Transitions in Conceptual Time Systems

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Abstract

Transitions have been studied in physics, mathematical system theory, automata theory, Petri nets, and many other theories - and they are always introduced with reference to the states of a given system. But usually the notion of 'state' is defined in an abstract way which is not related to a general formal notion of 'time' in the actual system description. That causes many problems in applications.
By contrast, in this paper the author introduces transitions in conceptual time systems mainly as pairs of time objects. Then transitions between situations, states, time states, and phases can be induced easily.
That leads to very effective temporal representations of processes, as demonstrated by, for example, applications in an air-conditioning plant.

Keywords: Transitions, Concepts, Time, States, Situations

1 Systems and Formal Concepts

Temporal phenomena have been studied extensively in nearly all areas of science, mainly in physics [2,9,17], mathematical system theory [3,11,12,13,14,17,31,32], Petri nets [15], automata theory [1,8,10], knowledge representation [18,19], and temporal logic [4].

To study 'real' systems like machines, animals, or societies it is useful to represent them formally. These formal representations may be protocols of the observations of a 'real' system or 'laws' abstracted from observations, sometimes mixtures of both. A simple example of a protocol is a data table which collects for a set of points in time the data observed at the 'real' system. A typical example of a 'law' description of a system is Galilei's law of a falling stone described by a fixed parabola, or more generally by a family of parabolas indexed by some parameters. This family can be described also by a differential equation which often is also considered as a description of a system. Clearly, a differential equation (or even a 'system' of differential equations) should be distinguished from its set of solutions and that from a single solution. What is a 'system'? Clearly, we like to use a general notion of 'system' in our scientific meta-language. Therefore many authors start with an intuitive notion of a system which leads to several difficulties. For example, if the notion of a system is not clear, then also the notions of 'states' and 'transitions' are not clear. Another difficulty arises from the fact that many systems have in some experiential sense 'subsystems', for example 'particles' or 'parts'; and without a formal definition of a system it is not clear how they are related to the given system.

Therefore we need clear basic definitions such as the definition of a vector space or the definition of an automaton. While vector spaces are used in system theory to describe the set of 'possible' states of a 'linear' system, the notion of a 'state of a system' needs some reference to the system description and can not be defined using only the definition of a vector space. By contrast, the definition of an automaton was introduced to describe transitions between states of a system. Indeed, the states of an automaton are introduced as the elements of an axiomatically given set, and transitions are described as labeled pairs of states, such that each input in a given state yields a uniquely determined transition into another state and a certain output. For applications in real systems this definition is not satisfactory since it does not include a formal time description (which should be related to the notion of 'state' and 'transition').

Such a general system definition was introduced by the author [25,27,29,30] using Formal Concept Analysis [6]. Now, before working with precise mathematical descriptions, we study an example of an air-conditioning plant.

2 Transitions in an Air-Conditioning Plant

In this section we start with some data on temperatures in an air-conditioning plant of a chemical firm. These data are represented in the form of a typical conceptual time system. The words 'states', 'situations', 'phases', and 'transitions' are introduced here at first intuitively, leading then to the precise definitions of states, situations, phases, and transitions in conceptual time systems as defined in the next sections.

2.1 Temporal Temperature Data of an Air-Conditioning Plant

The following data represent some temperature measurements taken for an air-conditioning plant in the chemical industry. For this example we focus on the hot water temperature, the temperatures in two production rooms, called room2 and room3, and the outdoor temperature, all of them measured daily each hour; here we study just three days. Figure 1 demonstrates graphically the temporal development of these four temperatures (in degree centigrades: °C).

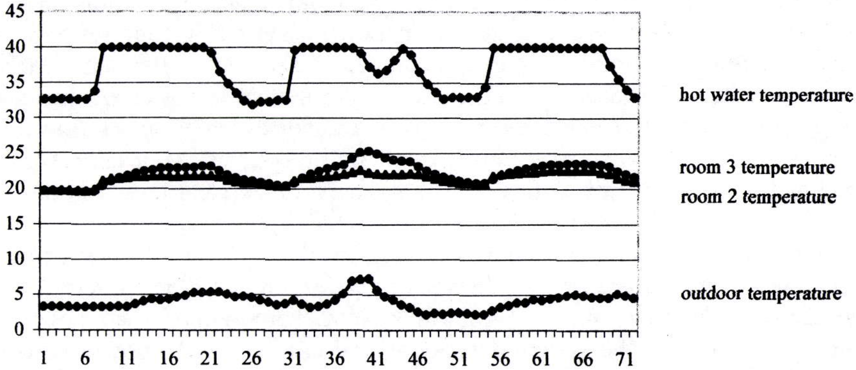

Figure 1: Temperatures (in°C) during three days (72 hours)

Using Figure 1 we now discuss some important ideas leading to the formal representation of states in conceptual time systems.

First of all, we write down some statements about the system described by Figure 1 using the words *states* and *situations* in the usual intuitive sense of a technical description:

Statement1: *We observe in Figure 1 that the hot water temperature changes within two states described by the values of about 33°C and 40°C.*
Statement2: *Between these two extremal states there are other states, for example the state where the hot water temperature has the value of about 35°C.*
Statement3: *The hot water temperature was in the state of about 35°C several times, for example during the evening of the second day and during the morning of the third day. That describes two different situations of the system.*

Secondly, we discuss these statements:
To Statement 1: Is 33°C a state of that system? Or maybe only a state of the subsystem described by the hot water temperature? Which system do we mean? The 'real' air-conditioning plant, or its graphic representation in Figure 1 or its representation as a data table, or what else? And what do we mean by a subsystem?
To Statement 2: In that statement an ordering among the states is used. Is the set of states of an arbitrary system always ordered in some natural way?
To Statement 3: If the same state 'occurs' several times one should distinguish these 'situations' by the times when they happen. That was done in Mathematical System Theory [11,14] by defining a 'phase' as a pair (t,s) of a time t and a state s. Clearly, that is more general than the often used idea that a phase is associated with an angle (which is usually defined in the much more special structure of a Euclidean vector space). Since we are interested in describing very general systems, including not only all the classical technical systems but also systems in psychology, linguistics, and many other areas we shall use the above mentioned general definition of a phase. Indeed, though there is no general definition of states in Mathematical System Theory [11,12,13,14,16], the author

has generalized the description of a 'phase' preserving the idea that a 'phase' should have a 'time part' and an 'event part' where the event part describes the states. But the two parts should be related to each other. The connection between these two parts are the 'time objects' or 'time granules' introduced by the author in [25,27].

In the example we take as time objects the 72 hours of three days, labeled from '100' to '323' where the first digit indicates the day (1 – 3) and the two last digits indicate the hour (00 – 23). The following Table 1 represents only the measurements during the first day. In this example the names of the time objects reflect the values of the time variables 'day' and 'hour', which is useful but not necessary.

Table 1: Data table of a conceptual time system

time object	day	hour	outdoor temp	room2 temp	room3 temp	hot water temp
100	1	0	3,3	19,7	19,6	32,6
101	1	1	3,3	19,7	19,6	32,6
102	1	2	3,3	19,7	19,5	32,6
103	1	3	3,3	19,7	19,5	32,6
104	1	4	3,3	19,7	19,5	32,6
105	1	5	3,3	19,6	19,5	32,6
106	1	6	3,3	19,8	19,5	33,8
107	1	7	3,3	21,1	20,3	39,9
108	1	8	3,3	21,4	21	40
109	1	9	3,4	21,4	21,5	40
110	1	10	3,3	21,5	21,8	40
111	1	11	3,7	21,6	22,1	40
112	1	12	4,1	21,6	22,4	40
113	1	13	4,4	21,7	22,6	40
114	1	14	4,2	21,7	22,8	40
115	1	15	4,4	21,7	22,9	40
116	1	16	4,7	21,7	22,9	40
117	1	17	4,9	21,8	23	40
118	1	18	5,3	21,8	23	40
119	1	19	5,3	21,8	23,1	40
120	1	20	5,4	21,8	23,1	39,2
121	1	21	5,4	21,6	22,5	36,5
122	1	22	5,1	21,2	21,9	34,8
123	1	23	4,7	20,9	21,5	33,5

It is clear that the representation in this table might lead us to use a finer granularity than Figure 1 – for example to distinguish between 19.6°C and 19.7°C which we would not do in the graphical representation of Figure 1. The choice of another granularity clearly leads to other states – in the intuitive sense that the state of 19.6°C is different from the state of 19.7°C (of the room2 temperature). This argument shows that it is necessary to introduce a notion of 'granularity' to define the notion of 'state' of a system.

Now the main idea in the definition of a conceptual time system can be described. As in Table 1 a conceptual time system is described by a data table where the set of columns ('variables', 'fields', 'many-valued attributes') is divided into a 'time part' (in this example described by 'day' and 'hour') and an 'event part' (in this example described by the temperature variables). In the formal description these two parts will be defined as two tables (many-valued contexts) on the same set of time objects.

Now the choice of the granularity is explained using the small subsystem of the time variable 'hour' and the event variable 'outdoor temperature' over all 72 hours. We assume that we are interested in the temporal trajectory of the 'outdoor temperature' only in a coarse granularity for the temperature and for the time.

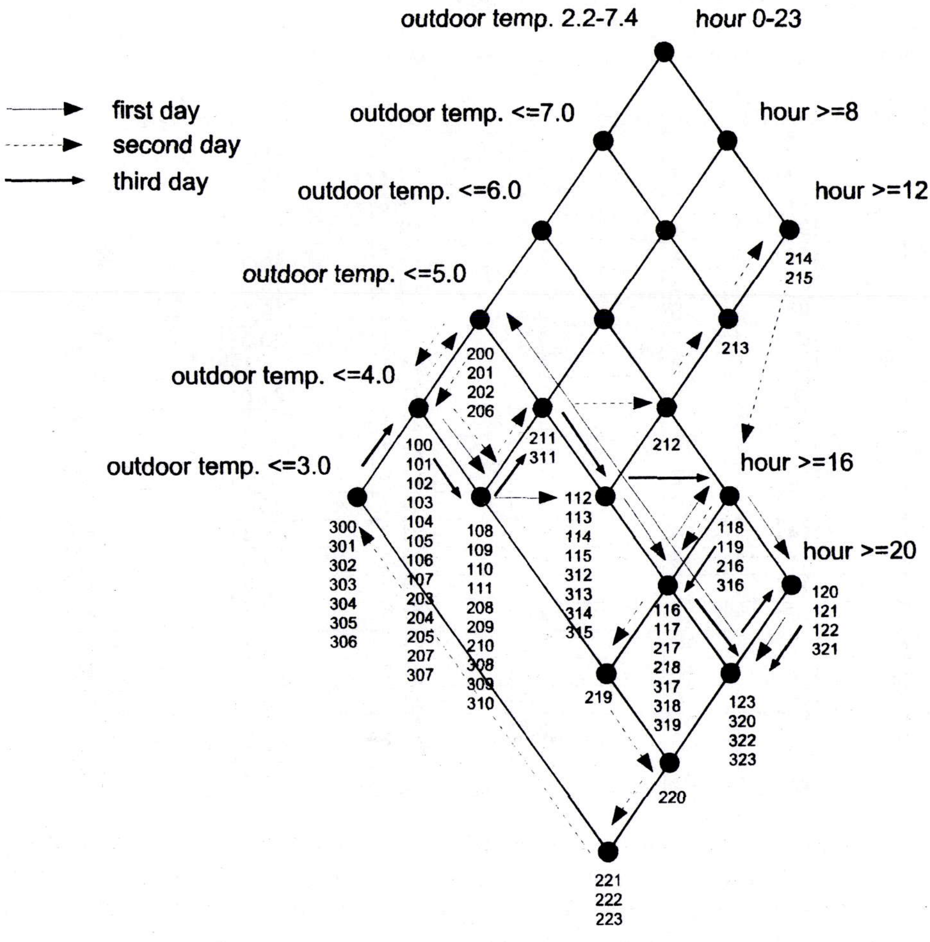

Figure 2: Transitions in a situation space for the outdoor temperature

First of all we discuss the meaning of Figure 2 without mathematical details. The main structure in Figure 2 is a grid spanned by two chains, one for the 'outdoor temperature' and one for 'hour'. Both chains result from our choice of a granularity. For example the chain for the 'outdoor temperature' is obtained from our choice of the attributes from "outdoor temp.<=3" to "outdoor temp.<=7" which partition the whole interval [2.2, 7.4] (indicated by the attribute at the top) into 6 intervals described by the chosen scale attributes. This partition of the temperature values induces a partition of the set of time objects. One of its classes described by 'outdoor temperature <=6.0' is {212, 118, 119, 216, 316, 120,121, 122, 321}.

Analogously the set of time objects is partitioned by our choice of the scale attributes for the variable 'hour'. That leads to the direct product of these two partitions which can be represented in the direct product of the two chains of length 6 and length 5. The black points in Figure 2 represent the concepts of the underlying concept lattice embedded in the direct product of the concept lattices of the two chosen scales.

At this point, instead of going into mathematical details we look at some examples. During the first hour '100' the 'outdoor temperature' has the original value of 3.3°C which is <=4.0°C, but not <=3.0°C. Since the time object '100' has the value '0' for the attribute 'hour' we say that the 'system' described by these two variables and the chosen granularity is in the 'situation' which is represented by the black point labeled '100'. The system stays in this situation until hour 7 and makes a transition to another situation labeled '108'. This transition is described by an arrow in Figure 2. Following the arrows of the first day the system reaches the situation labeled '123' and then the situation '200' of the midnight hour of the second day which is a little bit warmer than the first midnight hour. At the second day the outdoor temperature reaches its maximum during the afternoon, and in the late evening (hour >=20) it is colder than at the two other days during the same time.

Clearly Figure 2 combines the graphical representation of concept lattices with those of automata theory. But in contrast to automata theory the transitions are defined with respect to an explicitly given time description. The arrows in Figure 2 indicate transitions in the situation space. Factoring out the actual time variables (by omitting its columns) yields the usual transitions in the state space.

The air-conditioning plant example demonstrates the usefulness of the representation of time by time objects, time attributes, time values and time scales in conceptual time systems.

3 Conceptual Time Systems

Based on Formal Concept Analysis [6] the author has introduced conceptual time systems [25,27] as a system description which allows for *defining* the notions of 'states' and 'situations' in the framework of a general time description, in strong contrast to the introduction of 'states' as, for example, in automata theory.

We assume that the reader is familiar with the basic notions in Formal Concept Analysis and Conceptual Scaling Theory as described in [6,21,22]. For the other readers, using the examples, we try to give some short hints to grasp the main ideas.

What we really need here is the notion of a formal context $\mathbf{K} = (G, M, I)$ where $I \subseteq G \times M$ (describing the relation that 'an object $g \in G$ has an attribute $m \in M$'), its concept lattice $\mathbf{B}(\mathbf{K})$, its object concepts $\gamma(g)$ ($g \in G$), the notion of a scaled many-valued context as a pair $((G,M,W,I), (\mathbf{S}_m \mid m \in M))$ of a many-valued context (G, M, W, I) and a family of scales $\mathbf{S}_m := (G_m, M_m, I_m)$ which are formal contexts such that the values of m are contained in G_m. Such a scaled many-valued context represents a data table together with a suitable hierarchical granularity tool. Each scaled many-valued context yields a formal context $\mathbf{K} := (G, \{(m,n) \mid m \in M, n \in M_m\}, J)$ where $g\ J\ (m,n)$ iff $m(g)\ I_m\ n$, called the *derived context of* $((G,M,W,I), (\mathbf{S}_m \mid m \in M))$. The concept lattice of the derived context represents the knowledge about the given many-valued context in the hierarchical granularity of the chosen scales.

3.1 Definition of a Conceptual Time System

At first we recall the basic definitions for conceptual time systems and explain them using the previous example of temperature measurements.

Definition: 'conceptual time system'

Let $\mathbf{G}$ be an arbitrary set; the elements $g \in \mathbf{G}$ are called *time objects* or *time granules*. Let $\mathbf{T} := ((\mathbf{G}, \mathbf{M}, W, I_T), (\mathbf{S}_m \mid m \in \mathbf{M}))$ and $\mathbf{C} := ((\mathbf{G}, \mathbf{E}, V, I), (\mathbf{S}_e \mid e \in \mathbf{E}))$ be scaled many-valued contexts (on the same object set $\mathbf{G}$). Then the pair $(\mathbf{T}, \mathbf{C})$ is called a *conceptual time system on* $\boldsymbol{G}$. $\mathbf{T}$ is called the *time part* and $\mathbf{C}$ the *event part of* (T, C).

In the previous example with the variables 'hour' and 'outdoor temperature' the set $\mathbf{G}$ of time objects consists of the 72 hours labeled from '100' to '323'. We refer to the extension of Table 1 which contains all these 72 time objects as 'Table 1ext'. The time part $\mathbf{T}$ is described by the first and the third column of Table 1ext, the event part $\mathbf{C}$ by the first and the fourth column. The set $\mathbf{M}$ of 'many-valued attributes' (or often called 'variables') is the set {hour}, the set W is the set $\{0,1,...,23\}$, and I_T is the ternary relation $\{(g,m,w) \in G \times M \times W \mid w \text{ is the m-value of } g\}$ representing the 'time measurements', for example '12' is the hour-value of '112'.

3.2 The Derived Context of a Conceptual Time System

For a given conceptual time system $(\mathbf{T}, \mathbf{C})$ the derived context $\mathbf{K}_C$ represents the conceptual knowledge relating the time objects with the events while $\mathbf{K}_T$ describes the time objects using the chosen granularity of the scales for the time attributes. If we wish to represent that conceptual knowledge about the time part and the event part in a common formal context then the most simple construction is the apposition of $\mathbf{K}_T$ and $\mathbf{K}_C$ which can be described (in a tabular language) by affixing the columns of a table of $\mathbf{K}_C$ (say from the right) to the columns of a table of $\mathbf{K}_T$, which is always possible since both contexts have the same set $\mathbf{G}$ of objects. We denote this apposition by $\mathbf{K} := \mathbf{K}_T | \mathbf{K}_C$, for a precise definition the reader is referred to [6], p.40. The scales of the time and the event part generate a very useful granularity on the time objects since any two time objects g,h have the same object concept in **K** if and only if their values m(g) and m(h) have the same scale attributes in $\mathbf{S}_m$ (for each time or event attribute m). The choice of the scales depends heavily on the intended granularity and conceptual structure which seems to be appropriate for the purpose of the representation.

For example, the scale $\mathbf{S}_m$ for the attribute m := 'hour' is a formal context (G_m, M_m, I_m) where G_m is the set {0,1,2, ...,23} of the values of m, the set M_m is the set {0-23, >=8, >=12, >=16, >=20}, and $I_m \subseteq G_m \times M_m$ is the relation that an element of G_m is in the interval described by an attribute of M_m ; for example each element of G_m has the attribute '0-23' since each element lies in the interval [0,23], and the element '10' has also the attribute '>=8' since 10>=8. The concept lattice of this scale (G_m, M_m, I_m) is a chain of five concepts. Similarly the concept lattice of the scale for 'outdoor temperature' is a chain of 6 elements.

The direct product of these two lattices is a grid of 5×6 = 30 points. Not all of these 30 points are drawn as black points in Figure 2. Indeed, the black points represent the formal concepts of the derived context $\mathbf{K} = \mathbf{K}_T|\mathbf{K}_C$. In general the concept lattice of the derived context can be embedded (supremum preserving) in the direct product of the concept lattices of $\mathbf{K}_T$ and $\mathbf{K}_C$ ([6], p.77) Therefore it is necessary to distinguish between the lattices $\mathbf{B}(\mathbf{K}_T)$, $\mathbf{B}(\mathbf{K}_C)$, $\mathbf{B}(\mathbf{K}_T|\mathbf{K}_C)$ and $\mathbf{B}(\mathbf{K}_T) \times \mathbf{B}(\mathbf{K}_C)$.

These lattices can be visualized easily in the example in Figure 2. The line diagram in Figure 2 represents the concept lattice $\mathbf{B}(\mathbf{K}_T|\mathbf{K}_C)$. The concept lattice $\mathbf{B}(\mathbf{K}_T)$ is a chain of 5 elements which can be visualized by 'projecting' the line diagram in Figure 2 'to the upper right'. Similarly $\mathbf{B}(\mathbf{K}_C)$ is a chain of 6 elements obtained by projecting Figure 2 'to the upper left'. Therefore the direct product $\mathbf{B}(\mathbf{K}_T) \times \mathbf{B}(\mathbf{K}_C)$ is represented by a grid of 5×6 = 30 points. Clearly in these lattices the object concepts (labeled in the line diagram by at least one time object) play a prominent role. They will be studied in the next section.

3.3 Situations of a Conceptual Time System

As in colloquial speech we would like to say, for example, 'This morning I was in the same *situation* as yesterday: at 11 o'clock I saw an accident at our crossing.' The description of a situation usually contains a time and an event description. For example, in Figure 2 we would like to say 'The most frequent situation is the concept whose point is labeled by 'outdoor temp. <=4.0'. This situation actually happens 13 times, the first time at time object 100 and the last time at time object 307.' In the following definition we introduce the notion of a *situation of a conceptual time system* as an object concept of the concept lattice $\mathbf{B}(\mathbf{K}_T|\mathbf{K}_C)$.

Definition: 'situations of a conceptual time system'

Let (**T**, **C**) be a conceptual time system on **G**. For $g \in \mathbf{G}$ the object concept $\gamma(g)$ in the derived context $\mathbf{K} := \mathbf{K}_T|\mathbf{K}_C$ is called a *situation of the conceptual time system (**T**, **C**)* and we say that *the system (**T**, **C**) is at time object g in the situation s iff s =$\gamma(g)$*. The set of all situations of (**T**, **C**) is denoted by **S(T, C)**, called the *situation space*.

3.4 States and Time States

In a conceptual time system (**T**, **C**) we would like to say 'The system is at time object g in state s(g)'. That is introduced in the following definition where the object concepts of the derived context $\mathbf{K_C}$ are defined as the *states*. As to the time part we call the object concepts of the derived context $\mathbf{K_T}$ the *time states*.

Definition: 'states and time states of a conceptual time system'

Let (**T**, **C**) be a conceptual time system and $\mathbf{K_T}$ and $\mathbf{K_C}$ the derived contexts of **T** and **C**. For each time object g we define *the state s(g) of (**T**, **C**) at time object g* by $s(g) := \gamma_C(g)$:= the object concept of g in $\mathbf{K_C}$ and the *time state t(g) of (**T**, **C**) at time object g* by $t(g) := \gamma_T(g)$:= the object concept of g in $\mathbf{K_T}$. The set $\mathbf{S(C)} := \{s(g) \mid g \in \mathbf{G}\}$ is called *the state space of (**T**, **C**)*. The set $\mathbf{S(T)} := \{t(g) \mid g \in \mathbf{G}\}$ is called *the time state space of (**T**, **C**)*. We say that *the system (**T**, **C**) is at time object g in the state $s \in \mathbf{S(C)}$ iff $s = s(g)$*. We say that *the system (**T**, **C**) is at time object g in the time state $t \in \mathbf{S(T)}$ iff $t = t(g)$*.

This definition yields the 'partition meaning' of (time) states, namely, that the set **G** of time objects is partitioned by the extents of the (time) states, or, equivalently, that a system is at each time object in exactly one (time) state. Clearly, for any two time objects g and h, s(g) = s(h) iff for each event e the values e(g) and e(h) have the same scale attributes and therefore the same object concept in the scale S_e. Analogously for the time states and the situations.

The state space of the previous example can be seen in Figure 2 by 'projecting along the time axis' onto the chain for 'outdoor temperature'. In this example **S(C)** is exactly the set of all formal concepts of $\mathbf{K_C}$. Clearly, in general the state space is only a subset of the set of all formal concepts of $\mathbf{K_C}$. Analoguosly for the time states and the situations.

3.5 Phases of a Conceptual Time System

As in Mathematical System Theory [11,14] where the phases are defined as pairs (t,s) of a time point t and a state s we introduce phases of a conceptual time system as pairs (t(g), s(g)) of time states and states at the same time object.

Definition: 'phases of a conceptual time system'

Let (**T**, **C**) be a conceptual time system on **G** and $\mathbf{K_T}$ and $\mathbf{K_C}$ the derived contexts of **T** and **C**. For $g \in \mathbf{G}$ the pair (t(g), s(g)) is called *the phase of g in (**T**, **C**)*. The set $\mathbf{P(T, C)} := \{(t(g), s(g)) \mid g \in \mathbf{G}\}$ of all phases is called the *phase space of (**T**, **C**)*. The lattice $\mathbf{B(K_T)} \times \mathbf{B(K_C)}$ is called *the phase lattice of (**T**, **C**)*.

Clearly $\mathbf{P(T, C)} \subseteq \mathbf{S(T)} \times \mathbf{S(C)} \subseteq \mathbf{B(K_T)} \times \mathbf{B(K_C)}$.

3.6 Mappings between Situations, States, Time States, and Phases

The following Figure 3 demonstrates basic mappings between the lattices in which situations, states, time states and phases occur.

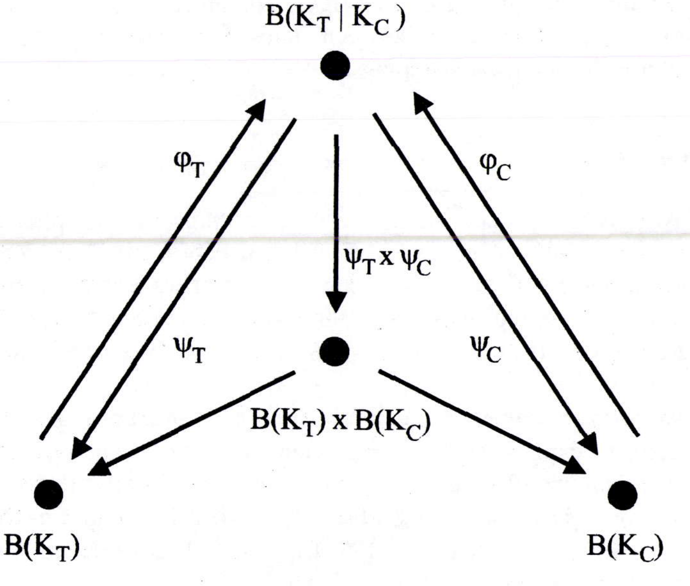

Figure 3: Basic mappings between situations, states, time states, and phases

Definition: 'basic mappings'

Let $\mathbf{K_T} := (G, M_T, I_T)$ and $\mathbf{K_C} := (G, M_C, I_C)$ be formal contexts on the same set G of objects and $\mathbf{K} := \mathbf{K_T|K_C} = (G, M, I)$ the apposition of them. Then the mapping $\psi_T : \mathbf{B(K_T|K_C)} \rightarrow \mathbf{B(K_T)}$ is defined by $\psi_T((A,B)) := ((B\cap M_T)^{\downarrow_T}, B\cap M_T)$ (for $(A,B)\in \mathbf{B(K_T|K_C)}$) where $(B\cap M_T)^{\downarrow_T} := \{g\in G \mid g\ I_T\ m \text{ for all } m \in B\cap M_T\}$ is the extent of $B\cap M_T$ in $\mathbf{K_T}$.

Analogously the mapping $\psi_C : \mathbf{B(K_T|K_C)} \rightarrow \mathbf{B(K_C)}$ is defined. The mapping $\psi : \mathbf{B(K_T|K_C)} \rightarrow \mathbf{B(K_T)} \times \mathbf{B(K_C)}$ is defined by $\psi := \psi_T \times \psi_C$, hence $\psi((A,B)) := (\psi_T((A,B)), \psi_C((A,B)))$.

The projections from $\mathbf{B(K_T)} \times \mathbf{B(K_C)}$ onto its first and second component are denoted by π_T and π_C. Hence $\pi_T((A_T, B_T), (A_C, B_C)) := (A_T, B_T)$ and $\pi_C((A_T, B_T), (A_C, B_C)) := (A_C, B_C)$.

The mapping $\varphi_T : \mathbf{B(K_T)} \rightarrow \mathbf{B(K_T|K_C)}$ is defined by $\varphi_T((A_T, B_T)) := (A_T, A_T^{\uparrow})$ where $A_T^{\uparrow} := \{m\in M \mid g\ I\ m \text{ for all } g \in A_T\}$ is the intent of A_T in $\mathbf{K_T|K_C}$.

The mapping ψ is a supremum-preserving order embedding ([6], p.77). Clearly $\psi_T = \pi_T\psi$ and $\psi_C = \pi_C\psi$. The mappings φ_T and φ_C are infimum-preserving order embeddings, called the *part embeddings*. Clearly $\varphi_T\psi_T$ is a projection from $\mathbf{B}(\mathbf{K}_T|\mathbf{K}_C)$ into itself which maps any concept (A,B) to the concept with the extent $(B\cap M_T)^{\downarrow}{}_T$.

The situations play a prominent role since ψ maps the situations from $\mathbf{B}(\mathbf{K}_T|\mathbf{K}_C)$ onto the phases in $\mathbf{B}(\mathbf{K}_T) \times \mathbf{B}(\mathbf{K}_C)$, and π_C maps the phases onto the states in $\mathbf{B}(\mathbf{K}_C)$, and π_T maps the phases onto the time states in $\mathbf{B}(\mathbf{K}_T)$. Therefore at first we study transitions between situations. Then transitions between states, between time states, and between phases can be defined using these mappings.

4 Transitions

In the following we introduce the notion of 'transitions' in conceptual time systems. Usually a transition is understood as a directed connection from one *state* to another, graphically represented by an arrow. In Figure 2 the arrows lead from one *situation* to another, for example from the situation $\gamma(100)$ to the situation $\gamma(108)$. Indeed there are three arrows from $\gamma(100)$ to the situation $\gamma(108)$. Are there three transitions or only one? *What is a transition?*

Clearly, if we wish to define a transition between two situations as a pair, say $(\gamma(g), \gamma(h))$ then we have to explain why that transition starts in $\gamma(g)$ and ends in $\gamma(h)$ and not vice versa. In the example of Figure 2 we would like to say that 'the system was in the beginning in situation $\gamma(100)$ until time object 107; when it changed to the time object 108 then a transition into the situation $\gamma(108)$ happened'. That explanation uses the usual ordering on the set of time objects where 107 is before 108.

But until now we have *not* introduced an ordering on the set $\mathbf{G}$ of time objects of an arbitrary conceptual time system. That also allowed for representing very simple time descriptions, for example just a nominal distinction that the time object 'day' is different from the time object 'night'. In the following section we shall introduce a linear ordering $\leq_G$ on the set $\mathbf{G}$ of time objects yielding the notion of a *life track* (or *life line*). A linear ordering $\leq_G$ on the set $\mathbf{G}$ of time objects is a nice and very often used structure which corresponds to the observation that our memory seems to arrange all sensual impressions in a sequence.

But sometimes we do not know or do not remember which one of two moments (time objects) was the first. Therefore, generalizing the linear ordering on the set $\mathbf{G}$ of time objects, we introduce an arbitrary binary *time relation* $\mathbf{R}$ on $\mathbf{G}$. There are also many other reasons for the introduction of such a time relation; for example that the time object 'today' is contained in the time object 'this week'. Clearly, we also wish to avoid the assumption that there is a uniquely determined linear time in the world which we all observe. Instead we want to construct a general mathematical framework which allows for investigating the formal conditions under which certain temporal notions can be introduced. In the following we introduce *transitions* which are induced by a time relation $\mathbf{R}$ on $\mathbf{G}$.

4.1 Time Relations on the Set of Time Objects

At first we extend the notion of a conceptual time system by introducing a relation on the set of time objects.

Definition: 'conceptual time systems with a time relation'
Let $(\mathbf{T}, \mathbf{C})$ be a conceptual time system on $\mathbf{G}$ and $\mathbf{R} \subseteq \mathbf{G} \times \mathbf{G}$. Then the triple $(\mathbf{T}, \mathbf{C}, \mathbf{R})$ is called a *conceptual time system (on $\mathbf{G}$) with a time relation.*
If the relational structure $(\mathbf{G}, \mathbf{R})$ is a chain (a linear ordered set) then $(\mathbf{T}, \mathbf{C}, \mathbf{R})$ is called a *chained conceptual time system*.

Clearly, in the example of Figure 2 the usual order relation $\leq_{\mathbf{G}}$ on the set $\mathbf{G}$ of the 72 time objects yields a chained conceptual time system. In a chained conceptual time system the concept lattice of $\mathbf{K_T}$ need not be a chain, it can be arbitrarily complicated.

4.2 Transitions

The main idea in the following definition is that a transition starts in a time object and ends in a time object. What happens at these time objects in the system is described by the object concepts of these time objects in the actually interesting context, for example by the situations in the context $\mathbf{K_T}|\mathbf{K_C}$.

Definition: 'transitions in conceptual time systems with a time relation'
Let $(\mathbf{T}, \mathbf{C}, \mathbf{R})$ be a *conceptual time system on $\mathbf{G}$ with a time relation.* Then any pair $(g,h) \in \mathbf{R}$ is called an ***R**-transition on **G***; g is called the *start* and h the *end of (g,h)*. The $(\mathbf{G}\times\mathbf{G})$-transitions are called *transitions on **G***. Let $\mathbf{X}$ be a set and $f\colon \mathbf{G} \to \mathbf{X}$, then the set $f = \{ (g, f(g)) \mid g \in \mathbf{G} \}$ is called the *f-life space* and f induces the mapping
$f_{\mathbf{R}} : \mathbf{R} \to f[\mathbf{R}] := \{ (f(g), f(h)) \mid (g,h) \in \mathbf{R} \}$ where $f_{\mathbf{R}}((g,h)) := (f(g), f(h))$, hence
$f_{\mathbf{R}} = \{((g,h), (f(g),f(h))) \mid (g,h) \in \mathbf{R} \}$. The element $((g,h), (f(g),f(h))) \in f_{\mathbf{R}}$ is called *the f-induced **R**-transition on **X** leading from the start point $(g, f(g))$ to the endpoint $(h, f(h))$ (either in the f-life space).*
A transition $((g,h), (f(g),f(h))) \in f_{\mathbf{R}}$ is called a *loop transitions of $f_{\mathbf{R}}$* iff $f(g) = f(h)$. A *proper (f-induced **R**-) transition* is a transition which is not a loop transition.

Now we are interested in some special choices of f.

For the object concept mapping $\gamma\colon \mathbf{G} \to \gamma\mathbf{G}$ of $\mathbf{K_T}|\mathbf{K_C}$ the γ-induced $\mathbf{R}$-transitions on the situation space $\mathbf{S(T,C)} = \gamma\mathbf{G}$ are called *the **R**-transitions on **S(T,C)***.

For the mapping $\psi_{\mathbf{T}}\gamma : \mathbf{G} \to \psi_{\mathbf{T}}\gamma\mathbf{G} = \mathbf{S(T)}$ the $\psi_{\mathbf{T}}\gamma$-induced $\mathbf{R}$-transitions on $\mathbf{S(T)}$ are called *the **R**-transitions on **S(T)***.

Analoguously *the **R**-transitions on **S(C)*** are the $\psi_{\mathbf{C}}\gamma$-induced $\mathbf{R}$-transitions on $\mathbf{S(C)}$ and the ***R**-transitions on **S(T)** $\times$ **S(C)*** are the $\psi\gamma$-induced $\mathbf{R}$-transitions on $\mathbf{S(T)} \times \mathbf{S(C)}$.

In the example of Figure 2 with the usual linear order relation $\leq_{\mathbf{G}}$ on the set $\mathbf{G}$ of the 72 time objects the arrows indicate the proper $\leq_{\mathbf{G}}$-transitions on the situation space

S(T,C). For example the transition ((107,108), (γ(107), γ(108))) is represented by the first-day-arrow from γ(107) to γ(108). The conceptual time system represented in Figure 2 'makes another transition' ((207,208), (γ(207), γ(208))) at the second day which is represented by the second-day-arrow from $\gamma(207) = \gamma(107)$ to $\gamma(208) = \gamma(108)$. That shows that it is useful to introduce transitions in the form ((g,h), (f(g),f(h))).

Following the transitions in Figure 2 in the usual linear order of the 72 time objects from 100 to 323 we get an idea of a *life track* (or *life line*) of a conceptual time system which will be introduced in the next section.

4.3 The Life Track of a Chained Conceptual Time System

In this section we introduce the *life track* (or *life line*) of a chained conceptual time system in each of its spaces of situations, time states, states, and phases.

Definition: 'life track'

Let $(\mathbf{T}, \mathbf{C}, \leq_G)$ be a chained conceptual time system on **G**. Let **X** be a set and f: $\mathbf{G} \to \mathbf{X}$; then the f-life space $f = \{ (g, f(g)) \mid g \in \mathbf{G} \}$ is called *the f-life track* (or *f-life line*) *on* ***X***. The elements $(g, f(g)) \in f$ are called *the points of the f-life track.*
For time objects g, h we say that the *point (g, f(g)) precedes ($\leq_f$) the point (h, f(h))* iff $g \leq_G h$. The ordered set $(f, \leq_f)$ is isomorphic to the ordered set $(\mathbf{G}, \leq_G)$.
For the object concept mapping γ: $\mathbf{G} \to \gamma\mathbf{G} = \mathbf{S(T,C)}$ of the formal context $\mathbf{K_T}|\mathbf{K_C}$ the γ-life track on **S(T,C)** is called *the situation life track.*
Using the mapping $\gamma_T = \psi_T\gamma$: $\mathbf{G} \to \psi_T\gamma\mathbf{G} = \mathbf{S(T)}$ we get the γ_T-life track on **S(T)** called *the time state life track.*
Using the mapping $\gamma_C = \psi_C\gamma$: $\mathbf{G} \to \psi_C\gamma\mathbf{G} = \mathbf{S(C)}$ we get the γ_C-life track on **S(C)** called *the state life track.*
Using the mapping $\psi\gamma$: $\mathbf{G} \to \psi\gamma\mathbf{G} \subseteq \mathbf{S(T)} \times \mathbf{S(C)}$ we get the $\psi\gamma$-life track on $\psi\gamma\mathbf{G}$ called *the phase life track.*

In Figure 2 the situation life track with its ordering can be reconstructed from the labeling of the situation points by the names of the time objects using $(\mathbf{G}, \leq_G)$. The same graphical representation is used if we draw a state life track of a particle moving in the plane and indicate the time points along that line.

5 Conclusion

We have introduced transitions in conceptual time systems with a binary time relation on the set of time objects. For chained time systems where the time relation is a chain we have defined the life tracks in the spaces of situations, time states, states, and phases. All these notions have been shown to be useful in practice by an example of temperature measurements in an air-conditioning plant.

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