Growth in Economic Systems with Delay

Marek Szydłowski
Complex Systems Research Centre, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland
email: *uoszydlo@cyf-kr.edu.pl*

Adam Krawiec
Institute of Public Affairs, Jagiellonian University
Rynek Główny 8, 31-004 Kraków, Poland
email: *uukrawie@cyf-kr.edu.pl*

Abstract We consider the role of the time delay parameter in basic models of economic growth. We study the Solow model with a time lag. We also consider the dynamical optimization in the Solow model. We demonstrate that, both for Solow's model and Cass' model with a delay, there appear cyclic fluctuations whose parameters we have calculated. These fluctuations have a constant period and appear through the mechanism of the single Hopf bifurcation.
Keywords : economic growth, delay differential equation, Hopf bifurcation

1 Introduction

Recently, there has been a renaissance of the deterministic approach to the economic fluctuations. This is undoubtedly because the so-called exogenous models are unable to explain cyclic fluctuations, whereas the latest achievements of dynamical systems theory (which model deterministic processes) hold out the possibility of describing complex chaotic behaviour. At this point the argument that the superposition of simple cyclic behaviour cannot reproduce the complexity of the cycle loses its basis, since the economic fluctuations can belong to a class of dynamically complex systems in which the internal mechanism of sensitive dependence on the initial conditions is a source of complexity which is simultaneously chaotic and deterministic. Moreover, according to the Rule-Takens scenario, the superposition of several periodic solutions leads to complex chaotic behaviour.

The situation is completely analogous to Landau's conception of turbulence, which for many years was treated as a superposition of infinite number of periodic solutions.

In this context, an obvious question enquires about a source which generates this behaviour. It is not, therefore, inappropriate to dust off M. Kalecki's idea of the "time to build" to explain the complexity of the fluctuations in the theory of the economic cycle.

International Journal of Computing Anticipatory Systems, Volume 15, 2004
Edited by D. M. Dubois, CHAOS, Liège, Belgium, ISSN 1373-5411 ISBN 2-930396-01-6

In this approach, the time to build becomes an alternative to the standard technology shock real business cycle model. In the simplest case, one assumes that the time to build is a constant parameter for various sectors of the economy.

Of course, other factors leading to complex behaviour can also exist. In this work we attempt to examine the forces leading to complexity in the simplest possible case where the time to build is included into basic models of growth theory.

We consider the problem of the influence of the delay of the time parameter on the delivery of new capital goods. This problem in context of economic growth was addressed by Zak et al. [9]. The assumptions of their model are as follows:

1. a production technology in which it takes constant periods $\tau \geq 0$ to receive and install capital before it becomes productive. 2. the output y at time t is given by $y(t) = f(k(t-\tau))$ where $f(k)$ is a standard neoclassical production function satisfying the Inada conditions. 3. production structure as in the Solow model [8] is embedded in the following way

$$\dot{k} = sf(k(t-\tau)) - \delta k(t-\tau) \tag{1}$$

where $k(t) = \phi(t)$ for $t \in [-\tau, 0]$.

Note that in this approach, Solow's equation for the change of capital per capita is the starting point for introducing the delay τ. Equation (1) can be put in the following way. The change in capital per capita is a function of the output y taken at an earlier time $(t-\tau)$; $s \in (0,1)$ is a constant rate of saving, $\delta \in [0,1]$ is a constant of depreciation.

Equation (1) is an example of an equation with a delayed parameter belonging to a wider class of functional equations for which many mathematical techniques have been developed, and which allow them to be treated analytically. For our aims it is important the existence of the generalization of Hopf's theorem [7] on the existence of periodic solutions. There are some applications of the Hopf theorem in economics, for the case of autonomous finite-dimensional systems. The generalized Hopf theorem (more properly called the Poincare-Andronov-Hopf theorem) has been applied in economics since the work of Benhabib and Nishimura [2].

Let us introduce the time delay and adopt Kalecki's ideas to Solow's equation (and other basic models of growth theory) for the change in capital

$$\dot{K}(t) = sY(t-\tau). \tag{2}$$

If we augment (2) with the equation for labour we obtain a closed dynamical system of the form

$$\frac{dL}{dt} = \dot{L}(t) = nL(t) \tag{3a}$$

$$\frac{dK}{dt} = \dot{K}(t) = sF(K(t-\tau), L(t-\tau)). \tag{3b}$$

Conventionally, we assume that $F(K, L)$ is a first-degree homogeneous production function, which means that the right-hand sides of system (3) are also first-degree homogeneous functions. It means that this kind of symmetry can be used to reduce the dimension of the system by one. It is sufficient to introduce the following projection coordinates

$$k = \frac{K}{L}, \qquad u = \frac{1}{L}. \tag{4}$$

Then the separated equations take the form

$$\dot{k}(t) = se^{-n\tau} f(k(t-\tau)) - (n+\delta)k(t) \tag{5a}$$
$$\dot{u}(t) = -nu. \tag{5b}$$

The difference in our and Zak et al.'s method of introducing the delay τ manifests itself through the presence of the factor $e^{-n\tau}$ in (5a) and the lack of a delay in the factor $(n+\delta)k(t)$ [9]. Note that even in the case of a constant population $n = 0$ equation (5a) does not coincide with (1).

It is possible to generalize the above results from the case of Solow's model to the case of basic models of growth theory classified in Jensen's monograph, where they are ordered in terms of the governing function $h(k)$. Then the equivalent of (5a) becomes the equation

$$\frac{dk}{dt} = e^{-n\tau} h(k(t-\tau)) - (n+\delta)k(t) \tag{6}$$

in which, if formally $h(k) \to sf(k)$ we obtain Solow's model.

2 Stability Analysis of Dynamical Systems with Lag

In order to study the stability of Solow's model with lag we can use the methods of local stability analysis by examining the characteristic equation of the system linearized about the critical points satisfying the condition

$$sf(k^*) = (n+\delta)k^*. \tag{7}$$

When the production function satisfies the standard Inada conditions the critical point k^* always exists. The system linearized about the critical point k^* has the form

$$\frac{dz}{dt}(t) = Az(t-\tau) - Bz(t) \tag{8}$$

where

$$A = se^{-n\tau} \left. \frac{df(k)}{dk} \right|_{k=k^*}, \qquad B = n+\delta$$

and $z(t) = k(t) - k^*$ is the deviation of the state variable of the system from the critical point k^*.

We can easily recognize Kalecki's equation in equation (8) [6]. In Solow's models with a zero delay the stable critical point k^* is for $A < 0$. In our case we have the same critical points however the analysis of their stability is more complicated.

Kalecki obtained the same equation as (8) for dynamics of his business cycle model. He put the following values of parameters $B = -1.6$ and $A = -1.72$.

Note that if we substitute $z(t) = e^{-Bt}x(t)$ into (8) we obtain a simpler equation for $x(t)$ of the type considered by Tinbergen

$$\frac{dx}{dt} = Ax(t-\tau). \tag{9}$$

Since $A < 0$ as in Tinbergen's case, we can interpret the above equation as the inverse proportionality of the rate of growth $x(t)$ to the value of x at an earlier time $t - \tau$. The explicit solution of (8) has the form of particular integrals of the type $e^{-Bt}\cos\omega t$, $e^{-Bt}\sin\omega t$, that is

$$z(t) = e^{-Bt}\left[\sum_{k=-\infty}^{k=+\infty} e^{\alpha_k t}(C_{1k}\cos\omega_k t + C_{2k}\sin\omega_k t)\right] \tag{10}$$

where $\lambda \equiv \alpha_k \pm i\omega_k$ are the solutions of the characteristic equation for (9); the summation is taken from minus infinity to plus infinity since if $\lambda = \alpha_k + i\omega_k$ is the solution of the characteristic equation then so is $\bar{\lambda} = \alpha_k - i\omega_k$.

The solutions of equation (9) can be grouped according to the relation of the parameter $A < 0$ to the delay τ. If $A < -\frac{1}{\tau e}$ then there are not any real solutions of the characteristic equation. In Kalecki's equation, this condition is always fulfilled because $-1.72 < -\frac{1}{\tau e} = -0.6$. The presence of real solutions $\lambda = r$ when $A > -\frac{1}{\tau e}$ implies that there exist additional modes of type ce^{rt} in the general solution.

In the example under consideration when all solutions are complex we can find their imaginary parts ω_k for a given delay τ as the intersection of the plots of $Y(X)$

$$Y = \omega\tau = X, \qquad Y = -A\tau\sin(X)e^{X\cot X}. \tag{11}$$

Figure 1 illustrates that there is an infinite number of solutions. The value of the k-th frequency ω_k decides the period of the k-th order cycle

$$P_k = \frac{2\pi}{\omega_k} \tag{12}$$

and determine the real parts of eigenvalues from the condition

$$\alpha_k = -\omega_k\cot(\omega_k\tau). \tag{13}$$

We obtain the characteristic equation for (8) by substituting the probe solution $e^{\lambda t}$. We then get an equation of the form

$$\lambda = Ae^{-\lambda\tau} - B. \tag{14}$$

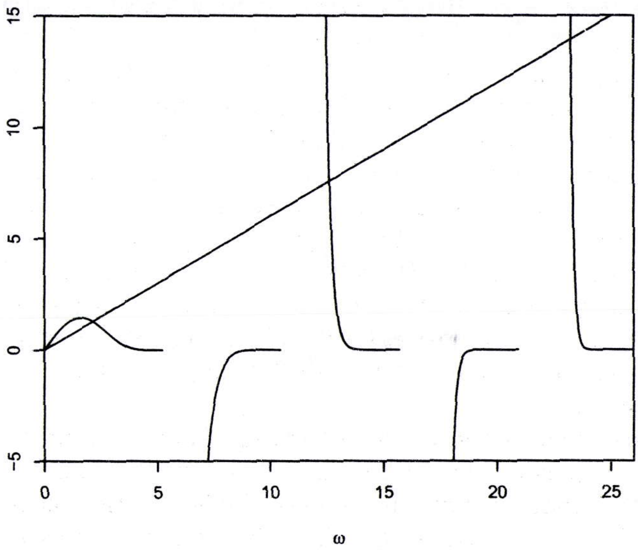

Fig. 1: Intersections of diagrams of functions (11) which show the existence of an infinite number of frequencies labelled by ω_k (for $\tau^* = 0.6$ and $A = -1.5$)

Of course we have to remember that the lag is also included in A, which is why for the discussion of $\lambda(t)$ it is more convenient to use the equation

$$\lambda = sf'(k)|_{k=k^*} e^{-(n+\lambda)\tau} - n - \delta \qquad (15)$$

where k^* is now determined from the condition $Ah(k^*) = (n+\delta)k^*$ since the critical point represents the stationary states of the system which are invariant with respect to the symmetry group (the homogeneity of time) $t \to t + \text{const}$. Of course, the Inada conditions remain sufficient for the existence of stationary points.

From the point of view of the theory of the stability of dynamical systems, two things are important. First, determining the regions of asymptotic stability in the system parameter space. Second, determining the values of τ which correspond to purely imaginary eigenvalues for which the Hopf bifurcation to a periodic orbit takes place. This occurs when the curves $\operatorname{Re}\lambda(\tau)$ cross the imaginary axis transversally, i.e., $\frac{d}{d\tau}\operatorname{Re}\lambda(\tau) > 0$ (the transversality condition).

2.1 Asymptotic Stability of the System

We can reach conclusions about the stability of a critical point by considering the signs of the real parts of eigenvalues in different regions of the system parameter space. This is a consequence of two facts. First, the roots of the characteristic equations are continuous functions of the parameters, and second, the number of characteristic roots with positive real parts can only change with the parameters of the system on passing through the imaginary axis.

If in equation (5a) we set the delay parameter τ then the parameter space (A, B) is the parameter space of the system. Then a useful way of visualizing the stability of the system is the construction of a D-partition with hypersurfaces whose points correspond to quasi-polynomials $\phi(\lambda)$ which have at least one zero on the imaginary axis ($\lambda = 0$ is not excluded). Of course, the points of each region d_k of the D-partition correspond to quasi-polynomials with same number of zeroes on the positive imaginary axis (we refer to the number of zeroes, ignoring their multiplicity), since a change in the number of zeroes with a positive real part can exist with a continuous change of coefficients only when the zero passes through the imaginary axis, that is when the point passes though the boundary of the D-partition in the coefficient space [5].

Thus, one can associate with each region d_k of the D-partition a number k of zeroes of the quasi-polynomials with positive real parts determined by the points of that region. Of course, amongst the regions of this partition one can find regions that correspond to quasi-polynomials which do not have any root with a positive real component. These regions are regions of asymptotic stability for solutions which correspond to the stationary characteristic polynomials under consideration. The use of D-partitions in studying the stability of systems with a delay parameter was presented in detail in El'sgol'ts and Norkin's monograph [5]. In the case of our characteristic polynomial of the form $\phi(\lambda) = \lambda + \tilde{A}e^{-\lambda\tau} + B = 0$ where $\tilde{A} = -Ae^{-n\tau} > 0$, $B = n + \delta$, this method allows the construction of a D-partition.

The results are summarized in Fig. 2. The D-partition in the first quarter ($\tilde{A} > 0, B > 0$) is significant from our point of view. The characteristic polynomial has a zero root when $\tilde{A} = -B$. A line in the parameter space $(\tilde{A}, B)$ of the system defined by this condition is, of course, one of the lines of the D-partition. The remaining lines can be found developing the case in which the eigenvalues of the characteristic polynomial are purely imaginary, $\lambda = i\omega$. Then, separating the real and imaginary parts of the characteristic equation $\phi(\lambda) = 0$ we obtain the parametric form of the equation of the curve, marked C on Fig. 2,

$$\tilde{A}\cos\omega\tau + B = 0, \qquad \omega - \tilde{A}\sin\omega\tau = 0. \tag{16}$$

For $B > 0$ and $\tilde{A} = 0$ the degenerate quasi-polynomial does not have any roots with positive real parts, from which it follows that region I is a region of asymptotic stability. If $\tilde{A} > \frac{1}{\tau}$, then for positive change of B and α ($dB > 0$ and $d\alpha > 0$) we find in region III two roots with positive real components. On the boundary curve $B + A\cos\omega\tau = 0$, marked on Fig. 2 as C_1, the roots are imaginary; from the equation of the curve we find that $dx\; dB < 0$ since $d\alpha = -\operatorname{Re}\frac{dB}{1-\tilde{A}\tau e^{-\lambda\tau}}$. Therefore, when crossing the boundary C_1 from region I to region III, real parts of a pair of complex conjugate roots become positive. Similarly by calculating the derivative $d\alpha = -\operatorname{Re}\sum_i^p \frac{\partial\phi}{\partial\alpha_i}d\alpha_i / \frac{\partial\phi}{\partial\lambda}$ on the boundaries of the D-partition (where

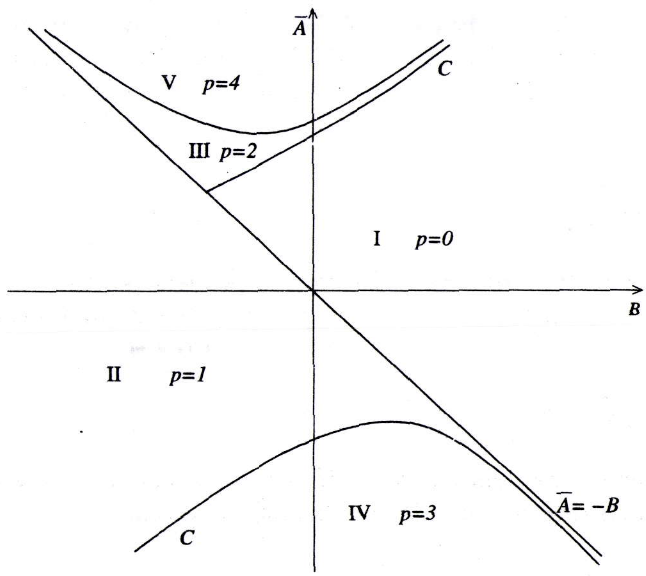

Fig. 2: The D-partition for equation (16)

$\phi(\lambda, \alpha_1, \ldots \alpha_p) = 0$ is the characteristic polynomial with parameter $\alpha_1, \ldots \alpha_\rho$), we can, changing only one parameter, obtain a complete picture of the stability of the trivial solution with fixed τ and the parameters $\tilde{A} \propto -sf'(k)|_{k=k^*}$ and $B = n + \delta$.

2.2 The Existence of Periodic Orbits in Solow's Model with Lag

In this section we prove the existence of periodic orbits in our model by using a useful method provided by the generalized PAH theorem of the Hopf bifurcation.

To this end, we fix all the parameters of the system except for the time to build τ and examine whether in such a system with varying τ there exists a $\tau = \tau_{\text{bi}}$ for which a Hopf bifurcation to a periodic orbit appears. The first step involves demonstrating the existence of one pair $(\lambda, \bar{\lambda})$ of purely imaginary eigenvalues for a certain $\tau = \tau_{\text{bi}}$ – called the bifurcation parameter of the time to build. To do this it is convenient to examine the characteristic equation (15) and to look for complex solutions in the form $\lambda = \alpha + i\omega$. Then we rewrite (15) separating real and imaginary parts to obtain the system of equations

$$B + \alpha = Ae^{-\alpha\tau} \cos \omega\tau \tag{17a}$$

$$\omega = -Ae^{-\alpha\tau} \sin \omega\tau. \tag{17b}$$

Note that system (17) has mirror symmetry with respect to the complex part, i.e., it is invariant with respect to the transformation $\omega \rightarrow -\omega$. This means that if

$\lambda = \alpha^* + i\omega^*$ is a solution then so is $\lambda = \alpha^* - i\omega^*$. Therefore, without loss of generality, we can consider the case $\omega > 0$. We obtain the parameters of the Hopf bifurcation when we substitute $\operatorname{Re}\lambda = \alpha = 0$ into (17). We then obtain the equation

$$B = \bar{A}e^{-n\tau}\cos\omega\tau \tag{18a}$$

$$\omega = -\bar{A}e^{-n\tau}\sin\omega\tau \tag{18b}$$

where $\bar{A}$ is that part of A from which the delay-dependent part has been separated, $\bar{A} = A(\tau = 0)$. Squaring both sides of equation (18) and adding we obtain the relation

$$e^{-2n\tau} = \frac{1}{\bar{A}^2}[\omega^2 + (n+\delta)^2]. \tag{19}$$

Alternatively, moving the appropriate coefficients in equation (18) containing sine and cosine functions over to one side and dividing we obtain

$$\tau = \frac{1}{\omega}\left[\arctan\left(-\frac{\omega}{n+\delta}\right) + k\pi\right] \tag{20}$$

where arctan is the branch of the inverse tan function in the range $(-\pi/2, \pi/2)$. Using formula (20) we can calculate the value of the bifurcation parameter τ when we know the frequency ω. Inserting (20) into (19) we obtain the implicit equation for ω

$$\frac{2n}{\omega}\left[\arctan\left(\frac{\omega}{n+\delta}\right) - k\pi\right] = \ln\frac{\omega^2 + (n+\delta)^2}{\bar{A}^2}, \qquad k \in \mathbb{Z}. \tag{21}$$

It is difficult to find analytic solutions for ω but we can examine the equation graphically as the intersection of the graphs of the function $\frac{\omega}{2n}\ln\frac{\omega^2+(n+\delta)^2}{A^2}$ with the function $\arctan\frac{\omega}{n+\delta}$ (see Fig. 3 and Fig. 4). The results give the following theorem

Theorem 1 *For system (5a) there exists exactly one pair of purely imaginary eigenvalues of the characteristic equation $\lambda = \pm i\omega^*$, where ω is a solution of equation (21) and the time to build parameter corresponding to a single Hopf bifurcation $\tau = \tau_{\mathrm{bi}}$ is given by (20). The period of cyclic behaviour for $\tau \simeq \tau_{bi}$ is equal to $P = \frac{2\pi}{\omega^*}$.*

Of course for a given τ the characteristic equation have an infinite number of solutions indexed by the parameter k – the order of the cycle. If τ is fixed then the first of equations (20) determines an infinite class of solutions for different k, but as $k = 0, 1, 2\ldots$ grows then so does ω_k which gives cycle periods shorter than the main period. It seems that only those cycle periods which are larger than the time to build have any economic relevance.

The equations which we have derived seem somewhat complicated to be able to understand them intuitively, but let us look more closely at the particular example

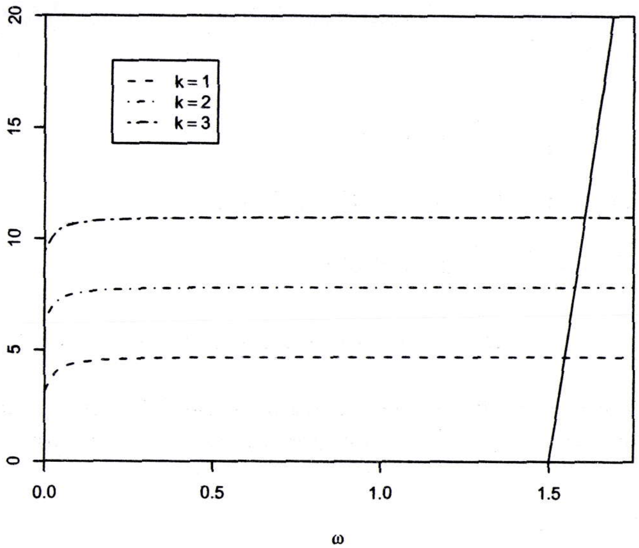

Fig. 3: Graphical solutions of equation (21) show the bifurcation value of ω (for $n = 0.01$, $\delta = 0.02$, $\tilde{A} = -1.5$)

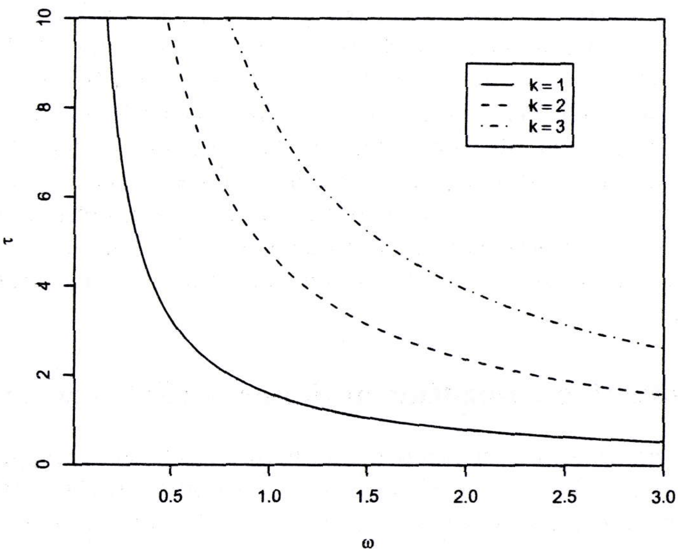

Fig. 4: The dependence of bifurcation value of τ on frequency ω, is presented for some values of k.

of constant labour when $n = 0$. Then from (19) we obtain $\omega = \sqrt{\bar{A}^2 - (n+\delta)^2}$ and the period of the cycle is

$$P \simeq \frac{2\pi}{\sqrt{(sf'(k)|_{k^*})^2 - (n+\delta)^2}}. \tag{22}$$

In a similar fashion we obtain for the Kalecki model

$$P \simeq \frac{2\pi}{\sqrt{\bar{A}^2 - \bar{B}^2}} = \frac{6.28}{0.67} \simeq 10 \text{ years}. \tag{23}$$

We can see from these simple estimates how effective a tool the bifurcation theory is.

For completeness of the proof one should check the fulfilment of the transversality condition. Differentiating the characteristic equation (15) we obtain

$$\frac{\partial \lambda}{\partial \tau} = \frac{-\lambda^2 - B\lambda}{1 + B\tau + \lambda\tau}$$

from which

$$\mathrm{Re}\left.\frac{\partial \lambda}{\partial \tau}\right|_{\alpha=0} = \omega^2 > 0.$$

Let us note that the methods we have applied can also be used for the other simple models of growth. These models have been classified in terms of the governing function $h(k)$. Our results apply directly to the linear case $h(k) = \sigma k + \epsilon$, where $\sigma, \epsilon = \text{const}$. Note that the constant term ϵ in $h(k)$ can always be eliminated by an appropriate translation of the coordinates. The formulae we have found remain valid if by $\bar{A}$ we understand $h'(k)|_{k=k^*}$, where if necessary we have earlier performed the transformation $k \to \bar{k} + \delta$ to eliminate the constant term present in $h(k)$.

In summary, we can see that the introduction of the time delay in the spirit of Kalecki's ideas generates cyclic fluctuations for a wide range of basic models of growth theory.

3 Dynamical Optimization in Solow's Model with Lag

In this section we will consider dynamical optimization of the system under discussion, into which we can easily introduce the growth of knowledge $A(t)$ increasing with a constant rate g, i.e., $\dot{A} = gA(t)$. Then we obtain an equation of the form

$$\frac{dk(t)}{dt} = se^{-(n+g)\tau} f(k(t-\tau)) - (n+g+\delta)k(t). \tag{24}$$

where $k(t) = \phi(t)$ for $t \in [-\tau, 0]$.

We base optimization on the classic work of Cass [3], where this procedure was carried out for zero delay and based on existing generalizations of Pontryagin's maximum principle for the case of dynamical systems with delay.

The fundamental problem which we wish to solve rests on answering the question whether the delay also induces cyclic behaviour in models of optimal growth and under what conditions? This question was posed by Asea and Zak and the answer was in the affirmative [1]. Now we must to answer this question in our case. It therefore seems appropriate to obtain a solution to the question as stated in the current approach. To this end we assume that consumers saving depends on their income from work, their wealth and the interest rate. The preferences of representative individuals are given by the continuous, strictly increasing and convex utility function $u(c(t))$ (which fulfils the standard Inada criteria) and the subjective discount rate $\rho > 0$. The problem of planning with an infinite horizon for this economy is defined by the extremals of the functional

$$\max_{c(t)} \int_0^\infty u(c(t))e^{-\rho t}dt \tag{25}$$

with the constraint

$$\dot{k}(t) = se^{-(n+g)\tau} f(k(t-\tau)) - (n+g+\delta)k(t) - c(t) \tag{26}$$

where consumption per capita $c(t)\colon 0 < c(t) \le se^{-(n+g)\tau} f(k(t-\tau))$, and $\delta \in [0,1)$.

We can develop the model using standard Hamiltonian methods for the generalized maximum principle. For this problem the Hamiltonian has the form

$$\mathcal{H} = u(c)e^{rt} + \lambda(t)\left[se^{-(n+g)\tau} f(k(t-\tau)) - (n+g+\delta)k(t) - c(t)\right] \tag{27}$$

and Hamilton's equations have the form,

$$\dot{k}(t) = \frac{\partial \mathcal{H}(\lambda(t), k(t), k(t-\tau), c(t))}{\partial \lambda(t)} \tag{28a}$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}(\lambda(t), k(t), k(t-\tau), c(t))}{\partial k(t)} \tag{28b}$$

where $r = \rho - n - g$, $L_0 = 1$, $A_0 = 1$.

Of course the first of Hamilton's equations (28a) reduces to a constraint condition (26) while (28b) along with the condition $\partial\mathcal{H}/\partial c = 0$ (or $u'(c) = \lambda e^{rt}$) determines the second equation of motion

$$\dot{k}(t) = se^{-(n+g)\tau} f(k(t-\tau)) - (n+g+\delta)k(t) - c(t) \tag{29a}$$

$$\dot{c}(t) = \frac{u'(c(t))}{u''(c(t))}[\rho + \delta - se^{-(n+g)\tau} f(k(t-\tau))] \tag{29b}$$

where a dot means the differentiation with respect to time t and a prime means the differentiation with respect to consumption per capita c.

The condition $\lim_{t\to\infty}\mathcal{H} = 0$ plays the role of the transversality condition for the problem with an infinite horizon, where the terminal time T is free, and so the variance ΔT is non-zero [4]. In order to test that the steady state solutions fulfil the transversality condition we note that the conditions of equilibrium determine the critical points which are somewhat different in comparison with Cass' case, that is

$$c^* = sf(k^*)e^{-(n+g)\tau} - (n+g+\delta)k^* \tag{30a}$$

$$sf'(k^*)e^{-(n+g)\tau} = (n+g+r) \tag{30b}$$

where (k^*, c^*) is the critical point of system (29). Just as in the case of Cass' model, this is a saddle point, since the eigenvalues of the linearization matrix are real and of different sign

$$\lambda_1\lambda_2 = \det J = -\frac{u'(c^*)}{u''(c^*)}sf''(k)e^{-(n+g)\tau} < 0$$

because $f(k)$ fulfils the Inada conditions.

To find the Hopf bifurcation appearing in system (29) we, as before, examine local instability through the characteristic equation. Linearization of system (29) about the critical point satisfying (30) gives

$$\dot{c}(t) = -s\tilde{f}''(k^*)\frac{u'(c)}{u''(c)}(k(t-\tau) - k^*) \tag{31a}$$

$$\dot{k}(t) = -(c(t) - c^*) + s\tilde{f}'(k^*)(k(t-\tau) - k^*) - (\delta+\rho)(k(t) - k^*) \tag{31b}$$

where $\tilde{f}(k)$ is a function which has absorbed the factor $e^{-(n+g)\tau}$, that is $\tilde{f}(k) = e^{-(n+g)\tau}f(k)$.

To find a critical point we redefine the phase variables $c(t) \to \bar{c}(t) = c - c^*$, $k(t) \to \bar{k}(t) = k - k^*$. System (31) can be written using the new variables as a single second-order equation

$$\ddot{\bar{k}}(t) = s\tilde{f}''(k^*)\frac{u'(c)}{u''(c)}\bar{k}(t-\tau) + s\tilde{f}'(k^*)\dot{\bar{k}}(t-\tau) - (\delta+\rho)\dot{\bar{k}}(t). \tag{32}$$

Formally, we obtain the characteristic polynomial for the linearized system by substituting a probe function $\bar{k} = e^{\lambda\tau}$ into (32). We then obtain

$$\lambda^2 - Ae^{-\lambda\tau} - B\lambda e^{-\lambda\tau} + C\lambda = 0 \tag{33}$$

where the constants

$$A = \frac{u'(0)}{u''(0)}sf''(0), \qquad B = sf'(0), \qquad C = \delta + \rho \tag{34}$$

are positive because both $u(c)$ and $sf(k)$ are convex.

We are interested in complex solutions of (33). Let $\lambda = \sigma + i\omega$ be the root of (33). Then separating the real and imaginary parts we obtain the system of equations

$$-\omega^2 + \sigma^2 - (A + B\sigma)e^{-\sigma\tau}\cos\omega\tau - B\omega e^{-\sigma\tau}\sin\omega\tau + C\sigma = 0 \tag{35a}$$
$$2\sigma\omega + (A + B\sigma)e^{-\sigma\tau}\sin\omega\tau - B\omega e^{-\sigma\tau}\cos\omega\tau + C\omega = 0. \tag{35b}$$

From the form of equations (35) one can see their invariance with respect to the reflection symmetry $\omega \to -\omega$, i.e., if $\lambda = \sigma + i\omega$ is a solution then so is $\bar{\lambda} = \sigma - i\omega$. Therefore we discuss only the case $\omega > 0$. The Hopf bifurcation occurs when a pair of complex conjugate eigenvalues $(\lambda(\tau), \bar{\lambda}(\tau))$ cross the imaginary axis $\mathrm{Re}\,\lambda(\tau) = \sigma = 0$ transversally such that $d\sigma/d\tau > 0$. Then (35) gives a solution for the bifurcation values of the delay parameter.

$$\omega^2 + A\cos\omega\tau + B\omega\sin\omega\tau = 0 \tag{36a}$$
$$A\sin\omega\tau - B\omega\cos\omega\tau + C\omega = 0. \tag{36b}$$

Moving terms ω^2 and $C\omega$ to one side of equations (36), then dividing sides of the first equation by the second one and after some rearrangement we obtain

$$\tan\omega\tau = \frac{B\omega^2 + AC}{\omega(A - BC)} = \frac{\bar{B}\omega^2 + \bar{A}C}{\omega(\bar{A} - \bar{B}C)}$$

where $\bar{A}$,$\bar{B}$ are $A(\tau = 0)$, $B(\tau = 0)$, i.e., they are no longer functions of τ.

The above relations allow us to calculate $\tau = \tau_{\mathrm{bi}}$ if we know ω

$$\tau_{\mathrm{bi}} = \frac{1}{\omega}\left[\arctan\left(\frac{\bar{B}\omega^2 + \bar{A}C}{\omega(\bar{B}C - \bar{A})}\right) + j\pi\right] \tag{37}$$

where arctan refers to the inverse tangent in the interval $(-\pi/2, \pi/2)$.

Moving terms ω^2 and $C\omega$ to one side of equations (36), squaring both sides and adding we obtain

$$\omega = \sqrt{\frac{1}{2}\left[(\bar{B}^2e^{-2(n+g)\tau} - C^2) + \sqrt{(\bar{B}^2e^{-2(n+g)\tau} - C^2)^2 + 4\bar{A}^2e^{-2(n+g)\tau}}\right]}. \tag{38}$$

Substituting (37) into (38) we obtain the implicit equation for ω

$$-\frac{\omega}{2(n+g)}\ln\left(\frac{\omega^4 + C\omega^2}{\bar{B}^2\omega + \bar{A}^2}\right) = \arctan\left(\frac{\bar{B}\omega^2 + \bar{A}C}{\omega(\bar{B}C - \bar{A}}\right) + j\pi. \tag{39}$$

Equation (39) is difficult to treat analytically, but it can easily be shown graphically that for all system parameter we can always find a corresponding value of ω.

We derive simpler relations for negligible rates of growth of labour and knowledge. Formally it is enough to substitute $C = 0$ into (37). We then obtain

$$\tau_{\mathrm{bi}} = \frac{1}{\omega}\left(\arctan\frac{B\omega}{A} + j\pi\right) \tag{40}$$

where ω is given by the positive root of the equation

$$\omega^4 - B^2\omega^2 - A^2 = 0 \quad \Rightarrow \quad \omega = \sqrt{\frac{B^2 + \sqrt{B^4 + 4A^2}}{2}}. \tag{41}$$

Of course an appropriate ω_{bi} always exists and then the intersection of the plots of the functions $y(\tau) = \tan \omega\tau_{\mathrm{bi}}$ and $y = \frac{B\omega}{A} = \mathrm{const}$ shows the existence of infinitely many values of τ. Similarly, setting $\tau \simeq \tau_{\mathrm{bi}}$, we obtain from a comparison of the plots of the functions (this time of ω) an infinite number of values ω_k and their corresponding periods $P = 2\pi/\omega_k$.

It only remains to test the transversality condition. Elementary calculations give

$$\mathrm{Re}\,\frac{\partial \lambda}{\partial \tau} = \frac{\omega^2 - C^2}{\omega^2 + C^2}. \tag{42}$$

4 Discussion

We showed the important role of the delay parameter can play in basic models of growth theory. Our conclusions are in agreement with those reached by Zak, although the basis on which we introduce the build time is fundamentally different. Even in the case of zero rates of growth of knowledge and labour the results of those two approaches do not coincide. Only if we were to also set the constant of depreciation of capital to zero the obtained equations are the same in Zak's and our approaches, but that is a special case.

We demonstrated that, both for the modified Solow's model and Cass' model with a delay, there are cyclic fluctuations whose parameters we have calculated. These fluctuations have a constant period and appear through the mechanism of the single Hopf bifurcation.

In this paper, we have not discussed more complex bifurcations which can appear as a result of interactions, for example the Hopf-Hopf second order bifurcation. We also considered a single constant delay of exogenous character. This is because of our aim to examine the simplest case before moving on to the analysis of more complex models in which, apart from the delay in the build time, there appears a delay in models of the growth of knowledge and population. We expect that this will be the next step towards understanding the dynamic complexity of the cycle, since the emergence of the Hopf cycle is, in some sense, a precursor of such behaviour.

As is well known, Solow's model does not explain the differentiation of income between countries since the estimated influence of savings and population growth is much greater than the model predicts. The question is whether the introduction of τ can improve the predictive ability of the model. Note that the effects of τ in Solow's model with a delay can be interpreted in terms of a change of the effective interest rate $s \longrightarrow \bar{s} = e^{-(n+g)\tau} s$, if, of course, we are on the path of balanced growth. This means that the path of balanced growth is achievable at a lower level of capital per unit of effective work $k \equiv \frac{K}{AL}$, or that increasing savings $e^{(n+g)\tau}$ times we find ourselves on the path of balanced growth only for $\tau = 0$.

$$e^{-(n+g)\tau} sf(k(t-\tau)) - (n+g+\delta)k(t) = 0.$$

The influence of the delay τ on the product in the long term is given by

$$\frac{\partial y^*}{\partial \tau} = f'(k)\frac{\partial k^*(\tau, n, s, g, \delta)}{\partial \tau}$$

where $y^* = f(k^*)$ is the level of the product per unit of effective work on the path of balanced growth. After several straightforward rearrangements we obtain the relation

$$\frac{\tau}{y^*}\frac{\partial y^*}{\partial \tau} = -(n+g)\tau\frac{\alpha_k(k^*)}{1-\alpha_k(k^*)} = \ln\frac{n+g+\delta}{sf'(k^*)}\frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}$$

where $\alpha_k(k^*)$ is the elasticity of the product with respect to the capital when $k = k^*$. In the most countries 1/3 of the income is assigned to capital. Therefore the elasticity of the product with respect to the delay time is

$$\frac{\tau}{y^*}\frac{\partial y^*}{\partial \tau} = -\frac{1}{2}(n+g)\tau. \tag{43}$$

Substituting into (43) the typical values $n = 0.01$, $g = 0.01$ and $\delta = 0.01$ we obtain

$$\frac{\tau}{y^*}\frac{\partial y^*}{\partial \tau} = -1\%\tau.$$

Thus, for example, a 10% growth in τ produces a drop in the product in the long term of only one-thousandth τ.

In summary, the effects of significant changes of τ on the path of balanced growth are proportional to τ.

Substituting Cobb-Douglas type functions and taking the logarithm of the equation of critical points defining the path of balanced growth we obtain an equation which allows us to estimate the dependence of product on τ on the balanced growth path

$$\ln y^* = a + \frac{\alpha}{\alpha-1}(n+g+\delta) + \frac{\alpha}{1-\alpha}\ln s + \frac{(n+g)\tau}{\alpha-1}.$$

The presence of the last term is linked to τ, which implies that y will contain a damping term of the form $e^{-0.075\tau}$ (taking $g+\delta = 0.05$, $\alpha = 1/3$).

Acknowledgements

The paper was supported by KBN grant no 1 H02B 023 27.

References

[1] P.K. Asea and P. J. Zak. Time-to-build and cycles. *Journal of Economic Dynamic and Control*, 23:1155–1175, 1999.

[2] J. Benhabib and K. Nishimura. The Hopf bifurcation and the existence and stability of closed orbits in multisectorial models of optimal economic growth. *Journal of Economic Theory*, 21:431–444, 1979.

[3] D. Cass. Optimum growth in aggregative model of capital accumulation. *Review of Economic Studies*, 32:233–240, 1965.

[4] A. C. Chiang. *Elements of Dynamic Optimization.* McGraw-Hill, New York, 1992.

[5] L. E. El'sgol'ts and S. B. Norkin. *Introduction to the Theory of Differential Equations with Deviating Argument* (revised edition). Academic Press, New York, 1973.

[6] M. Kalecki. A macroeconomic theory of business cycles. *Econometrica*, 3:327–344, 1935.

[7] J. E. Marsden and M. McCracken. *The Hopf Bifurcation and its Applications.* Springer-Verlag, New York, 1976.

[8] R. M. Solow. A contribution to the theory of economic growth. *Quarterly Journal of Economics*, 70:65–94, 1956.

[9] P. J. Zak, L. Tampubolon and D. Young. Is time-to-build model empirically viable? In: Computation in Economics, Finance and Engineering: Economic Systems, ed. by S. Holly, Elsevier, 2000.