Realizability of Anticipatory Feedbacks

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Abstract
Different methods of system design lead to closed-loop systems which property of stability is unrobust. In the linear time-invariant case it is proved that this fact can be frequently connected with anticipatory feedbacks. Given are the robustness conditions with respect to negligible small time–delay and other parametric perturbations.
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1 Introduction

Representation of control systems in a state space was predominant in the 60's literature [1] (of the last century). During the 70's, the researchers' attention returned to system analyses based on the stability notion of the input-output relations [2] and so on. At that time many books were published in which the concept of a state space is not employed. The main goal of these books and other similar works [3, 4, 5, 6, 7, 8, 9] was to extend the field of application of the classical frequency approach well checked in practice and characterized by the set of rationally defined useful aims and problems. During the 80's this approach was enriched by studying the robust problems arisen when internal and external uncertainty takes place [10]. It resulted in $\mathcal{H}_\infty$–design theory [11], although their state–space consideration was not late [12]. Here it is necessary to stress that the robustness problem has been worked out in other settings in Russian academicians' works (with respect to small non–linearity [13], internal parameters and initial data [14, 15, 16], small time–delay and discreteness in feedbacks [17]).

Behavior of closed-loop systems depends on their transfer matrices and disturbances (among them, internal perturbations and initial data; any part of closed-loop systems can posses properties that it is frequently considered as dangerous or unwanted [18], *e.g.* it can be unstable). This dependence can be continuous or not. In the first case there is preservation of some system properties, and then it is said that the system under consideration is (parametrically or structurally) robust (more exactly, the corresponding properties are robust with respect to some class of parametric perturbations). In most cases natural plant models possess this property of continuous dependence, but demand of the last is often absent in applied problem settings when idealized feedbacks are synthesized. As a result the system properties can fail for some small (small, but it is not necessary to be arbitrary) perturbation of its parameters.

We will work with Laplace transform because the stability condition of linear time–invariant differential systems is expressed as the property of its spectrum to be in the open left half–plane and in the same domain there are poles of the system transfer functions (when this transform is used). Note that we might use Fourier transform too because in the complex domain it differs from Laplace one with counterclockwise rotation on the angle $\pi/2$, *i.e.* $s = i\nu$, where s and ν are the variables of Laplace and Fourier transforms, respectively.

There are many design methods which leads to the loss of continuity (modal control, Wiener–Hopf method, the separation theorem and so on). Often, as the unique condition which closed loop systems must satisfy it is used that their characteristic equations must be Hurwitz, *i.e.* no their zero is in the closed right half–plane. As the result of such approach to design we may obtain feedback transfer functions, which are not physical realizable. Such feedbacks are called idealized. Remember that a system (more exactly, its model) is physical realizable if its transfer functions (defined with the help of Laplace transform) is bounded in the closed right half–plane and becomes zero in point of infinity (such function is called also strongly proper). As a rule models of free (from control) plants are physical realizable. The use of idealized feedbacks can lead to anticipatory closed loop systems but this possibility depends on plants where such feedback are applied. Let us clarify the problems connected with using idealized feedbacks and give the corresponding examples.

2 Anticipatory Feedbacks

Consider the simplest linear control system

$$\frac{d}{dt}y = u + f(t) \tag{1}$$

with the negative unit feedback

$$u = -y \tag{2}$$

where u is the scalar input, y is the scalar output, $f(t)$ is the disturbance.

Although the use of feedback (2) is widespread and ensures the stability of system (1), this feedback is only idealization. By opinion of A. Letov (one of the most known authors in analytical design of closed-loop systems and the first vice-president of IFAC (in the 60's of the last century)), there exists no technical device that can realize feedback (2) while for any small positive τ the following feedback

$$\tau\frac{d}{dt}u + u = -y \tag{3}$$

is realizable [19].

Feedback (3) is the simplest finite–dimensional differential approximation of the next feedback [20]

$$u = -y(t - \tau) \tag{4}$$

By this reason the idealized feedbacks are usual to be approximated by finite–dimensional differential equations (in particular, it is of very importance for systems with proportional–differential feedback [21]). However finite–dimensional approximation is not able to remove completely the control inertia, and a negligible small time–delay takes a place in the approximated feedbacks. This time–delay can be reason why the approximated closed loop systems fail to be stable.

The concept of a negligible small time–delay follows also from the correct definition of differential equations [22, 23]. Let us consider again system (1) where $y = y_o$ for $t = 0$ and the input u is given as any integrable function on the set $[0, t]$. Then it follows from equation (1) that

$$y(\vartheta) = y_o + \int_0^{\vartheta} (u + f)d\theta, \vartheta \in [0, t)$$

and

$$y_-(t) = \lim_{\vartheta \to t-0} y(\vartheta) = y_o + \int_0^t (u + f)d\theta$$

Any feedback is a functional of the output, but in time t the output is known only as the left-side limit. That is why for putting in practice feedback (2) we replace $y(t)$ with $y_-(t)$ or $y(t - \tau)$ where τ is the negligible small time–delay.

It is of importance to stress that in the case when synthesized systems are robust to negligible small time–delay and involve themselves idealized feedbacks the concept of time–delay gives the natural way of physical realization for such feedbacks: they can be realized with artificial time–delay or with the help of its approximation by finite–dimensional differential equations [20].

As an example let us consider system (1) with the proportional–differential feedback

$$u = 2\frac{d}{dt}y(t) + y(t) \tag{5}$$

By the definition of anticipatory transfer functions [24], feedback (5) is anticipatory. To realize this feedback, it needs calculating $\frac{d}{dt}y(t)$ in the instant t. But it is impossible as in result of the closed-loop system evolution we know y only on $[0, t]$ and we have no possibility to define $\lim_{\vartheta \to t+0} \frac{y(\vartheta) - y(t)}{\vartheta - t}$. Instead of feedback (5) let us use the following one

$$u(t) = (2\frac{d}{dt} + 1)y(t - \tau) \tag{6}$$

with the small time–delay $\tau > 0$. Show that the closed loop system (1) and (6) has (asymptotic) eigenvalues in the neighborhood of the straight line $\Re p = \frac{ln2}{2\tau}$ for sufficiently small τ, and therefore it is not stable. Indeed, its characteristic equation takes the form

$$\lambda[1 - 2\exp(-\lambda\tau)] - \exp(-\lambda\tau) = 0$$

and has zeros in the form $\lambda = \frac{\ln 2 + 2\pi ki + s}{\tau}$. The last follows from the characteristic equation that the complex number s satisfies the equation $f(s) = 0$ where $f(s) = 1 - \exp(-s\tau) - 0.5\tau\exp(-s\tau)/(\ln 2 + 2\pi ki + s)$, k is the integer number. The condition of using the Newton–Raphson method $|\frac{df(s)}{ds}| \leq q < 1$ is fulfilled in the neighborhood of $\Re s = 0$ for τ such that $\tau + 0.5\tau^2/[\ln 2 + (\ln 2)^2] < q$. At the same time ignoring the time–delay τ we have the "undelayed" system as being stable.

In the case where system (1) has the following feedback

$$u(t) = (0.5\frac{d}{dt} - 1)y(t) \tag{7}$$

let us use the following approximation

$$u(t) = (0.5\frac{d}{dt} - 1)y(t - \tau) \tag{8}$$

It is easy to see by using the method above that the closed loop system has asymptotic eigenvalues in the left half–plane (*i.e.* eigenvalues connected with small τ). As proved in the theory of difference differential equations this system has not other asymptotic eigenvalues. Thus the property of stability is reserved when we pass from the anticipatory feedback (7) to the causal one (8) (or its finite–dimensional approximations).

Thus we see that "undelayed" idealized feedback can be used if the resultant system with negligible small time–delay feedback is (asymptotically) stable. If this is not the case the corresponding closed loop system is not robust with respect to the negligible small time–delay, and we may call this feedback "bad" (for details – see [18]). Similarly to the stability theory here it is not of very importance what properties of parts of some system are. It is more important what the properties of the system are as a whole when an arbitrarily small time–delay presents in its feedback.

3 Singular Systems

Different methods of closed loop system design leads not only to idealized and anticipatory feedbacks but and to closed loop systems of the kind

$$(A - B\frac{d}{dt})z = f \tag{9}$$

where A and B are square scalar matrices of dimension n; $0 < rank B = k < n$; f is any disturbance. System (9) is called singular as that its characteristic equation has the degree less than the state vector dimension. Assume that system (9) is regular, *i.e.*

$$\det(A - \lambda B) \not\equiv 0 \tag{10}$$

Let us show that system (9) is solvable for by far not all initial data, and therefore there is no continuous dependence from initial data. To this end system (9) is represented in the equivalent form

$$(A_1 - E_k \frac{d}{dt}) z_1 + A_2 z_2 = f_1 \tag{11}$$

$$A_3 z_2 = f_2 \tag{12}$$

where the known square scalar matrices A_1, A_2, and A_3 have the dimensions $k \times k$, $k \times (n-k)$ and $(n-k) \times (n-k)$; $f = col(f_1, f_2)$; $z = col(z_1, z_2)$; E_k is the identity matrix.

Hence $z_2 = A_3^{-1} f_2$ as system (9) is regular. Thus the next statement is true.

Lemma 1 *In the regular singular system (9) there is not continuous dependence from initial data, i.e. this system is not robust with respect to initial data.*

The characteristic polynomial $\det(A - \lambda B)$ can be Hurwitz, but for the system stability it means nothing because there is the conflict with the fact that the set of initial data is open according to control objective setting (here initial data for the variable z_2 are fixed). By this reason the singular systems cannot be used.

The situation with system (9) is complicated by the fact that the matrix B proves to be singular due to the idealization which connects with ignorance of some parameters of the system under consideration (*e.g.* proportional feedback (2) is the idealization of the physical realizable feedback (3)). If this ignorance is not made then their stability has not already followd from the old characteristic polynomial $\det(A - \lambda B)$. New characteristic polynomial can be Hurwitz or not in the dependence from the idealization which has led to the singular matrix B.

Often, synthesized systems are obtained in the form

$$F(\frac{d}{dt}) z(t) = f(t) \tag{13}$$

where $z(t) \in \mathbf{R}^n$ is the system vector in the time $t \in \mathbf{R}^+ = [0, \infty)$; $f(t)$ is the external disturbance; $F(\cdot)$ is the polynomial with square scalar matrix coefficients

$$F(s) = \sum_{k=0}^{m} s^k a_k \tag{14}$$

where s is the variable of Laplace transform, m is some natural number.

In the case of system (13) we might use the above result after the corresponding transform into the form of system (9). Let us show how the degeneracy of system (13) is discovered without this transform. The degree of the polynomial matrix $F(s)$ is termed as the degree of the polynomial entry of highest degree in $F(s)$. Hence it is obviously what the degree of the i–th column of $F(s)$ (as one–column matrix) is. Let it be noted by d_i. Below we shall need also in the regulating matrix

$$F_r(s) = diag[(s+1)^{d_1}, (s+1)^{d_2}, \dots, (s+1)^{d_n}] \tag{15}$$

It is easy to define the column D_i consisting of the coefficients of the highest degree s terms in the i–th column. The matrix

$$D(F) = [D_1, D_2, \dots, D_n]$$

is called pivot.

Theorem 1 *In system (13) there is not continuous dependence from initial data if* $\det D(F) = 0$.

It follows from the fact verified by induction that

$$\det F(\lambda) = \det D(F)\ \lambda^d + \text{lower degree terms in } \lambda \tag{16}$$

where $d = \sum d_i$.

As said above some negligible small time–delay τ is present always in synthesized systems. Consider the characteristic quasi–polynomial $\det F(s, \mu)$ where

$$F(s, \mu) = F^{(1)}(s) + F^{(2)}(s)\mu, \ \ \mu = \exp(-s\tau) \tag{17}$$

$F^{(1)}(s)$ and $F^{(2)}(s)$ are matrix polynomials of kind (14).

Assume that matrices (14) and (17) have the same degrees (in s) of their corresponding columns and $D(\mu) = D(F(s, \mu))$ is the pivot matrix in the variable s. Then

$$\det F(\lambda, \mu) = \det D(\mu)\ \lambda^d + \text{lower degree terms in } \lambda \tag{18}$$

Theorem 2 *Let the polynomial* $\det F(\lambda, 0)$ *be Hurwitz then quasi-polynomial (18) is Hurwitz for sufficiently small* τ *if no zero of the polynomial* $\det[D(\mu)]$ *is in the closed unit circle.*

The theorem proof can be easily be obtained with Newton–Raphson method and Rouche theorem (see [25]).

In the case where $\det D(\mu) \equiv 0$, the polynomial $\det F(\lambda, 0)$ is singular and thus there is not the continuous dependence of system solutions from initial data for $\tau = 0$. But, in general, the system workability for small time–delay τ can have no in common with lack of the continuous dependence of "undelayed" system solutions from initial data.

Singular closed-loop systems can be considered in such way as in the classical mechanics (the Lagrange formalism). To this end the feedbacks (which lead to equations of the kind (12)) must be considered as constraints [26]. Then we may preserve the continuous dependence from initial data.

4 Estimation of Robustness Domain

For robust systems the estimation of their robustness domain is of great importance. Assume that polynomial (14) is obtained the polynomial increment $\Delta F(\cdot)$ with sufficiently small square scalar matrix coefficients of its terms. Then system (13) takes the form

$$F_p(\frac{d}{dt})z(t) = f(t) \tag{19}$$

where $F_p(\cdot) = F(\cdot) + \Delta F(\cdot)$.

Is system (19) stable if non-perturbed system (13) possesses this property? Does system (19) depend continuously on initial data if it is true for system (13)? We give the positive answers with using some properties of transfer functions.

To this end we write the matrix function $F_p(s)$ in the form $F_p(s) = F(s)[I_n + F^{-1}(s)\Delta F(s)]$. Hence the transfer function $W_p(s)$ of the perturbed system takes the form

$$W_p(s) = F_p^{-1}(s) = \sum_{k=0}^{\infty}(-1)^k[F^{-1}(s)\Delta F(s)]^k W(s) \tag{20}$$

which converges uniformly over the imaginary axis if the next condition

$$\sup_{\Re s=0} \|F^{-1}(s)\Delta F(s)\| = q < 1 \tag{21}$$

is fulfilled. If this is the case for any sufficiently small coefficients of the terms of the increment $\Delta F(s)$ then formula (20) defines the function $W_p(s)$ satisfying the inequality

$$\mathop{ess\ sup}_{\Re s=0} |W_p(s)| \leq \frac{1}{1-q} \mathop{ess\ sup}_{\Re s=0} \|W(s)\| = \frac{1}{1-q}\|W\|_{H_\infty} \tag{22}$$

where H_∞ is Hardy space of matrices analytical in the right half-plane.

Thus due to relation (22) the transfer matrix W_p of the perturbed system is bounded. Function (20) uniformly converging consists of functions analytical in the right half–plane. That is why the function $W_p(s)$ is analytical in the same half–plane and it defines the stable transfer matrix W_p.

Condition (21) guarantees the non-degeneracy of the characteristic polynomial $\det F_p(\lambda)$ in the sense that its degree does not change for any small coefficients of the increment terms. Indeed, in order to be robust the transfer matrix W_p must be bounded, *i.e.* the transfer matrix $W_p(s)$ must be bounded in the right half-plane. If polynomial $\det F_p(\lambda)$ is singular then the transfer matrix of system (19) is not bounded as in any its neighborhood there are unbounded transfer matrices. Thus condition (21) ensures also the solution continuous dependence of equation (13) from initial data.

Let us change the perturbed system assuming that the corresponding increment $\Delta F(s)$ takes the form $F_r(s)\Delta F_1(s)$ where the matrix $F_r(s)$ is defined by relation

(15), the matrix $\Delta F_1(s)$ is analytical and bounded (small) in the closed right half–plane. Then expansion (20) converges uniformly over the imaginary axis if the next condition

$$\sup_{\Re s=0} \|F^{-1}(s)F_r(s)\Delta F(s)\| = q < 1 \tag{23}$$

is fulfilled. This is the case for any sufficiently small matrix $\Delta F_1(s)$ if the regularized matrix $F^{-1}(s)F_r(s)$ is bounded over the imaginary axis (and admits to be extended analytically in the right half-plane). It is easy to see that the system (9) does not satisfy this condition as in this case $F_r(s) = (s+1)E_n$ and the matrix $F^{-1}(s)F_r(s)$ is similar to the matrix

$$\begin{bmatrix} (A_1 - E_k s)^{-1} & -(A_1 - E_k s)^{-1} A_2 A_3^{-1} \\ O & A_3^{-1} \end{bmatrix} (s+1) \tag{24}$$

unbounded over the imaginary axis.

For the case when the negligible small time–delay τ is in the perturbed system its increment is $F_2(s)[1 - \exp(-s\tau)]$ where $F_2(s)$ is the matrix polynomial. Assume that the transfer matrix of the non-perturbed system is bounded over the imaginary axis. Then the transfer matrix of the perturbed system is bounded for the sufficiently small time–delay τ too if the next condition

$$\sup_{\Re s=0} \|[F(s) + F_2(s)]^{-1} F_2(s)[1 - \exp(-s\tau)]\| \leq q < 1 \tag{25}$$

is fulfilled. This is the case when

$$\sup_{\Re s=0} \|[F(s) + F_2(s)]^{-1} F_2(s)\| < 1 \tag{26}$$

The last condition is more restricted one that the condition of the theorem 2.

Thus, although causality of the systems under consideration is equivalent to analyticity of their transfer matrices in the closed right half-plane $\Re s \geq 0$, and their boundness in this half-plane means the systems stability, but this boundness does not automatically ensure the parametric robustness.

5 Conclusions

In the time-invariant linear case with the help of complex analysis method and Laplace transform it has above been shown that idealized feedbacks being anticipatory can be put in practice if their time-delay approximations lead to (asymptotically) stable closed loop systems. The anticipation of "bad" idealized feedbacks might be eliminated only if the control problem setting would be changed. That is why the approach proposed in [24] is seem to be artificial: any part of a system can be unstable or anticipatory but it does not define the system properties at a whole. And therefore any efforts to do them stable or causal are superfluous in general.

Realizability of anticipatory feedbacks is considered as a part of the common problem of robustness. Above given are some new results connected with the following aspects of this problem in the time-invariant linear case, namely: singular systems and estimation of their robustness domains.

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