Reactive Colimit: a View of Internal Measurement Based on a Hamiltonian System

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Abstract Consistent structure of a Hamiltonian dynamical system with constant energy is shown in terms of category theory. A colimit of the dynamical system corresponds to a set of pairs of an initial state and a final state. Expansion of the colimit based on a concept of internal measurement induces heterarchical structure in the dynamical system and derives interaction between the system and the other one. Dynamical change of potential functions derived from that expansion is relevant to a concept of emergence based on the viewpoint of the Hamiltonian system.
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1 Introduction

Matsuno showed that a conventional viewpoint of physics restricts an understanding of a concept of emergence that is found on complex systems such as biological or economic systems, and he proposed a concept of internal measurement [1]. Internal measurement is explained as a motion that carries on canceling conflicts between particles with local perspective and correspond to a process of transformation from intensity into an extensive quantity. The extensive quantity satisfies each physical law in hindsight.

In conventional science, if one analyzes a physical system, he replaces intensity with extensive quantities. For example, given a Newtonian equation $m\dot{\mathbf{v}} = \mathbf{F}$, one replaces the force (i.e. intensity) $\mathbf{F}$ with an extensive quantity such as $-k\mathbf{x}$ since he cannot directly deal with the intensity. Such replacement is a means to deal with the intensity analytically. If the intensity and the extensive quantity have an one-to-one correspondence between them, you have only to deal with the extensive quantity. Conventional quantitative science is based on this viewpoint, but fixed experimental environment is necessary for such an one-to-one correspondence.

A model for a physical system is constructed based on several hypothesis and experimental facts. Specified experimental environment is necessary to keep reproducibility and ability to control the system. And reproducibility and control ability are necessary to fit a mathematical form to the system. When the system is modeled in a deterministic mathematical form, we obtain sets of data represnting behavior of the system at an arbitrary time. The model that is empirically constructed is regarded as a real rule to control the system.

Modeling the system in a specified mathematical form needs actively ignoring unexpected influence that is not included in the model. For example, if one hooks his foot on an experimental setup (i.e. unexpected influence), he works over the experiment to obtain correct results. He does not rewrite the model but experiments again. Such unexpected influence is the same kind of frame problem on artificial intelligence [2][3]. Ignoring unexpected influence means separating the system from indefiniteness and expressing the model by a closed form. A Hamiltonian system is one of the strictest form in such a veiwpoint. A Hamiltonian constrains the structure of the phase space for the dynamical system, and is generally dealt with as a ruler of a physical system.

On the other hand, reproducibility and control ability are incompatible with emergence. If deviation of the data is observed in the system under the reproducibility and the control ability, we conclude that the deviation is derived from fluctuation or improper experimental conditions and the rules that administer the system are invariable. I.e. the system is regarded as a machine that works by the rules. By contrast, if the deviation of the data is regarded as not fluctuation but a result of change of the rules, we find emergence by the system itself.

The change of the rules corresponds to rewriting the model of the system. I.e. the frame problem and the emergence are two sides of the same coin. If we find the invariable rules and the fluctuation in the system, unexpected change of the system means the frame problem. By contrast, if we find the variable rules in the system, the unexpected change means the emergence. The concept of emergence obviously is not properties of specific systems but lies between the system and its observer that assumes the specified rules.

Such change of the rules corresponds to change of potential functions and/or interaction terms in Hamiltonian dynamical system. Thus, when we try to understand a concept of emergence on a Hamiltonian system, one of the most important aim is a formalization of dynamical change of the potential function. Such change of the potential is equal to transformation of a manifold on a phase space.

Outline of our model is the following: A manifold on a phase space of a Hamiltonian system is a trajectory of the system and is the whole of the probable states. We suppose a quasi-Hamiltonian system that has an infinite number of small gaps on the manifold. It means there are discontinuous points throughout the manifold that is regarded as continuous under an approximation. The manifold with gaps corresponds to a system with the frame problem. It is incompatible with the energy conservation law. When we emphasize heterarchical structure of the system between a macro-level layer (i.e. the energy conservation law) and a micro-level layer (i.e. a set of the vectors on the phase space), interaction between the system and the outside of the system and transformation of the manifold are required so that the system satisfies the energy conservation law. If change of the rule (i.e., transformation of the manifold) is permitted, the frame problem evolves into emergence of the system. A concept of heterarchy was presented by McCulloch [4] and is relevant to emergence and robustness [5]-[8]. A heterarchical system is generally characterized by a hierarchical system with interacting or switching between its each layer.

In the present paper, a dynamical system is expressed in terms of category theory[13]-[15]. Using two slice category induced from a colimit and a cocone, we show a structure of a Hamiltonian dynamical system with constant energy. In a conventional Hamiltonian dynamical system, the state satisfies the energy conservation law at an arbitrary time, thus it has a static manifold. By contrast, our extended Hamiltonian system has a heterarchical structure that is derived from inconsistency between the macro-level layer and the micro-level layer, thus the system has a manifold with dynamical change along time.

2 Category of a dynamical system

In this section, we survey terms of category theory that is useful for an expansion of a Hamiltonian dynamical system.

Definition 2.1 (coproduct) Suppose a category $\mathbf{C}$ and an index set Λ. A coproduct $\coprod C_i$ is defined as an object with arrows $\{\iota_i : C_i \to \coprod C_i\}_{i\in\Lambda}$ such that it satisfies the following condition: given $g_i : C_i \to A$, there is an unique arrow $h : \coprod C_i \to A$ such that $h \circ \iota_i = g_i$ is commutateive for each $i \in \Lambda$. Also h is expressed by $[\iota_i]_{i\in\Lambda} : \coprod C_i \to A$.

Definition 2.2 (coequalizer) A coequalizer of arrows $g, h : A \to B$ in a category $\mathbf{C}$ is defined by an allow $e : B \to X$ such that it satisfies the following two conditions: i) $e \circ g = e \circ h$; ii) For an arbitrary arrow $e' : B \to X'$ that satisfies $e' \circ g = e' \circ h$, there is an unique arrow $k : X \to X'$ such that it satisfies $k \circ e = e'$.

Definition 2.3 (cocone) Given a category $\mathbf{C}$, its diagram $\mathbf{C}'$ and its set of vertexes V, a cocone of $\mathbf{C}'$ is an object X with a family of arrows $\nu = \{\nu_i : C_i \to X\}_{i\in V}$ (expressed by $\nu : \mathbf{C}' \to X$) that satisfies the following: For arbitrary arrows $g : C_i \to C_j$ on $\mathbf{C}$, ν satisfies $\nu_j \circ g = \nu_i$.

Definition 2.4 (colimit) Given a category $\mathbf{C}$ and its diagram $\mathbf{C}'$, a colimit of $\mathbf{C}'$ is a cocone $\mu : \mathbf{C}' \to M$ that satisfies the following: For arbitrary cocone $\nu : \mathbf{C}' \to X$ of $\mathbf{C}'$, there is an unique arrow $k : M \to X$ such that $k \circ \mu = \nu$.

Definition 2.5 (pullback) In any category $\mathbf{C}$, pullback of arrows $g : A \to C$ and $h : B \to C$ is a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$, that satisfies the following i) and ii): i) $g \circ p_1 = h \circ p_2$, ii) Given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $g \circ z_1 = h \circ z_2$, there is a unique map $u : Z \to P$ with $z_1 = p_1 \circ u$ and $z_2 = p_2 \circ u$.

Remark 2.6 $A \times_C B$ that is a subobject of $A \times B$ with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ is a pullback of $g : A \to C$ and $h : B \to C$, where $A \times_C B = \{(a, b) | a \in A, b \in B, g(a) = h(b)\}$.

Definition 2.7 (slice category) A slice category $\mathbf{C}/X$ of a category $\mathbf{C}$ over an object $X \in \mathbf{C}$ consists of the following objects and arrows: objects are all arrows $\nu_i \in \mathbf{C}$ such that $\mathrm{cod}(\nu_i) = X$, and an arrow g from $\nu_i : C_i \to X$ to $\nu_j : C_j \to X$ is $g : C_i \to C_j$ in $\mathbf{C}$ such that $\nu_j \circ g = \nu_i$.

Remark 2.8 (composition functor) Composition induces a functor. For any slice category $\mathbf{C}/A$ with objects $\{\mu_i : C_i \to A\}$ and arrows $g : C_i \to C_j$, an arrow $k : A \to B$ induces a slice category $\mathbf{C}/B$ with objects $\{\nu_i = k \circ \mu_i : C_i \to B\}$ and arrows $g : C_i \to C_j$ and a functor $K : \mathbf{C}/A \to \mathbf{C}/\mathbf{B}$ such that $K(\mu_i) = k \circ \mu_i = \nu_i$ and $K(g) = g$.

Remark 2.9 (pullback functor) Pullback induces a functor. For $k : A \to B$ in a category $\mathbf{C}$ with pullbacks, there is a functor $K^* : \mathbf{C}/B \to \mathbf{C}/A$ defined by $(\nu_i : C_i \to B) \mapsto (\mu'_i : C_i \times_B A \to A)$ where μ'_i is the pullback of ν_i along k.

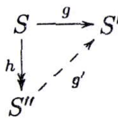
Definition 2.10 (dynamical system and its category) Suppose a topological space D and a continuous map $f : D \times \mathbb{R} \to D$. For each $t \in \mathbb{R}$, a map $f_t : D \to D$ is defined by $f_t(x) = f(x,t)$ $(x \in D)$. If a family of the maps $\{f_t\}_{t \in \mathbb{R}}$ satisfies the following conditions i) and ii), then (D, f) is called a continuous dynamical system on D: i) $f_t \circ f_{t'} = f_{t+t'}$ for all $t, t' \in \mathbb{R}$; ii) $f_0 = 1_D$. The map f_t means a time evolution operator of the dynamical system. And for each $x \in D$, $Gx = \{f_t(x) | t \in \mathbb{R}\}$ is called a trajectory or an orbit through x.

The composition i) satisfies associative law, thus we obtain a category of the dynamical system $\mathbf{D}$ that has the phase space D as its object and the map f_t as its arrow. $\mathbf{D}$ is obviously a subcategory of **Top**, thus a functor F :$\mathbf{D} \to$ **Top** is defined by an inclusion mapping.

We use the following lemma to construct a colimit of a dynamical system.

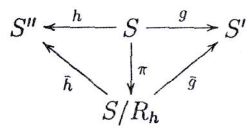
Lemma 2.11 Given a map $g : S \to S'$ and a surjective map $h : S \to S''$, the following two condition are equivalent.

1. For $x, y \in S$, $h(x) = h(y) \Rightarrow g(x) = g(y)$.
2. There is a unique map $g' : S'' \to S'$ such that $g = g' \circ h$.



(1)

Proof (2. $\Rightarrow$ 1.) $h(x) = h(y) \Rightarrow g'(h(x)) = g'(h(y)) \Rightarrow g(x) = g(y)$. h is a surjective map, thus an arbitrary element in S'' is expressed by $h(x)$ $(x \in S)$ and its image of g' is $g'(h(x)) = g(x)$. The fact is independent of x, thus g' is unique.
(1. $\Rightarrow$ 2.) h induces a injection $\bar{h} : S/R_h \to S''$ and $h = \bar{h} \circ \pi$ ($\pi : S \to S/R_h$ is a canonical mapping). h is a surjection, thus $\bar{h}$ is a bijection.

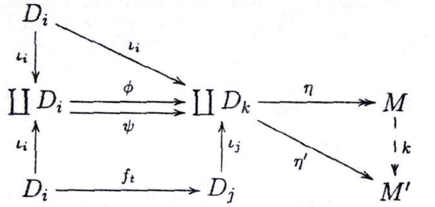


(2)

By the condition 1., g induces a map $\bar{g} : S/R_h \to S'$ and $g = \bar{g} \circ \pi$. If g' is defined by $g' = \bar{g} \circ \bar{h}^{-1}$, we obtain $g = \bar{g} \circ \pi = \bar{g} \circ \bar{h}^{-1} \circ h = g' \circ h$ (i.e. the condition 2.). ■

We construct a colimit of a dynamical system $\mathbf{D}$ using the coproduct and the coequalizer in $\mathbf{D}$.

Construction 2.12 (colimit of diagram for a dynamical system) Suppose a category of a Hamiltonian dynamical system $\mathbf{D}$ with constant energy. For arbitrary $t \in \mathbb{R}$, objects in $\mathbf{D}$ are $\text{dom}(f_t) = D_i$ and $\text{cod}(f_t) = D_j$. And an arbitrary vertex of the diagram of $\mathbf{D}$ is represented by D_k. $\mathbf{D}$ is a subcategory of **Top**, thus there are coproducts $\coprod D_i$ and $\coprod D_k$ in **Top**. For canonical injections $\iota_i : D_i \to \coprod D_i$ and $\iota_i : D_i \to \coprod D_k$, there is an unique arrow $\phi : \coprod D_i \to \coprod D_k$ such that $\phi \circ \iota_i = \iota_i$ because of a definition of a coproduct. Again, for $\iota_i : D_i \to \coprod D_i$, $\iota_j : D_j \to \coprod D_k$ and $\iota_j \circ f_t : D_i \to \coprod D_k$, there is an unique arrow $\psi : \coprod D_i \to \coprod D_k$ such that $\psi \circ \iota_i = \iota_j \circ f_t$. ϕ and ψ stand for $\phi = [\iota_i]_{i \in \{\text{dom}(f_t)\}}$ and $\psi = [\iota_j \circ f_t]_{j \in \{\text{cod}(f_t)\}, t \in \mathbb{R}}$. We can construct $M = \{\{x_i, x_j\} | x_i \in D_i, x_j \in D_j, x_j = f_t(x_i), t \in \mathbb{R}\}$. A surjection $\eta : \coprod D_k \to M; x_i, x_j \mapsto \{x_i, x_j\}$ satisfies $\eta \circ \phi = \eta \circ \psi$ where $x_j = f_t(x_i)$. Given a cocone M' with $\nu : \mathbf{D} \to M'$, it induces an unique arrow $\eta' : \coprod D_k \to M'$ such that $\eta' \circ \iota_i = \nu$, $\rho \circ \iota_i = \nu$ and $\rho = \eta' \circ \phi = \eta' \circ \psi$ because of the definition of coproduct. And there is an unique arrow $k : M \to M'$ because of Lemma 2.11 (note that η is a surjection). Therefore, η is a coequalizer of ϕ and ψ and M with $\mu = \eta \circ \iota$ is a colimit of $\mathbf{D}$. The above facts are expressed by the following diagram:



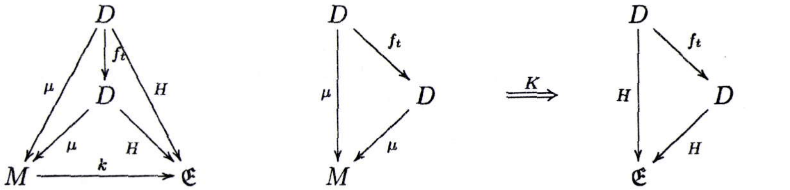
(3)

The colimit M corresponds to a set of pairs of an initial state and a final state (i.e. extent of a set of arrows $\{f_t\}_{t \in \mathbb{R}}$). If $D_i = D_j = D$ is satisfied, M corresponds to a quotient set D/R_G where R_G is a equivalence relation defined by $x R_G y :\iff Gx = Gy$ and $Gx = \{f_t(x) | t \in \mathbb{R}\}$.

We show static structure between a micro-level layer (i.e. a set of the vectors on the phase space) and a macro-level layer (i.e. the energy conservation law) in a Hamiltonian dynamical system.

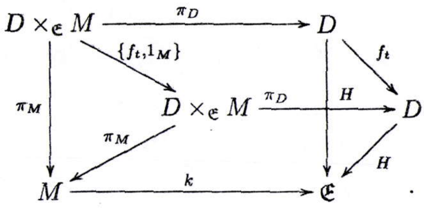
Suppose a Hamiltonian of a n-dimensional system with constant energy $H(\mathbf{p}, \mathbf{q}) = E$. A category of the Hamiltonian dynamical system $\mathbf{D}$ consists of objects $D = \mathbb{R}^{2n} \ni (\mathbf{p}, \mathbf{q})$ and allows $\{f_t = e^{-\mathcal{L}t} : D \to D\}$ where $\mathcal{L} = \sum_{i=0}^{n-1} \left(\frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} \right)$ is a Liouville operator. We obtain a colimit M of a diagram $\mathbf{D}'$ of $\mathbf{D}$ by the construction 2.12. And if $H(\mathbf{p}, \mathbf{q}) = E$ then $H(f_t(\mathbf{p}, \mathbf{q})) = E$ for arbitrary t, thus a set of energy values $\mathfrak{E} = \{E | E \geq 0, E \in \mathbb{R}\}$ with arrows $\{H : D \to \mathfrak{E}\}$ is a cocone of $\mathbf{D}'$.

Two slice categories $\mathbf{D}/M$ and $\mathbf{D}/\mathfrak{E}$ are induced from $\mathbf{D}$, M and $\mathfrak{E}$. And a composition functor $K : \mathbf{D}/M \to \mathbf{D}/\mathfrak{E}$ is induced from $k : M \to \mathfrak{E}$ that uniquely exists.



(4)

And the pullback of H along k induces a functor $K^* : \mathbf{D}/\mathfrak{E} \to \mathbf{D}/M$. it is expressed by the following diagram:



(5)

Thus a consistency between the micro-level layer and the macro-level layer in the Hamiltonian dynamical system is expressed by $K : \mathbf{D}/M \to \mathbf{D}/\mathfrak{E}$ and $K^* : \mathbf{D}/\mathfrak{E} \to \mathbf{D}/M$.

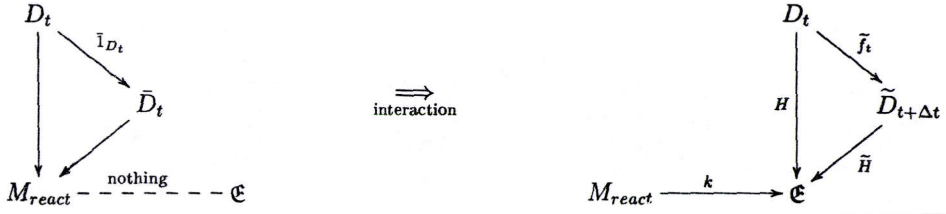
3 Extended harmonic oscillators system

3.1 Reactive Colimit

Introducing a concept of *reactive colimit*, we extend the diagram (2) and (3) into a heterarchical structure. In the previous section, a colimit of the diagram of $\mathbf{D}$ was expressed by $M = \{(\mathbf{x}, f_{\Delta t}(\mathbf{x})) | \mathbf{x} \in D, \Delta t \in \mathbb{R}\}$. Now, replacing $f_{\Delta t}(\mathbf{x})$ of the colimit with a indefinite symbol "$\Box$", we define $M_{react} = \{(\mathbf{x}, \Box) | \mathbf{x} \in D\}$. We call M_{react} a reactive colimit. A concrete value of $\Box$ of the reactive colimit is determined by the following way: Suppose a phase space $D_t = \mathbb{R}^2 \ni \mathbf{x}_t = (p_t, q_t)$ with a Hamiltonian H and $\bar{D}_t \ni \bar{\mathbf{x}}_t = (\bar{p}_t, \bar{q}_t)$ that is a phase space with an infinite number of small gaps. It means

$$H(p_t, q_t) = E, \quad H(\bar{p}_t, \bar{q}_t) = \bar{E} \neq E \quad \text{and} \quad (p_t, q_t) \approx (\bar{p}_t, \bar{q}_t) \tag{6}$$

for arbitrary t. First, we replace $1_D : D \to D; (p, q) \mapsto (p, q)$ with $\bar{1}_D : D \to \bar{D}; (p, q) \mapsto (\bar{p}, \bar{q})$ and substitute $\bar{\mathbf{x}}_t = (\bar{p}_t, \bar{q}_t)$ for $\Box$. Then the left triangle of the diagram (5) commutes. But there is no $k : M_{react} \to \mathfrak{E}$ such that the diagram (2) commutes because of Eq.(4). Secondly, through "interaction" between the system and another system (the interaction is defined in the following section), we obtain $\widetilde{\mathbf{x}}_{t+\Delta t} = (\widetilde{p}_{t+\Delta t}, \widetilde{q}_{t+\Delta t})$ and $\widetilde{H}$ such that $H(p_t, q_t) = \widetilde{H}(\widetilde{p}_{t+\Delta t}, \widetilde{q}_{t+\Delta t}) = E$. And $\widetilde{f}_t : D_t \to \widetilde{D}_{t+\Delta t}$ is defined by $(p_t, q_t) \mapsto (\widetilde{p}_{t+\Delta t}, \widetilde{q}_{t+\Delta t})$ and $\widetilde{H} \circ \widetilde{f}_t = H$. Finally, Substituting $(\widetilde{p}_{t+\Delta t}, \widetilde{q}_{t+\Delta t})$ for $\Box$ again, we obtain $k : M_{react} \to \mathfrak{E}; (\mathbf{x}_t, \widetilde{\mathbf{x}}_{t+\Delta t}) \mapsto E$. The right triangle of the diagram (5) commutes, thus we can construct its pullback. I.e. consistency between a micro-level layer and a macro-level layer is restored by the above process.



(7)

3.2 Dynamical change of angular frequencies

In this section, we define an extended harmonic oscillators system based on the viewpoint of reactive colimit. The system has dynamical change of potential functions (i.e. dynamical change of the angular frequencies).

Our extended harmonic oscillators system consists of N sites. Each site is addressed by the index l and has the following hamiltonian;

$$H_{(l)t} = \frac{1}{2}(p^2_{(l)t} + \omega^2_{(l)t} q^2_{(l)t}) \tag{8}$$

where $(p_{(l)t}, q_{(l)t})$ is a state and $\omega_{(l)t}$ is an angular frequency at t. Each of the harmonic oscillators has a form of isolated one, but particular interaction between them is defined later.

procedure on micro-level layer We define a replacement;

$$(p_{(l)t}, q_{(l)t}) \mapsto (\bar{p}_{(l)t}, \bar{q}_{(l)t}) = (p_{(l)t} + \epsilon_{(l)t}\Delta p_{(l)t}, q_{(l)t} + \epsilon_{(l)t}\Delta q_{(l)t}) \tag{9}$$

where $\epsilon_{(l)t}$ and $(\Delta p_{(l)t}, \Delta q_{(l)t})$ are defined by the following steps:

(i) Given Δt and $\{(\Delta p_{(l)\tau}, \Delta q_{(l)\tau}) | 0 \leq \tau \leq t - \Delta t\}$, $\epsilon_{(l)t}$ is defined by the following;

$$\epsilon_{(l)t} = \begin{cases} i & (R_{(l)t} \geq 0) \\ 1 & (R_{(l)t} \leq 0) \end{cases}$$

where i is imaginary unit,

$$R_{(l)t} = \int_0^t \Delta R_{(l)t} dt \tag{10}$$

and

$$\Delta R_{(l)t} = \frac{\Delta p_{(l)t-\Delta t} \Delta q_{(l)t-\Delta t}}{\Delta t}. \tag{11}$$

Moreover, we define a condition that is required for interaction between the sites j and k by

$$\epsilon_{(j)t} = \epsilon_{(k)t}. \tag{12}$$

(ii) Suppose that the site j interacts with the site k. $(\Delta p_{(j)t}, \Delta q_{(j)t})$ and $(\Delta p_{(k)t}, \Delta q_{(k)t})$ are defined by the following;

$$\begin{aligned} \Delta p_{(k)t} &= -\frac{p_{(j)t}}{p_{(k)t}}\Delta p_{(j)t}, \qquad \Delta p_{(j)t} = \alpha_{(p,jk)} \qquad (\text{ if } p_{(j)t} \leq p_{(k)t}) \\ \Delta p_{(j)t} &= -\frac{p_{(k)t}}{p_{(j)t}}\Delta p_{(k)t}, \qquad \Delta p_{(k)t} = \alpha_{(p,jk)} \qquad (\text{ if } p_{(k)t} \leq p_{(j)t}) \end{aligned} \tag{13}$$

$$\begin{aligned} \Delta q_{(k)t} &= -\frac{\omega_{(j)t}^2 q_{(j)t}}{\omega_{(k)t}^2 q_{(k)t}}\Delta q_{(j)t}, \quad \Delta q_{(j)t} = \alpha_{(q,jk)} \qquad (\text{ if } \omega_{(j)t}^2 q_{(j)t} \leq \omega_{(k)t}^2 q_{(k)t}) \\ \Delta q_{(j)t} &= -\frac{\omega_{(k)t}^2 q_{(k)t}}{\omega_{(j)t}^2 q_{(j)t}}\Delta q_{(k)t}, \quad \Delta q_{(k)t} = \alpha_{(q,jk)} \qquad (\text{ if } \omega_{(k)t}^2 q_{(k)t} \leq \omega_{(j)t}^2 q_{(j)t}) \end{aligned} \tag{14}$$

where $\alpha_{(p,jk)}$ and $\alpha_{(q,jk)}$ are Gaussian-distributed random numbers with mean zero and small variance.

A pair $(\Delta p_{(l)t}, \Delta q_{(l)t})$ satisfies the following;

$$\Delta E_{(l)t} = \frac{1}{2}(\Delta p_{(l)t}^2 + \omega_{(l)t}^2 \Delta q_{(l)t}^2) \ll E_{(l)t} = \frac{1}{2}(p_{(l)t}^2 + \omega_{(l)t}^2 q_{(l)t}^2) \tag{15}$$

thus we can approximately ignore $\epsilon_{(l)}{}_t^2 \Delta E_{(l)t}$ and obtain the following;

$$\frac{1}{2}(\bar{p}_{(l)}{}_t^2 + \omega_{(l)t}^2 \bar{q}_{(l)}{}_t^2) \approx E_{(l)t} + \epsilon_{(l)}\frac{1}{2}W_{(l)t} \tag{16}$$

where

$$W_{(l)t} = 2(p_{(l)t}\Delta p_{(l)t} + \omega_{(l)t}^2 q_{(l)t}\Delta q_{(l)t}). \tag{17}$$

And the conditions (i) and (ii) derive;

$$\epsilon_{(j)t}W_{(j)t} + \epsilon_{(k)t}W_{(k)t} = \epsilon_{(j)t}(W_{(j)t} + W_{(k)t}) = 0. \tag{18}$$

procedure of macro-level layer First, we calculate an entropy of N' sites isolated harmonic oscillators system based on a micro-canonical ensemble of classical statistical mechanics. Suppose a Hamiltonian of the system;

$$H(\mathbf{p}, \mathbf{q}) = \sum_{j=1}^{N'} \frac{1}{2}(p_{(j)}^2 + \omega_{(j)}^2 q_{(j)}^2). \tag{19}$$

Volume Σ of the N'-dimensional hypersphere of the phase space such that $H(\mathbf{p}, \mathbf{q}) \leq E_{total}$ is;

$$\Sigma = \frac{1}{h^{N'}} \int_{H(\mathbf{p},\mathbf{q}) \leq E_{total}} dp^{N'} dq^{N'} \tag{20}$$

thus the surface area $\Omega(E, N')$ of the hypersphere such that $H(\mathbf{p}, \mathbf{q}) = E_{total}$ is;

$$\Omega(E, N') \approx \frac{\partial \Sigma}{\partial E_{total}} E_{total} = \frac{1}{\Gamma(N')} \left(\frac{E_{total}}{\hbar\omega} \right)^{N'} \tag{21}$$

where $h = 2\pi\hbar$ is the Planck constant and Γ is the gamma function. In a conventional method, one calculates the entropy

$$S = k_B \ln \Omega \tag{22}$$

under $N' \gg 1$ and Stirling's Approximation $\ln \Gamma(N') \approx N' \ln N' - N'$. But now, using the Eq.(19) and Eq.(20) under $N' = 1$, we define an entropy-like quantity S for one harmonic oscillator;

$$S = k_B \ln \Omega(E, 1) = k_B \ln \frac{E}{\hbar\omega}. \tag{23}$$

Note that there is an one-to-one correspondence between E and S for a fixed ω. S is no longer an extensive variable, but we still call S entropy.

Now, we suppose that angular frequencies $\omega_{(j)t}$ and $\omega_{(k)t}$ are transformed into $\bar{\omega}_{(j)t}$ and $\bar{\omega}_{(k)t}$ by interaction between the sites j and k. With this process, entropy $S_{(j)t}$ and $S_{(k)t}$ and energy $E_{(j)t}$ and $E_{(k)t}$ of the sites j and k are transformed into $\bar{S}_{(j)t}$, $\bar{S}_{(k)t}$, $\bar{E}_{(j)t}$ and $\bar{E}_{(k)t}$.

Suppose these energy and entropy satisfy the conservation law;

$$\bar{E}_{(j)t} + \bar{E}_{(k)t} = E_{(j)t} + E_{(k)t} \tag{24}$$

$$\bar{S}_{(j)t} + \bar{S}_{(k)t} = S_{(j)t} + S_{(k)t}. \tag{25}$$

Because of Eq.(21) and Eq.(23), we obtain;

$$\ln \frac{\bar{E}_{(j)t}}{\hbar\bar{\omega}_{(j)t}} \frac{\hbar\omega_{(j)t}}{E_{(j)t}} \frac{\bar{E}_{(k)t}}{\hbar\bar{\omega}_{(k)t}} \frac{\hbar\omega_{(k)t}}{E_{(k)t}} = 0. \tag{26}$$

Eq.(22) and Eq.(24) derive the following second-degree equation for $\bar{E}_{(l)t}$;

$$\bar{E}_{(l)t}^2 - (E_{(j)t} + E_{(k)t})\bar{E}_{(l)t} + \frac{\bar{\omega}_{(j)t}\bar{\omega}_{(k)t}}{\omega_{(j)t}\omega_{(k)t}} E_{(j)t}E_{(k)t} = 0 \tag{27}$$

where $l = j, k$. The solutions of Eq.(25) are;

$$\bar{E}_{(l)t} = \frac{1}{2}(E_{(j)t} + E_{(k)t}) \pm U_{(l)t} \tag{28}$$

where

$$U_{(l)t} = \sqrt{(E_{(j)t} + E_{(k)t})^2 - 4\frac{\bar{\omega}_{(j)t}\bar{\omega}_{(k)t}}{\omega_{(j)t}\omega_{(k)t}} E_{(j)t}E_{(k)t}}. \tag{29}$$

If $\bar{\omega}_{(j)t}\bar{\omega}_{(k)t} = \omega_{(j)t}\omega_{(k)t}$, then $\bar{E}_{(l)t} = E_{(j)t}, E_{(k)t}$.

mixture of the micro-level and the macro-level The energy conservation law including $\bar{E}_{(l)t}$ and $W_{(l)t}$ is the following;

$$\begin{aligned}
E_{(j)t} + E_{(k)t} &= E_{(j)t} + \epsilon_{(j)t}\frac{1}{2}W_{(j)t} + E_{(k)t} + \epsilon_{(k)}\frac{1}{2}W_{(k)t} \\
&= \bar{E}_{(j)t} + \epsilon_{(j)t}\frac{1}{2}W_{(j)t} + \bar{E}_{(k)t} + \epsilon_{(k)}\frac{1}{2}W_{(k)t} \\
&= \frac{1}{2}(E_{(j)t} + E_{(k)t}) \pm \frac{1}{2}U_{(j)t} + \epsilon_{(j)t}\frac{1}{2}W_{(j)t} \\
&\quad + \frac{1}{2}(E_{(j)t} + E_{(k)t}) \pm \frac{1}{2}U_{(k)t} + \epsilon_{(k)t}\frac{1}{2}W_{(k)t}.
\end{aligned} \tag{30}$$

This is a strict equation because of Eq.(16).

Eq.(28) derives a mechanism such that gaps on the phase space derives transformation of the manifold as the following;

$$\epsilon_{(l)t}W_{(l)t} = \pm U_{(l)t}. \tag{31}$$

Solving Eq.(29) for $\bar{\omega}_{(j)t}\bar{\omega}_{(k)t}$,

$$\bar{\omega}_{(j)t}\bar{\omega}_{(k)t} = \frac{\omega_{(j)t}\omega_{(k)t}\{(E_{(j)t} + E_{(k)t})^2 - \epsilon_{(l)t}^2 W_{(l)t}^2\}}{4E_{(j)t}E_{(k)t}} \tag{32}$$

and separating $\bar{\omega}_{(j)t}\bar{\omega}_{(k)t}$ into $\bar{\omega}_{(j)t}$ and $\bar{\omega}_{(k)t}$, we obtain the following;

$$\bar{\omega}_{(l)t} = \omega_{(l)t}\frac{\sqrt{(E_{(j)t} + E_{(k)t})^2 - \epsilon_{(l)t}^2 W_{(l)t}^2}}{2E_{(l)t}}. \tag{33}$$

And we define $E_{(l)t+\Delta t}$ and $\omega_{(l)t+\Delta t}$ by the following;

$$E_{(l)t+\Delta t} = \frac{1}{2}(E_{(j)t} + E_{(k)t}) \tag{34}$$

$$\omega_{(l)t+\Delta t} = \bar{\omega}_{(l)t} \tag{35}$$

thus we obtain the state at $t + \Delta t$;

$$p_{(l)t+\Delta t} = -\sqrt{2E_{(l)t+\Delta t}}\sin\omega_{(l)t+\Delta t}(t + \Delta t) \tag{36}$$

$$q_{(l)t+\Delta t} = \frac{\sqrt{2E_{(l)t+\Delta t}}}{\omega_{(l)t+\Delta t}}\cos\omega_{(l)t+\Delta t}(t + \Delta t). \tag{37}$$

We call the system defined in this section *EHO system* (*extended harmonic oscillators system*), and call the conventional harmonic oscillators system *HO system* (*harmonic oscillators system*).

4 Results

We calculate time evolution of the EHO system defined in the section 3.2 under the following conditions: i) The number of sites is $N = 20$. ii) Time interval for the calculation is defined by $\Delta t = 0.005$. iii) For each site, $E_{(l)0} = 50.0$, $\omega_{(l)0} = 1.0$ and $(p_{(l)0}, q_{(l)0}) = (10.0, 0.0)$ are defined as the same initial conditions. iv) At each time step, an site j is coupled with the other site k that $|q_{(j)t} - q_{(k)t}|$ becomes the minimum, and interacts by the procedure defined in the previous section. v) Each of the variances of $\alpha_{(p,jk)}$ and $\alpha_{(q,jk)}$ is given by $\sigma^2 = 2.5 \times 10^{-3}$.

Fig.1 shows the typical time series of $q_{(l)t}$ of one site on the EHO system. The vertical axis is on a range $-50.0 \leq q \leq 50.0$ and the iteration is calculated for 4×10^5 steps. Each site has various time series of $q_{(l)t}$ and shows intermittent motion of the angular frequency $\omega_{(l)t}$. We assume that a mechanism of Eq.(31) derives the intermittent motion of $\omega_{(l)t}$ and is related to multiplicative noise [11].

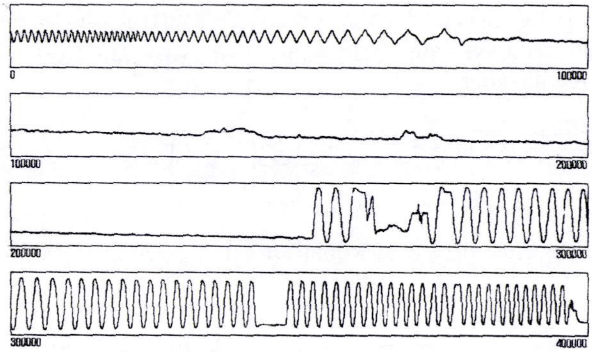


Fig. 1: Time series of $q_{(j)t}$ of one site on the EHO system. The vertical axis is on a range $-50.0 \leq q \leq 50.0$ and the iteration is calculated for 4×10^5 steps.

The EHO system has time evolution of the potential functions (i.e. of the angular frequencies). Fig.2 shows time series of $\omega_{(l)t}$ of N sites. The vertical axis is $\log_{10}(\omega_{(l)t})$ and the iteration is calculated for 5×10^5 steps. In a HO system, the state satisfies the energy conservation law at an arbitrary time, thus it has a static manifold. By contrast, The EHO system has a heterarchical structure that is derived from inconsistency between the macro-level layer and the micro-level layer, thus the system has a manifold with dynamical change along time.

Moreover, in a HO system, its time evolution operator is a canonical transformation. It is important to see whether the time evolution of the EHO system is a canonical transformation. In a HO system, its time evolution operator $f_{\Delta t}$ is expressed by the following matrix;

$$f_{\Delta t} = \begin{pmatrix} e^{-\mathcal{L}\Delta t} & 0 \\ 0 & e^{-\mathcal{L}\Delta t} \end{pmatrix} = \begin{pmatrix} e^{i\omega\Delta t} & 0 \\ 0 & e^{i\omega\Delta t} \end{pmatrix} \tag{38}$$

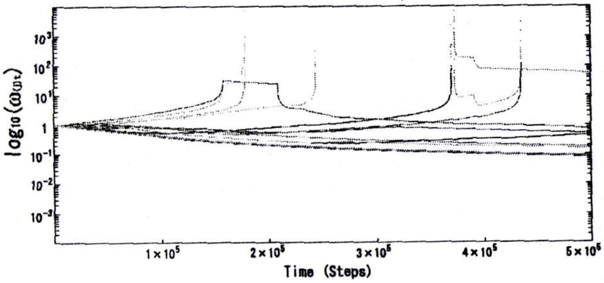


Fig. 2: Time series of $\omega_{(l)t}$ of 20 sites. The vertical axis is $\log_{10}(\omega_{(l)t})$ and the iteration is calculated for 5×10^5 steps.

where $\mathcal{L}$ is the Liouville operator for the system. By contrast, the time evolution operator for each site in the EHO system, $\widetilde{f}_{(l)\Delta t} : D_{(l)t} \to \widetilde{D}_{(l)t+\Delta t}; (q_{(l)t}, p_{(l)t}) \to (q_{(l)t+\Delta t}, p_{(l)t+\Delta t})$, is expressed by the following;

$$\widetilde{f}_{(l)\Delta t} = \begin{pmatrix} \frac{\omega_{(l)t}}{\omega_{(l)t+\Delta t}} \sqrt{\frac{E_{(l)t+\Delta t}}{E_{(l)t}}} e^{i\omega\Delta t} & 0 \\ 0 & \sqrt{\frac{E_{(l)t+\Delta t}}{E_{(l)t}}} e^{i\omega\Delta t} \end{pmatrix}. \tag{39}$$

The conditions of the canonical transformation for $\widetilde{f}_{(l)\Delta t}$ are expressed by the following;

$$\widetilde{f}_{(l)\Delta t} \text{ is a canonical transformation} \iff \begin{cases} \frac{\partial q}{\partial Q} = \frac{\partial P}{\partial p}, & \frac{\partial p}{\partial Q} = -\frac{\partial P}{\partial q} \\ \frac{\partial q}{\partial P} = -\frac{\partial Q}{\partial p}, & \frac{\partial p}{\partial P} = \frac{\partial Q}{\partial q} \end{cases}$$
$$\iff E_{(l)t+\Delta t}\omega_{(l)t+\Delta t} = E_{(l)t}\omega_{(l)t}$$

where $(q, p) = (q_{(l)t}, p_{(l)t})$ and $(Q, P) = (q_{(l)t+\Delta t}, p_{(l)t+\Delta t})$. Therefore, we can evaluate the degree of the satisfaction of the conditions quantitatively. The assessment function $\Delta C_{(l)t}$ is defined by the following;

$$\Delta C_{(l)t} = \left| \frac{E_{(l)t+\Delta t}\omega_{(l)t+\Delta t} - E_{(l)t}\omega_{(l)t}}{\Delta t} \right|. \tag{40}$$

The more the value of $\Delta C_{(l)t}$ is small, the more $\widetilde{f}_{(l)\Delta t}$ approaches the canonical transformation. If $\Delta C_{(l)t} = 0$ then $\widetilde{f}_{(l)\Delta t}$ is the canonical transformation strictly.

Fig.3 shows time series for $\langle \Delta C_{(l)t} \rangle$ of N sites where

$$\langle \Delta C_{(l)t} \rangle = \sum_{\tau=t-n\Delta t}^{t} \frac{1}{n} \Delta C_{(l)\tau} \tag{41}$$

and we set $n = 10^3$ in this case. The vertical axis shows the values of $\log_{10}(\langle \Delta C_{(l)t} \rangle)$ and the iteration is calculated for 5×10^5 steps. Time evolution of some sites satisfies the canonical transformation, and that of the others breaks the canonical transformation; i.e. we can see the differentiation into the motions in the classical dynamics and that ones in the non classical dynamics.

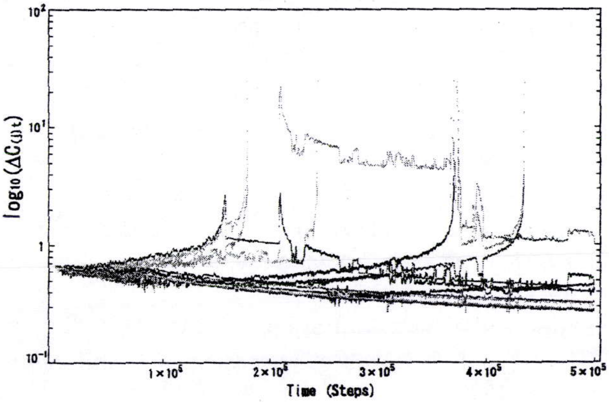


Fig. 3: Time series of $\langle \Delta C_{(l)t} \rangle$ of 20 sites for 5×10^5 steps. The vertical axis shows the values of $\log_{10}(\langle \Delta C_{(l)t} \rangle)$. The more $\Delta C_{(l)t}$ is small, the more time evolution of the site l, $\widetilde{f}_{(l)\Delta t}$, approaches the canonical transformation.

5 Conclusion

In a dynamical system, intent means a function defining time evolution of states and extent means a pair of an initial state/input and a final state/output. Such a pair of the intent and the extent constructs a micro-level layer of the dynamical system. We proposed a system including dynamical change of the intent that is derived from extent with indefiniteness. If we can approximately ignore the indefiniteness of the extent, the sequence of $\langle$input-function-output$\rangle$ is consistent. By contrast, if we cannot ignore the indefiniteness of the extent, inconsistency between the intent and the extent influences the macro-level layer such as energy or entropy, and the process of the change derives the dynamical change of the intent. We introduced a concept of reactive colimit to express such extent with indefiniteness. In addition, such a process of reconciliation against the inconsistency requires interaction between the system and the outside of the system. We applied it to a harmonic oscillators system and expressed emergence as dynamical change of angular frequencies of the system.

References

[1] K.Matsuno, *Protobiology: Physical Basis for Biology*, CRC Press, Boca Raton, 1989.

[2] J.McCarthy and P.J.Hayes, Machine Intelligence 4(1969)463.

[3] D.C.Dennet, *Cognitive Wheels: The Frame Problem of AI*, In Christopher Hookway, ed., *Minds, Machines and Evolution: Philosophical Studies*, pp.129-151, Cambridge Univ. Press, Cambridge, 1984.

[4] W.S.McCulloch, Bull.Math.Biophys. **7**(1945)89.

[5] E.Jen, Complexity **8**(2003)12.

[6] M.de Mey, *The Cognitive Paradigm: Cognitive Science, a Newly Explored Approach to the Study of Cognition Applied in an Analysis of Science and Scientific Knowledge*, D.Reidel Pub.Co.,1982.

[7] Y.-P.Gunji and M.Kamiura, Physica D **198**(2004)74.

[8] M.Kamiura and Y.-P.Gunji, *Robust and Ubiquitous On-Off Intermittency in Active Coupling*, Physica D (accepted).

[9] K.Matsuno and A.Nemoto, Phys.Life.Rev. **2**(2005)227.

[10] Y.-P. Gunji, T. Haruna, T. Shirakawa and K. Sonoda, Proceedings of the ZiF Interdisciplinary Research Workshop: Endophysics, Time, Quantum and the Subjective. (2005)57.

[11] A.Sato, H.Takayasu and Y.Sawada, Phys.Rev.E **61**(2000)1081.

[12] W.Greiner, L.Neise and H.Stoker, *Thermodynamics and Statistical Mechanics*, Springer, New York, 1995.

[13] S.MacLane, *Categories for Working Mathematician*, Springer, New York, 1971, 1997(2^{nd})

[14] S.Awodey, *Category Theory*, http://www.andrew.cmu.edu/course/80-413-713/notes/, 2005.

[15] J.Lambek and P.J.Scott, *Introduction to high order categorical logic*, Cambridge Univ. Press, Cambridge, 1986.