-10}z_2^{-1} + T_{-01}z_1^{-1} + T_{00}z_1^{-1}z_2^{-1}$$

where

$$T_{-2-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, T_{-20} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_{-1-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

$$T_{-1,0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_{0-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, T_{00} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence $\mu_1 = rank\, E_1 - r_1 + 1 = 2$ and $\mu_2 = rank\, E_2 - r_2 + 1 = 1$

Using (40) we obtain

$$x_{ij} = \sum_{k=0}^{i+1}\sum_{l=0}^{j} T_{i-k-1,j-l-1}Bu_{kl} + \sum_{l=0}^{j} T_{i,j-l-1}E_1 x_{0l}^h + \sum_{k=0}^{i+1} T_{i-k-1,j}E_2 x_{k0}^v = \begin{bmatrix} u_{ij} + u_{i-1,j} \\ -u_{ij} \\ -u_{i+1,j-1} + u_{i,j-1} - u_{i-1,j-1} \\ -u_{i+1,j} + u_{ij} \end{bmatrix} \text{for } i,j > 0$$

CONCLUDING REMARKS

The standard linear continuous-time and discrete-time systems are not anticipatory systems (theorems 1 and 3). The singular linear continuous-time systems are anticipatory systems if and only if the condition (15) is satisfied (theorem 4) and the singular linear discrete-time systems are anticipatory systems if and only if the condition (7) is satisfied (theorem 2). It has been shown that the singular electrical circuits are not anticipatory systems (theorem 5) and that the discrete-time system obtained by discretization from continuous-time one is anticipatory for any value of the discretization step $\Delta t > 0$ if and only if the continuous-time is also anticipatory (theorem 6). Necessary and sufficient conditions for the anticipation of singular 2D discrete linear systems have been established (theorem 7 and 8). In [18] an analysis of the influence of value of the discretization step on internal and external positivities and asymptotic stability of discrete-time system obtained by discretization from continuous-time one has been presented. An open problem is an analysis of the influence of value of the discretization step on the reachability, controllability and observability of a positive discrete-time system obtained by discretization from continuous-time one [12,20]. An other open problem is also an extension of the considerations for singular 2D continuous-discrete linear systems [19].

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