A Computational Path to the Nilpotent Dirac Equation

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Abstract
Using a rewrite approach we introduce a computational path to a nilpotent form of the Dirac equation. The system is novel in allowing new symbols to be added to the initial alphabet and starts with just one symbol, representing 'nothing', and two fundamental rules: create, a process which adds news symbols, and conserve, a process which examines the effect of any new symbol on those that currently exist. With each step a new sub-alphabet of an infinite universal alphabet is created. The implementation may be iterative, where a sequence of algebraic properties is required of the emerging sub-alphabets. The path proceeds from nothing through conjugation, complexification, and dimensionalisation to a steady (nilpotent) state in which no fundamentally new symbol is needed. Many simple ways of implementing the computational path exist.
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1 Rewrite Systems

Rewrite systems are synonymous with computing in the sense that most software is written in a language that must be rewritten as characters for some hardware to interpret. Formal rewrite (substitution or production) systems are pieces of software that take an object usually represented as a string of characters and using a set of *rewrite rules* (which define the system) generate a new string representing an altered state of the object. If required, a second *realisation system* takes the string and produces a visualisation or manifestation of the objects being represented.

Each step of such rewrite systems sees one or more character entities of the complex object, defined in terms of symbols drawn from a finite alphabet Σ, being mapped using rewrite rules of the form L→R, into other character entities. Some stopping mechanism is defined to identify the end of one step and the start of the next (for example we can define that for each character entity or group of entities in a string, and working in a specific order, we will apply every rule that applies). It is usual in such systems to halt the execution of the entire system if some goal state is reached (e.g. all the character entities are in some normal form); if no changes are generated; if changes are cycling; or after a specified number of iterations. The objects being rewritten and differing stopping mechanisms determine different families of rewrite system, and in each family, alternative rules and halting conditions may result in strings representing differing species of object. Allowing new rules to be added dynamically to the existing set and allowing rules to be invoked in a stochastic fashion are means whereby more complexity may be introduced. For examples of various types of rewrite system see: von Koch (1905), Chomsky (1956), Naur et al (1960), Mandelbrot (1982), Wolfram (1985), Prusinkiewicz and Lindenmayer (1990), Dershowitz and Plaisted (2001), Marti-Oliet and Meseguer (2002), etc.

In this paper we seek to extend the applicability and power of rewriting by examining how rewrite systems work at a fundamental level, and by creating a rewrite system from which other rewrite systems may be constructed. We will show that a rewrite system can represent mathematics and foundational aspects of physics and can lead to a fundamental basis for quantum computing. The application to physics is particularly important because it is a strong test of the worth of a fundamental idea. Mathematics can be structured on fundamental principles in a large variety of ways, but physics has to survive the test of observation and experiment under many different conditions. The mathematical foundations of physics may also be expected to provide a route to understanding principles that are important in establishing a route to quantum computing.

Deutsch, Ekert and Lupacchini (1999) have stated that: 'Though the truths of logic and pure mathematics are objective and independent of any contingent facts or laws of nature, our *knowledge* of these truths depends entirely on our knowledge of the laws of physics.' According to these authors we have been forced by 'recent progress in the theory of computation', 'to abandon the classical view that computation, and hence mathematical proof, are purely logical notions independent of that of computation as a physical process'. Mathematical structures, however autonomous, 'are revealed to us only through the physical world'. We will, in fact, go further and state that that mathematical structure which is most fundamental in understanding the physical world is also likely to be the structure which is most fundamental to understanding mathematics itself.

The key concepts here seem to be those of nothingness and duality. Many authors have claimed that the physical universe presents a zero totality, at least in energy terms. Atkins (1994), for example, writes that 'the seemingly something is elegantly reorganized nothing, and ... the net content of the universe is ... nothing'. According to our reasoning, however, it is not just matter and the universe that appear to be nothing, but the entire conceptual scheme of which these are merely components; and we believe that the way to preserve this overall nothingness is via duality, another concept which is widely held to be fundamental, and which, as Young (1968) points out, seemingly 'pervades mathematics'.

A zero totality certainly provides a valuable constraint on the possible mathematical and physical structures that can be generated by a universal rewrite system, but we believe that it also allows us to specify them uniquely. A key step in rewriting is the fact that there is an initial state. If we assume a string representation of 0, and begin with the idea that only 0 is unique, and that everything that is not 0 is undefined, we can begin rewriting by denying that we have a non-0 starting-point. That is, we assume that we are not entitled to posit anything other than 0, and are forced to rewrite when we start from any other position. We can then show how a universal alphabet that encompasses duality and nothingness can be developed using such a system.

In the process we will stress the significance of the concept of hierarchy, and of the difference between recursion and iteration, the two methods by which the elements of this alphabet may be discovered. One of these methods yields an infinite number of subset alphabets each of which has properties that can be exploited, for example using further rewrite systems based on the subset alphabet.

2 Evolving Alphabets and the Functions Create and Conserve

Although some rewrite systems assume an infinite alphabet, e.g. of integers, it is more usual to consider the alphabet both static and finite. To relax this constraint and provide *evolving alphabets* we must consider a rewrite rule $\Sigma \rightarrow \Sigma'$ where Σ is the original rewrite alphabet extended by the symbols in Σ'. Adding a rule of this form does not restrict the other rules that comprise the rewrite system nor does it restrict addition of rules that include symbols introduced in Σ'. However, for this to be a valid rewrite system an initial state (that can be re-written) must exist as well as an alphabet containing at least one symbol. The rule $\Sigma \rightarrow \Sigma'$ may be implemented in a number of ways but requires that Σ appears, or is inserted at some time, in the object being rewritten. We do not consider further the full implications of this, requiring only that the process or processes that are invoked have the ability to determine the symbols inserted.

Given an evolving alphabet of this form we may constrain the process to ensure that the alphabet remains balanced with respect to the previous state. Thus for example, given '0' as the symbol representing the character null, empty, or zero in the initial alphabet we use a function *create* to generate the symbol '*a*' and the function *conserve* to generate a conjugate symbol '*A*', where '*a*' and '*A*' together yield 0. The functions *create* and *conserve* requires that a process such as conjugation exist that re-balances the emerging alphabet. However, although it is possible to construct algorithms for create and conserve (for example that select symbols from an infinite set) the specification of the balancing process is an arbitrary one and provides, potentially, an infinite number of balanced evolving alphabets. By selecting the processes that have some natural progression we can either impose desirable properties as outlined below or further constrain the evolving alphabet.

In the limit the alphabet generated will be universal in the sense that it provides all properties and every symbol. Furthermore, the rewrite system too may be considered universal in the different sense that all elements of the system are amenable to reformulation.

3 A Universal Alphabet and Rewriting System

A minimal evolving rewrite system must have an initial state (usually called the ω-state) that contains at least one symbol that we can use to identify that the universe is empty. However, any symbol we choose is immediately (and simultaneously) a symbol, a character of the final alphabet, a subset alphabet and full alphabet in its own right. It is perfectly reasonable to choose, arbitrarily, the single symbol 0 (zero) for Σ, and also to set it as the string representing the complex object in the ω-state {0}. We are obliged to make an arbitrary choice here because we cannot use *create* without the ω-state – the *minimum* rewrite system condition for a universal system. If we were to use *conserve* now it would simply return that 0 is unique, fixed, and consistent and no change from the ω-state would be generated. We now invoke *create* supplying the ω–state as parameter, or source, string.

If we presume that *create* is an algorithm with stopping criteria, it returns a result target string containing a new symbol. If the paradigm for the algorithm were recursive, the resulting symbol (we use *E*) would represent every character of the alphabet at the first step. To create any refining character, a specific e_x, using the recursive paradigm would be impractical because of the implied infinity and storage requirement. We may not use an iterative paradigm at this stage because we would have to supply an upper limit and/or need to identify which of the infinite characters we are creating. Both of these actions require a character not yet in the character set (alphabet) so far defined.

The pair of symbols, the string {0, *E*} is our new object (alphabet) and is now submitted to *conserve* which examines all combinations of possible symbols:

Table 1: Recursive paradigm for universal rewrite alphabet.

	0	***E***
0	00	0*E*
E	*E*0	*EE*

We note that 00, the 'transition' from 0 to 0, conserves 0. The combination 0*E* is the transition from 0 to *E* and is balanced, for all *E*, by its conjugate partner *E*0 which is the transition back from *E* to 0, thereby conserving 0. The combination *EE*, the transition from every symbol *E* to every other, is anomalous and must be returned by *conserve* as unexplained or 'inconsistent' as it does not appear to conserve 0. However, at infinity all transitions represented by *EE* will have been examined, *EE* will be declared 'nilpotent' in that it delivers 0, and we will be left with three generic combinations:

(00, 0*E*, *E*0)

However, it is impractical to use the recursive version of *conserve* to examine further the elements of *E* because of the implied infinite number of iterations.

We return to the *create* process and accept that we must postulate symbols Δ_a, Δ_b, … Δ_n drawn from *E* such that they are in an arbitrary ordinal sequence. We note that there is an infinite number of such sequences because choice of Δ_a is arbitrary. However, we may now use an iterative paradigm for *create* and because *n* is specified, an iterative (or recursive) *conserve* can be constructed. However, at the end of each invocation we are presented with a symmetrical table of transitions that represent the simplest set of properties for the current set of *n* symbols (Table 2).

Table 2: Iterative paradigm for universal rewrite alphabet.

	0	Δ_a	Δ_b	Δ_c	$\dots$	Δ_n
0	00	$0\Delta_a$	$0\Delta_b$	$0\Delta_c$		$0\Delta_n$
Δ_a	$\Delta_a 0$	$\Delta_a\Delta_a$	$\Delta_a\Delta_b$	$\Delta_a\Delta_c$		$\Delta_a\Delta_n$
Δ_b	$\Delta_b 0$	$\Delta_b\Delta_a$	$\Delta_b\Delta_b$	$\Delta_b\Delta_c$		$\Delta_b\Delta_n$
Δ_c	$\Delta_c 0$	$\Delta_c\Delta_a$	$\Delta_c\Delta_b$	$\Delta_c\Delta_c$		$\Delta_c\Delta_n$
$\vdots$						
Δ_n	$\Delta_n 0$	$\Delta_n\Delta_a$	$\Delta_n\Delta_b$	$\Delta_n\Delta_c$		$\Delta_n\Delta_n$

The Δ_a row and Δ_a column illustrate the conjugate pair structure observed earlier. The remaining cells of Table 2 identify explicitly each Δ symbol to Δ symbol transition observed generically in Table 1. Off diagonal there are symmetrical conjugate pairs, for example when $n = b$ there are three such cancelling pairs and six when $n = c$. The diagonal cells of the table contain transitions from each symbol to itself and do not cancel out in this way.

We now invoke the *conserve* process noting that it does not define the transition property but merely identifies those novel transition combinations that appear not to conserve 0. When $n = a$, the symbol Δ_a is added to the alphabet and the transition $0\Delta_a$ is introduced. We need $\Delta_a 0$ (and the idea that this is a conjugate form) to conserve 0. However, this leaves the combination $\Delta_a\Delta_a$ unexplained (novel) and to conserve 0 we must conjecture that whatever it is, is balanced by whatever is to come – or both are 'nilpotent' in the sense introduced above. To discover this we invoke *create* to add a new symbol to the alphabet which then defines (arbitrarily) the $n = b$ row and column. At $n = b$ (in *conserve*) we continue to require the conjugate explanation for all off diagonal elements in the table. In addition, we have non-0 to non-0 symbol transitions, each of which has a cancelling conjugate, and which must ultimately yield a symbol already in the alphabet. However, when these transitions are explained we still have $\Delta_b\Delta_b$ as novel, and require the method of explaining the novelty used earlier. We see that at every invocation of *conserve* we define the need for an additional symbol, delivered by *create* – it is inherent that both processes are obligatory. Other processes may now be conjectured within the rewrite system that impart meaning to 'transition' and also to each transition from Δ_n to Δ_n; however, in each case all of what is to come must balance the $\Delta_n\Delta_n$ in the diagonal position. 'Balance' in this explanation assumes that the 00 transition yields 0, however, we could consider it to yield a conjugate of some form. Where this is the case we may consider each newly created diagonal element as 'balancing' that conjugate by delivering the unconjugated form.

Finally, we note that the symbol 0, the existence of the ω-state, and the processes *create* and *conserve* are outside the rewrite system in that they must exist before the system can function. If we can allow these assumptions, we may also presume the existence of some natural machine that will deliver, for a set of appropriate rewrite rules, a corresponding alphabet where the symbols themselves map to specific rules.

4 Mathematical Properties Required

The properties and symbols required for the universal alphabet we have proposed emerge from the application of the two rewrite rules and would have been equally valid for any of the infinite alternative selections. Significantly, since the ultimate aim is to recover the zero state (ω-state) through an infinite series of processes, the emergence should be seen as being of a *supervenient* nature, that is, without temporal connotation. Furthermore, the symbol delivered at each step has all the properties of all the symbols previously delivered and in a hierarchical and orthogonal fashion.

We may now adopt a dual methodology that simultaneously produces a realisation of the universal alphabet in terms of recognised mathematical procedures and provides a route to the fundamental structures underlying both mathematics and physics. This can be regarded as both a symbolic description of the universal algebra and a specific application of it. Mathematical and physical structures will be shown to effectively rewrite themselves. Because of the fundamental nature of the rewrite approach, further applications in such diverse fields as quantum computing and applied biology immediately suggest themselves, though they will not be dealt with in detail in this paper. It is significant that the process outlined here makes no prior assumption about the existence of the concept of number or any specific mathematical structure. Numbers and their relationships are shown rather to be a *result* of the more fundamental process. Fundamental terms such as 'ordinality', 'numbering', 'negativity', 'conjugation', 'complexity', 'group', 'category', etc., will *emerge* from the evolving structure rather than be predefined to create it. A more specific form of category theory may be applied subsequently, but is not needed as a prior condition.

It has become a standard procedure to derive mathematical structures from the process of counting using the natural numbers, 1, 2, 3, ..., and then progress by successively extending the set to incorporate negative, rational, algebraic, real, and complex numbers, before proceeding to higher algebraic structures involving, say, quaternions, vectors, Grassmann and Clifford algebras, Hilbert spaces, and even higher structures. However, to begin mathematics with the integers, though natural to our human perceptions, is to start from a position already beyond the beginning. The integers are loaded with a mass of assumptions about mathematics. They are not fundamentally simple but already contain packaged information about things beyond the integer series itself. This makes them a convenient codification of mathematics, but not a simple starting-point. The number 1 is not the most obvious initial step from 0 because it contains, for example, the notion of discreteness, as well as ordinality. In addition, there is no obvious route of progression from natural numbers to reals. It would seem to be more logical, in terms of rewrite procedures, to begin with the real 'numbers'.

However, when we first conceive of the real 'numbers', they are not numbers at all. They are not related to anything concerned with counting, because counting does not yet exist. The set of reals ($\mathfrak{R}$) is simply one of things unspecified. Our starting-point must be non-specific, and could be anything. We don't define it at all, not even as a set, and certainly not as numeric. In terms of the rewrite procedures we have adopted, such an assumption of any non-zero category must immediately lead to the return to zero, which, in mathematical terms, becomes equivalent to supposing a 'negative' category or 'conjugate' corresponding to the original assumption. At this point we have created ordinality, or ordering, though not yet counting, as there is no discreteness or anything fixed involved in the procedure; and, although we will use the sign '–', used for the algebraic negative, our process of 'conjugation' is in no way limited to the concept of algebraic subtraction, which, in our terms, does not yet exist. In terms of Table 2, this stage is the recognition that $\Delta_a\Delta_a$ leads to the creation of the new symbol Δ_b.

It is the next application of the *create* procedure ($\Delta_b\Delta_b \rightarrow \Delta_c$) which leads to the number system as we know it, for now we have an undifferentiated 'set' of possible origins for the 'negative' ordinal category or conjugate. We describe these as complex forms ($\mathcal{C}$), and each must have its own conjugate. In mathematical terms, the complex category remains completely undefined in respect to the real category, and has no ordinal relation to it. There are infinitely possible or indefinitely possible systems that are represented by the mathematical $\mathcal{C}$, even for a seemingly specified real category. It is only when we express this fact in the next creation stage that we are able to begin to extend ordinality towards enumeration, for this stage leads to what become mathematical 'combinations' of complex categories. We find here that to every conceivable $\mathcal{C}$, e.g. $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{C}'''$, ..., there are indefinitely possible (commutative) combinations leading to the original real category (e.g. $\mathcal{C}'\mathcal{C}'' \times \mathcal{C}'\mathcal{C}'' = \mathfrak{R}$), but very definite (anticommutative) ones leading to the conjugate (e.g. $\mathcal{C}'\mathcal{C}'' \times \mathcal{C}'\mathcal{C}'' = -\mathfrak{R}$). Again, although we use the algebraic multiplication sign to represent the process of 'combination', the operation of multiplication does not yet exist, and the 'combination' need not imply anything more definite than concatenation.

The alternative commutative and noncommutative possibilities relate to the respective mathematical structures which we call Grassmann and Hamilton algebras. The Grassmann algebra leads to the infinite Hilbert vector spaces, while the Hamilton algebra is responsible for the cyclic system of quaternions. It is the cyclicity of the latter which introduces discreteness or closure, and the concept of 'unity'. We can choose the default position of taking the conjugate combination to create a regular ordinal sequence. We now find that only 'one' independent $\mathcal{C}$-type concept (say $\mathcal{C}'$) is associated with each conceivable $\mathcal{C}$, and we can sequence the terms ordinally by choosing indistinguishability between the $\mathcal{C}$s in every conceivable respect. So the sequence, although arbitrary, becomes a series of integral binary enumerations, which we can also apply to ordinality in the real categories. With the reals, integers, and complexity as fundamental aspects of the system, the remaining mathematical number categories (and higher algebras) can be defined by applying the ordinality condition in a variety of ways, as in conventional mathematics. No new principle is required.

The key proposition at all times, as derived from the rewrite scheme, is that only a combination of an alphabet with itself, along the diagonal of Table 2 ($\Delta_n\Delta_n$), will yield a new ('higher') alphabet to extend the table. Any combination with a component (sub-) alphabet (e.g. $\Delta_a\Delta_n$) will yield only the alphabet itself (Δ_n). It is the application of this last aspect of the rewrite procedure to all possible subalphabets that determines the composition of the new alphabet that is created by $\Delta_n\Delta_n$.

In effect, the hierarchical and orthogonal mathematical structure suggested by the rewrite mechanism is the following:

Table 3: The hierarchical mathematical structure.

$\mathfrak{R}$	undefined	Δ_a
$\mathfrak{R}, -\mathfrak{R}$	conjugation	Δ_b
$\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C}$	complexification	Δ_c
$\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C}, \mathcal{C}', -\mathcal{C}', \mathcal{C}\mathcal{C}', -\mathcal{C}\mathcal{C}'$	dimensionalization	Δ_d
$\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C}, \mathcal{C}', -\mathcal{C}', \mathcal{C}\mathcal{C}', -\mathcal{C}\mathcal{C}'$, $\mathcal{C}'', -\mathcal{C}'', \mathcal{C}\mathcal{C}'', -\mathcal{C}\mathcal{C}'', \mathcal{C}'\mathcal{C}'', -\mathcal{C}'\mathcal{C}'', \mathcal{C}\mathcal{C}'\mathcal{C}'', -\mathcal{C}\mathcal{C}'\mathcal{C}''$	repetition	Δ_e

The subset alphabets at each step represent all those, including $\mathfrak{R}$, $-\mathfrak{R}$ which are generated by operating on themselves:

$$\begin{aligned}
&(\mathfrak{R}) \times (\mathfrak{R}) = (\mathfrak{R}) \qquad (1)\\
&(\mathfrak{R}, -\mathfrak{R}) \times (\mathfrak{R}, -\mathfrak{R}) = (\mathfrak{R}, -\mathfrak{R})\\
&(\mathfrak{R}, -\mathfrak{R}, \mathbb{C}, -\mathcal{C}) \times (\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C}) = (\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C})\\
&(\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C}, \mathcal{C}', -\mathcal{C}', \mathcal{C}\mathcal{C}', -\mathcal{C}\mathcal{C}') \times (\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C}, \mathcal{C}', -\mathcal{C}', \mathcal{C}\mathcal{C}', -\mathcal{C}\mathcal{C}')\\
&\qquad = (\mathfrak{R}, -\mathfrak{R}, \mathcal{C}, -\mathcal{C}, \mathcal{C}', -\mathcal{C}', \mathcal{C}\mathcal{C}', -\mathcal{C}\mathcal{C}'), \text{ etc.}
\end{aligned}$$

Of course, all these processes are of the form $(\Delta_n\Delta_n)$ and will simultaneously generate the next stage in the hierarchical structure. However, the full alphabet Δ_n will also be produced where the process is $(\Delta_a\Delta_n)$, with Δ_a a subalphabet of Δ_n, or any component of one, and, from the general rule that a character set operating on itself or any set or symbol contained within it produces itself, before producing the new alphabet, we may obtain rules between the individual characters, $\mathfrak{R}$, $\mathcal{C}$, etc., of the form:

$$\begin{aligned}
&\mathfrak{R} \times \mathfrak{R} = -\mathfrak{R} \times -\mathfrak{R} = \mathfrak{R} && (2)\\
&\mathfrak{R} \times -\mathfrak{R} = -\mathfrak{R} \times \mathfrak{R} = -\mathfrak{R}\\
&\mathfrak{R} \times \mathcal{C} = \mathcal{C} \times \mathfrak{R} = \mathcal{C}\\
&\mathcal{C} \times \mathcal{C} = -\mathcal{C} \times -\mathcal{C} = -\mathfrak{R}\\
&\mathcal{C} \times -\mathcal{C} = -\mathcal{C} \times \mathcal{C} = \mathfrak{R}\\
&\mathcal{C}' \times \mathcal{C}' = -\mathcal{C}' \times -\mathcal{C}' = -\mathfrak{R}\\
&\mathcal{C}\mathcal{C}' \times \mathcal{C}\mathcal{C}' = -\mathcal{C}\mathcal{C}' \times -\mathcal{C}\mathcal{C}' = -\mathfrak{R} && \text{closed (anticommutative)}\\
&\mathcal{C}'\mathcal{C}'' \times \mathcal{C}'\mathcal{C}'' = -\mathcal{C}'\mathcal{C}'' \times -\mathcal{C}'\mathcal{C}'' = \mathfrak{R} && \text{unlimited (commutative)}
\end{aligned}$$

The choice between the last two procedures is not determined by the algebra. Both are true infinitely and an infinite number of each would be contained within E. However, since we consider the generating mechanism to be supervenient, we can structure it to default at the first option, and so generate an infinite number of identically closed systems, from which we derive an infinite integral sequence. Such a default position additionally represents a condition of minimum use of the available options.

With this procedure, we establish for the first time the meaning of both the number 1 and the binary symbol 1 as it appears in classical Boolean logic. We identify the logical 1 as potentially a conjugation state of 0, that is, a subset alphabet defined within the system; and we can say that 1 appears as a possibility at the point where we *choose* the anticommutative option as a default.

5 Group Properties of Subset Alphabets

Once we have chosen the default which creates the integral sequence, we are free to reinterpret the more general structures we have already created by direct application of the sequence, although this does not retrospectively make the structures less general. Prior to this stage, for example, we may restructure the subset alphabets as a series of finite groups, the order of which doubles at every stage, producing an ordinal binary enumeration. The succession, allowing for conjugation (±) within each group, becomes:

Table 4: Subset alphabets as finite groups.

order 2	real scalar
order 4	complex scalar (pseudoscalar)
order 8	quaternions
order 16	complex quaternions or multivariate vectors
order 32	double quaternions
order 64	complex double quaternions or multivariate vector quaternions

Further stages would extend to triple and higher multiple quaternions, with alternately 'real' and 'complex' coefficients.

Defining closure in terms of enumeration further allows us to understand $\mathfrak{R}$ in terms of the set of real numbers (defined by the Cantor continuum), with + and × now understood as the processes of mathematical addition and multiplication. The dimensional or constructible 'real' numbers represented by terms such as $\mathcal{C}'\mathcal{C}''$ (with countable units squaring to 1) would then be equivalent to those of Robinson's non-standard analysis or Skolem's non-standard arithmetic. From this particular interpretation, it is possible to develop new types of mathematics by combining different aspects of the overall structure in novel ways, as has been the usual procedure in mathematics.

There are, effectively, only three processes at work: conjugation, which produces the alternative + and – values; complexification, which introduces a new complex factor of the form $\mathcal{C} = \boldsymbol{i}$; and dimensionalization, which introduces a complementary complex factor of the form $\mathcal{C}' = \boldsymbol{j}$, converting the $\boldsymbol{i}$ into an element of a quaternion set. The sequence proceeds through an infinite series of quaternionic structures by repeated processes of complexification and dimensionalization. (It is significant that further applications of conjugation does not affect the structure of the elements in the groups.)

In terms of 'units' (once we have established their existence), we could express the structures in the form:

Table 5: Algebraic units for the subset alphabets.

order 2	± 1
order 4	$\pm 1, \pm \boldsymbol{i}_1$
order 8	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1$
order 16	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_2, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$
order 32	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_2, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1,$
	$\pm \boldsymbol{j}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$
order 64	$\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1,$
	$\pm \boldsymbol{j}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$
	$\pm \boldsymbol{i}_3, \pm \boldsymbol{i}_3\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_2, \pm \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1,$
	$\pm \boldsymbol{i}_3\boldsymbol{j}_2, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_3\boldsymbol{j}_2\boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$

Usually, of course, $\boldsymbol{i}_1\boldsymbol{j}_1$ would be written $\boldsymbol{k}_1$, but no new independent unit is created by this notation. An alternative expression could be in terms of multiplying factors:

Table 6: Multiplying factors for the subset alphabets.

order 2	$(1, -1)$
order 4	$(1, -1) \times (1, \boldsymbol{i}_1)$
order 8	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1)$
order 16	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) \times (1, \boldsymbol{i}_2)$
order 32	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) \times (1, \boldsymbol{i}_2) \times (1, \boldsymbol{j}_2)$
order 64	$(1, -1) \times (1, \boldsymbol{i}_1) \times (1, \boldsymbol{j}_1) \times (1, \boldsymbol{i}_2) \times (1, \boldsymbol{j}_2) \times (1, \boldsymbol{i}_3)$,

with the series repeating for an endless succession of indistinguishable $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ values. It is the potentially infinite sequence of $\boldsymbol{i}_n$ values, with commutativity between $\boldsymbol{i}_m$ and $\boldsymbol{i}_n$ or $\boldsymbol{j}_n$ $(m \neq n)$, which creates the possibility of a Grassmann or infinite-dimensional vector algebra, while the anticommutativity between $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ ensures the finite- and, specifically, three-dimensionality of each of the quaternion systems. The commutativity of $\boldsymbol{i}_m$ and $\boldsymbol{i}_n$ is equivalent to defining $(\boldsymbol{i}_m\boldsymbol{i}_n)$ $(\boldsymbol{i}_m\boldsymbol{i}_n)$ as 1, while the anticommutativity of $\boldsymbol{i}_n$ and $\boldsymbol{j}_n$ defines $(\boldsymbol{i}_n\boldsymbol{j}_n)$ $(\boldsymbol{i}_n\boldsymbol{j}_n)$ as the conjugate, or -1. It is notable from this that there is no such thing, in principle, as a pure complex number, only an incomplete representation of a quaternion set.

The order 16 group is of special interest as creating what is effectively a 'real' dimensional structure of the kind observed in normal 3-dimensional vector space. The components, $\pm 1, \pm \boldsymbol{i}_1, \pm \boldsymbol{j}_1, \pm \boldsymbol{i}_1\boldsymbol{j}_1, \pm \boldsymbol{i}_2, \pm \boldsymbol{i}_2\boldsymbol{i}_1, \pm \boldsymbol{i}_2\boldsymbol{j}_1, \pm \boldsymbol{i}_2\boldsymbol{i}_1\boldsymbol{j}_1$, could be more conveniently rearranged and written in the form $\pm 1, \pm i, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}, \pm i\mathbf{i}, \pm i\mathbf{j}, \pm i\mathbf{k}$, where $\pm 1, \pm i$, become the respective scalar and pseudoscalar, and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and $i\mathbf{i}$, $i\mathbf{j}$, $i\mathbf{k}$ the respective vector and pseudovector terms of the multivariate algebra, explored by Hestenes and others (Hestenes 1966, Gough 1990), and applied by them to the algebra of physical space and time, to generate electron spin as a natural consequence of spatial three-dimensionality. This is the algebra of Pauli matrices, in which the 'total' product of two multivariate vectors $\mathbf{a}$ and $\mathbf{b}$ is of the form $\mathbf{a}.\mathbf{b} + i\mathbf{a} \times \mathbf{b}$, and the 'total' products of the vector units becomes $\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 1$; and $\mathbf{ij} = -\mathbf{ji} = i\mathbf{k}$; $\mathbf{jk} = -\mathbf{kj} = i\mathbf{i}$; and $\mathbf{ki} = -\mathbf{ik} = i\mathbf{j}$.

The order 16 group also (if we are to retain the maximum indistinguishability by avoiding octonion-type nonassociativity) is the point at which the extension of the sequence becomes one of repetition, and so a complete specification of an interative procedure could be made by using the groups of order 2, 4, 8 and 16. Taken as independent entities, these may be combined in the group of order 64, using the symbols $\pm 1, \pm i, \pm \boldsymbol{i}, \pm \boldsymbol{j}, \pm \boldsymbol{k}, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}$, to represent the respective units required by the scalar, pseudoscalar, quaternion and multivariate vector groups. This takes on physical significance when we realize that the algebra of this group is that of the gamma matrices used in the Dirac equation – the quantum equation determining the behaviour of the most fundamental components of matter – and that these matrices may be represented as the terms $i\boldsymbol{k}$, $i\mathbf{i}$, $i\mathbf{j}$, $i\mathbf{k}$, $\boldsymbol{j}$, whose binomial combinations are sufficient to generate the entire group (Rowlands 1998, Rowlands and Cullerne 2001).

It would appear that the minimal mathematical structure which most closely corresponds to the 'unit' required to generate the iterative procedure of our rewrite mechanism is significant to physics at the foundational level. Mathematical analysis also shows that the reduction of the group elements to a smaller number of composite generating units is only possible in a pentad or 5-fold form either identical or isomorphic to Dirac matrices in the Rowlands formulation. It is significant for physics that this creates a naturally broken symmetry.

6 The Creation of the Dirac State

Each of the processes involved in the generation of the sequence of mathematical structures by the rewrite mechanism – conjugation, complexification, and dimensionalization – would appear to have a realization in physics, which seemingly contrives to use the minimum possible structure for returning to zero without privileging any of the component processes. A structure previously proposed as foundational to physics suggests that the only truly fundamental parameters are space, time, mass(-energy) and charge, which are respectively represented as multivariate vector, pseudoscalar, real scalar and quaternion (see Rowlands 1983, 1997, 2001 and Rowlands *et al* 2001). The quaternion nature of charge is indicated by its existence in three types (electric, strong and weak), and the fact that interactions between identical charges are of opposite sign to those between identical charges. The parameters also have an internal group symmetry, which, for the purposes of this discussion, can be expressed in the following form (Table 7):

Table 7: Group properties of the fundamental physical parameters.

space	nonconjugated	real	dimensional
time	nonconjugated	complex	nondimensional
mass	conjugated	real	nondimensional
charge	conjugated	complex	dimensional

Conjugated here is equivalent to conserved, so a positive charge (or source of mass-energy) cannot be created without also creating a negative one. Significantly, only the (3-)dimensional quantities, space and charge, are countable, and, physically, one cannot imagine a mechanism for dividing the units in a single dimension. (This is why time is physically irreversible and mass-energy is physically unipolar; neither quantity allows a discontinuity or zero state.) In addition, the mathematical processes which allow for the continual recreation of new non-integral structures in 1-to-1 correspondence with the integers would be inconceivable in a system without dimensionality. As in conventional mathematics, two versions of the 'real' numbers are required: the uncountable ones of the Cantor continuum and standard analysis (for mass), and the countable ones of the Löwenheim-Skolem arithmetic and Robinson's non-standard analysis (for space).

In terms of the structures produced by the rewrite mechanism, the algebras required by the four fundamental parameters occur at the first four levels:

Table 8: Algebras of the fundamental parameters.

order 2	real scalar	mass
order 4	complex scalar (pseudoscalar)	time
order 8	quaternions	charge
order 16	complex quaternions or multivariate vectors	space

Empirically, it appears that these four parameters are all that are required to construct a physical universe, and that, in order to conceive them as such, we need to package the information in such a way as to produce repetition, exactly as we obtain in the rewrite procedure at order 64 with the four mathematical involved.

Now, if the combination of these parameters, or of the real scalar, pseudoscalar, multivariate vector, and quaternion units by which they are realized, is to become itself a 'unit' of the rewrite procedure, we should expect to find some degree of 'closure' or cyclicity, parallel to that which produces the pure quaternion system. However, a fundamental aspect of the quaternion algebra, which, in our system, introduces discreteness, enumeration, or countability, is that it is anticommutative, and it is this very anticommutativity which causes the cyclicity which leads to discreteness. It is significant, in this context, that the presence of anticommutativity allows physics to create a more direct route to the zeroing or conjugation of an act of 'creation', at the level of the 64-element Dirac algebra.

By packaging the algebraic units of mass (1), time (i), charge ($\boldsymbol{k}, \boldsymbol{i}, \boldsymbol{j}$) and space (**i**, **j**, **k**) into Dirac pentads we create units of the form $\boldsymbol{ik}$, $\boldsymbol{i}$**i**, $\boldsymbol{i}$**j**, $\boldsymbol{i}$**k**, $\boldsymbol{j}$, whose physical manifestations are derived from combinations of the three units of charge with, respectively, the unit of time, the three units of space and the unit of space. Because charge is a conserved and quantized (discrete or dimensional) quantity, the new composite quantities are conserved and quantized, but they also retain the respective characteristics of their other parent quantities, time, space, and mass, which are, respectively pseudoscalar, vector and scalar. The new composite pseudoscalar quantity is described as Dirac energy (iE); the new vector quantity is Dirac momentum (**p**); and the new scalar quantity is Dirac rest mass (m). Since the original time, space and mass have the full range of real number values (with those of space being of the Löwenheim-Skolem type and those of time and mass being Cantorian), then it is possible to find values of E, $\mathbf{p}$ and m, after collecting the vector components of $\mathbf{p}$, such that

$$(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m) = 0\,. \tag{3}$$

Also, if we allow the nonconserved parameters space and time to take the full range of possible values for a free particle, while conserving E and $\mathbf{p}$, then we can recover the 'eigenvalue' or 'amplitude' $(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ by the action of a differential operator $(\mp \boldsymbol{k}\partial/\partial t \pm \boldsymbol{i}\nabla + \boldsymbol{j}m)$ on a 'phase term', $exp\ i(-Et + \mathbf{p.r})$, which, for a free fermion, or fundamental particle state, simply expresses the full set of space-time translations and rotations. This version of (3) is the *nilpotent Dirac equation*:

$$(\mp \boldsymbol{k}\partial/\partial t \pm \boldsymbol{i}\nabla + \boldsymbol{j}m)\,(\pm \boldsymbol{ik}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\ exp\ i(-Et + \mathbf{p.r}) = 0\,, \tag{4}$$

which, in principle, is nothing more than a definition of a nilpotent state with specific incorporation of space and time 'nonconjugation'.

In parameterizing the physical world using the nilpotent algebra, then, we create a structure which zeros itself by being a nilpotent or square root of zero, so producing a cyclicity at a higher level which incorporates the whole range of procedures required for the rewrite mechanism. This, in fact, appears to be the packaged structure on which physics is based, for it describes the fermionic or antifermionic state, whether free or bound, and whether pure or existing in bosonic combination. The next stage is then simply to make infinitely or indefinitely many applications of this closed system or 'unit' structure to construct the entire physical universe, as a kind of 'fermionic space', in the same way as we iterate applications of the quaternion system to construct a system of mathematics. Here, the remaining terms in the rewrite algebra form an infinite number of undefined coefficients for individual nilpotents, ψ_1, ψ_2, ψ_3, ..., which are commutative with the nilpotent algebra. That is, they create an infinite-dimensional Grassmann algebra (equivalent to Hilbert space), with successive outer products defined by the Slater determinant, and so requiring $\psi_1 \wedge \psi_1 = 0$ and $\psi_1 \wedge \psi_2 = -\ \psi_2 \wedge \psi_1$, etc., and an immediate and nonlocal algebraic superposition of all fermionic states. It is only in this form that they will be accessible from within the nilpotent system. Any *inner* structure such terms have with respect to each other will remain completely unknown, and only those terms compatible with preserving the infinite nilpotent system will be 'naturally selected'.

An interesting observation to be made in this connection is that the nilpotent algebra introduced here and used in the Dirac formalism provides a mathematics of uniqueness previously unexplored. The formalism is only possible because the terms E, $\mathbf{p}$ and m, like the original parameters time, space and mass, from which they were derived, have the full range of real number values. In principle, then, each individual nilpotent can be unique; and must be if, as we believe, the entire universe can be structured as a superposition of fermionic states, with any nonuniqueness in the components producing immediate zeroing (manifested physically as Pauli exclusion). The algebra that provides this, which we have created by our rewrite mechanism, can be extended to infinity, through the physical property of fermionic wavefunctions being nonlocally connected throughout the entire universe. In principle, it is the mathematical interconnectedness of the nilpotent operators that allows us to group its components as a 'unit' of the even higher (Grassmann) algebra, which may be in the form of the conventional complex Hilbert space or the equivalent geometric algebra as demonstrated by Matzke (or even a complex version of the latter) (Matzke 2002). It is also the principle that allows us to return to the fundamental connection between recursive and iterative expressions of the universal rewrite alphabet, in a classic realization of the possibility of an anticipatory system. The recursive nature is shown by the immediate mathematical connection between the infinity of fermionic states, while the iterative nature is apparent in the uniqueness of each. In effect, as soon as we define one finite fermionic state, we have immediate and specific knowledge about the infinite totality of other states.

7 Conclusion

We believe that much of mathematics can be shown to be constructible using the rewrite mechanism outlined in this paper, with an order which is more coherent than one produced by starting with integers. By rejecting the 'loaded information' that the integers represent, and by basing our mathematics on an immediate zero totality, we believe that we are able to produce a mathematical structure that has the potential of avoiding incompleteness in the Gödel sense. (Conventional approaches, based on the primacy of the number system, have necessarily led to the discovery that a more primitive structure cannot be recovered than the one initially assumed.)

The structure may be found relevant also to many aspects of theoretical computation especially abstract machine specification where notation and the needs of rewriting (substitution) languages are explicitly required (Abrial 1996). The universal rewrite system proposed may be mapped to a Turing machine, very close to Turing's original assumptions where every operation 'consists of some change in the physical system consisting of the computer and his tape' (Turing 1936), and every subset alphabet can be used in such an environment. For example at a simple level the subset alphabet with just conjugation when appropriately wrapped provides an exact mapping to a Boolean encoding. Similarly, when a symbolism for the conjugate character is added the alphabet maps to a ternary encoding similar to Booth encoding of two's complement numbers as used in floating point processors for speeding up multiplication (Booth, 1951).

A physical universe composed of a potentially infinite series of unique (but changeable) nilpotents, originating in the supervenient dualistic processes needed to maintain the zero total state, has itself all the characteristics of a Turing machine. The description of physical systems in these terms allows a mapping of Turing systems to other physical processes and suggests a novel approach to investigating such systems. Here the algebraic and rewrite structure that underlies the mapping can be used to simulate and demonstrate such systems.

We believe that the approach has possible practical application in parallel computation; this is especially the case when cast as parallel agents having autonomous actions mediated by message passing within a well defined spatial and temporal set of constraints. The required properties of this processing environment are captured by the concept of a subset alphabet, and process steps and communication mechanisms are represented as rewrite rules. It is likely that this sort of parallel processing environment will provide a metaphor that has application to our understanding of the complexity of biological and biotechnological systems. And, because such systems are easily implemented will allow direct simulation of complex natural and synthetic biological processes.

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