Infinitesimal Parametrical Families of Distributions and the Statistical Foundations of Quantum Mechanics

Vadim F. Krotov
Institute of Control Sciences, Russian Academy of Sciences
117997 Profsoyuznaya 65 Moscow Russian Federation,
Fax-(495) 334-9340, e-mail:vfkrotov@ipu.rssi.ru

Abstract
Parametrical families of distributions are considered, characterised by Borel's measure, infinitisimal in parameters space. The latter property defines the construction of the family and each its member. The properties of this construction are studied, as well as techniques of synthesis of probabilistic distributions, and application areas, first of all quantun systems. They are defined as infinitisimal dynamic systems, observation results of which are described by above mentioned families with the wave function as a parameter. We have demonstrated that axiomatic system of the quantum mechanics is overdefined ("redundant" axioms are restated as theorems), as well as showed new statistical properties of relativistic quantum mechanics. We also proposed non-contradictive model to explain some paradoxal properties of quantum field (paradox of the boson field energy).
Keywords: Parametrical families of distributions; Quadratic Forms in Hilbert space; measures of Borel; Statistical foundations of Quantum Mechanics; Boson.

Introduction

The **mathematical matter** for our enquiry is the parametrical families of the distributions described by Borel's measures. In themselves, a transition from individual distributions to a family do not give new possibilities for its investigation. But some natural conditions for concrete classes of the systems limit their choice to the point of its total fixing. The requirement of the infinitesimality of measures in the parameters space, which is true for infinitesimal physical systems, possess this property. Namely, infinitesimal parametrical families of the distributions are analized here: their properties, the conditions and mechanisms of the realization, spheres of the applications. These properties prove to be similar to lows which form the statistical bases of the quantum mechanics, that motivate investigation of the latter from these positions. These bases of the quantum mechanics are a **physical matter** for our enquiry. Let us note, that their analysis is the subject for interdisciplinary studies starting from [1], and it includes, becides mathematical analysis of axioms, the study of "quantum logic" and physical phenomenology [2], [3], as well as theory of probability, and even its nontraditional postulates [4], [5], [3], etc. In the quantum system there are synthesised the deterministic dynamics of the wave function and statistical link of the latter with observed values. Genesis of wave function and its dynamics in not discussed here, niether the genesis of randomisation of the observed values. In this paper we consider:

International Journal of Computing Anticipatory Systems, Volume 17, 2006
Edited by D. M. Dubois, CHAOS, Liège, Belgium, ISSN 1373-5411 ISBN 2-930396-03-2

the distribution laws of the quantum mechanics, including probabilistic interpretation of the wave function, the superposition principle, the von Neumann's postulates, the problems of their minimal description, the problems of its additional definition for relativistic quantum mechanics and for quantum field theory. It is analized here in the context of understanding quantum system as an infinitesimal family of distributions with the wave function as a parameter.

The following results are obtained:

- the redundancy axioms are revealed in the statistical foundations of quantum mechanics and are translated in the theorems;
- the new statistical properties of the quantum mechanics (relativistic) are predicted, that give a theoretical matter for an experimental proof;
- an explanation for some paradoxal properties of the quantum fields in terms of a noncontradictive model (energy paradox of the boson quantum field).

Some of these results was described in a brief and simplified form in [6], [7]. In this paper, mathematical sections 1 and 2 contain a preliminary survey of related results. New results in quantum mechanics (section 3) are described in more detail.

We will use the following **Abbreviations:** FD – Family of Distributions; IFD – Infinitesimal Family of Distributions; QM – Quantum Mechanics; MP – Material Point; QS – Quantum System; WF – Wave Function; EQF – Ermit Quadratic Form; QP – Quantum Particle; QF – Quantum Field; SR – Special Relativity;

1 Infinitisimal Families of Distributions. Definitions.

Let us introduce into consideration the parametrical families of distributions, defined in the Hilbert complex space of parameters H, with elements u and product (u_1,u_2). With fixed parameters u they are described by nonnegative function of subsets Q on the numerical axis Y, namely, by generalised Borel's measure **P**(Q,u) on the Σ-ring K. The functional **P**(Q,u): $K \times M \dashrightarrow R_+$ will be named the family of distributions (FD), the variable $y \in Y$ is its argument, and the vector u is its parameter.
This construction describes the following physical situation: there is a system with the state u, Y is its charachteristic with values $y \in R$; and **P**(Q) is defined as a measure of distributions of such values, which depends on the state u. Of special importance is the case of probabilistic measure **P**(Q,u), adequate to description of statistical results of observation of the physical system on the set of fixed state values.

Let us name FD as IFD in the space H if values u are sufficiently small and functionals **P**(Q,u), $H \rightarrow R_+$, are sufficiently smooth. Finally, these functionals can be represented as Ermit Quadratic Forms (EQF) with the set-parameter $Q \subset Y$.

$$\mathbf{P}(Q,u) = (u, L_p(Q)u) = \|u\|^2\, P(Q,e),\ \forall\, Q \qquad (1)$$

here $L_p(Q)$ is the family of linear Ermit nonnegative operators, which will be called operators of measure; $e = u/\|u\|$. Let us normalize the family with the condition: $\mathbf{P}(Y,e) = 1, \forall e$.

Average value <Y>(u) of the IFD argument (first momentum of measure), obviously, is also an EQF: $\langle Y\rangle(u) = (u, L_y u)$. Operator L_y will be called the operator of the argument. The values u in IFD are defined accurate to the sign in real space H and to the multiplier $\exp(i\alpha)$, $\alpha \in R$, in complex space. Appropriate technique for analysis of forms eq.1 is the spectral theory of selfadjoint operators [8].

Let us denote: $P(\xi,u)$ measure of set-point, $P(\xi,u) = P(Q: y = \xi,u)$. A pair argument-paramenter (ξ,u) will be called full-measured if $P(\xi,u) = \mathbf{P}(Y,u)$. Full-measured basis of the family will denote a set of full-measured pairs (ξ,u), such as their components u form a basis U of the space H.

2 Properties of infinitisimal families of distributions

For further clarity let us describe these properties for countable full-measured basis $\{(y_k,u_k)\}$, although similar results take place for even more complex constructions of the full-measured basis. Its main property is a decomposition on full-measured basis $\{(y_k,u_k)\}$ and representability of the measure $\mathbf{P}(Q,u)$ as a sum of the values $|(u_k,u)|^2$, $y_k \in Q$ (if it has a simple discreet basis). The average is also an EQF $\langle Y\rangle(u) = (u,L_y u)$. Its eigen numbers λ_k and vectors v_k are equal to the components of the elements (y_k,u_k) of the full-measured basis. Similar results has been obtained also for more complex spectra. Therefore, if the measure is understood stochastically, the measurement results are concentrated only in eigen values λ_k. And only in eigen states $u = v_k$ the measurement result is certain. One can observe also the vectors $y \in R^n$ under condition that the eigen states of the components are identical.

Let the parameter of the IFD to be the function $u = \varphi(x)$, $x \in R$, its argument is congruent with the agrument of the measure: $y = x$, and the space H has the norm L_2. Then the density of the measure is defined as: $p(x,\varphi) = |\varphi(x)|^2$.

Let $u = \sum_k a_k v_k$, where a_k – normalized complex coefficients, $\{v_k\}$ – eigen basis of the operator L_y (unnormalized). Then the measure is distributed over the values $y_k = \lambda_k \|v_k\|^2$ in quantities $|a_k|^2$. To fix the scales $\|v_k\|$ additional information is required.

Various interpretations of the IFD components are considered, such as: measure, material and stochastic, and parameter u. The latter, in particular, is defined as the state of the dynamic system. From its equations we can derive the expression for the characteristic Y in the quadratic form $y = J(u) = (u,Lu)$. We will call it dynamic functional. If the distribution is centered around this value: $\langle Y\rangle(u) = J(u)$, then the system in the state u obtains an additional meaning of the combination of such systems in eigen values v_k of this Quadratic Form in quantities defined by the properties of the

Infinitesimal Family. In stochastic case the characteristics of the deterministic system is observed in conditions of random perturbations, and the result of observations is the distribution of probability on the eigen values y_k in corresponding proportions.

We have defined the statistical techniques for observation of the material Infinitesimal Family, which allow the probabilistic measure to reproduce the material one, which give us a mechanism of synthesis of these laws, nontraditional for the theory of probability.

Infinitesimal Family is defined accurate to the full-measured basis. Two ways of its fixation are feasible:

1. It is defined if the state of the system is the function $u = \varphi(x)$, $x \in R^m$, and its argument: $y = x$ is observed.
2. Parameter u is the state of dynamic system, as described above. This yeilds in fixation of the operator L_y, and thus of the full-measured pairs (y_k,u_k), accurate to the scale coefficients $\|u_k\|^2$, and, in general, of the common addendum b of the eigen values y_k. For their fixation more information about the system is required. With their variation the values y_k change, but the values $\mathbf{P}(y_k,u) = |a_k|^2$ stay invariant.

3 Quantum Mechanics as Theory of Infinitisimal Families of Distributions

The foundations of QM include:

1. Dynamic postulate: the WF $\varphi(.)$ is dynamic state of QS;
2. Statistical postulate: observed values of the system's characteristics are random and are described by special parametrical family of distributions (quantum principle of superposition), in which the role of parameter is played by dynamic state $\varphi(.)$.

There are attempts to stipulate the wave function genesis by statistical factors, and to consider its equations as the equations of Planck-Chapmen-Kolmogorov type under different assumptions [9], [3]. But most popular opinion is that dynamics and statistics of QM have different physical nature. In any case, such formal segregation of the QM foundations is non-contradictive and is not discussed here.

In general, the QS is a dynamic system in which deterministic dynamics of the WF is synthesised with its statistical relation to observed values. The analogy between the properties of IFD and postulates of QM is evident. The postulates 1 and 2 define, respectively, dynamic and statistical parts of the system. WF is considered as the element of Hilbert space with the norm L_2. Dynamic part is deterministic dynamic system with WF φ as the state variable. This state does not have clear physical interpretation, but it has defined energy, impulse, and other characteristics (dynamic functionals), represented by EQF $J(u)$. Impulse, energy, angle moment, spin, and charge are the dynamic invariants of the system. Representations of the coordinates of the system and functions of these coordinates follow from statistical interpretation of the WF. Functionals $J(u)$ are interpreted as averages of physical values.

The main statistical postulate of quantum mechanics is the principle of superposition, usually understood as the law of addition of probabilistic amplitudes or the law of distribution over eigen values of the operator L_y. Generalised representation of these laws is comprised in the two postulates of von Neumann [1]:

1) Average of the observed values y is a quadratic form $\langle Y\rangle(u) = (u, L_y u)$;

2) If the function $z = f(y)$ is observed, then $\langle Z\rangle(u) = (u, f(L_y)u)$.

These equations are meaningful if the operator L_y is defined, which is provided by additional postulate about the centerring distributions around dynamic functionals J(u). This is the case of observation of energy and other dynamic invariants. When the coordinates are observed the distribution is stated explicitely: the function $|\varphi(.)|^2$ is the density of probability in the space of configurations (Born). The first postulate of von Neumann follows from the postulate of centerring distributions and from dynamic properties of quantum equations (J(u) is a quadratic functional) when energy and other dynamic invariants are observed. When the coordinates are observed this postulate follows from the the Born's postulate. The second postulate of von Neumenn serves as a "bridge" from the first one to the quadratic measure P(Q,u), and it is not as obvious.

Minimal desription of statistical quantum mechanics axioms, related to princile of superposition, thus, is as follows:

1) Born's postulate with related generalisations;
2) Centerring around values J(u);
3) Second postulate of von Neumann.

Note: From superposition principle and concretisation of eigen wave functions of impulse and coordinates there follows the relation of uncertainty. It is important to emphasise that the use of the terms "adjoint or simultaneous observation of several characteristics" is not really precise here. We mean independent measurement of characteristics for given WF, not disturbed by this measurement. If these terms are understood literally, there appear difficulies in synthesis of logically closed theory of such observation [11]. A vast series of studies suggests this type of synthesis based on new non-commutative theory of probability, [3,4,5].

3.1 New Axiomatics of Statistical Foundations of Quantum Mechanics.

Let us formulate the

A) Postulate of Infinitisimality of the Quantum Systems. QS is an infinitisimal dynamic system with WF as the state, observation results of which are described by probabilistic IFD.

This postulate minimises a priori information about statistical laws of QM: among above mentioned axioms only axiom 2) hold. Von Neumann's and Born's postulates become theorems; equation $p(.) = |\varphi(.)|^2$ is excluded from the definition of the WF.

Postulate of infinitisimality makes all wave equations linear and all functionals, which describe the system, quadratic, including the probabilistic measure $P(Q,\varphi)$. Thus, we establish the fundamental property of all characteristics of quantum system, which is conditionned by smallness of its dynamic state. This property restricts the area of application of quantum mechanics laws to linear systems.

Described postulates will be called general postulates of quantization. In their framework the distributions remain undefined. To fix them additional information is necessary, which will be defined in the form of special postulates. In distinction to general ones, these postulates are different for various QS.

3.2 Special Postulates of Energy-Impulse

.

Energy and impulse are dynamic invariants of quantum system, corresponding to known symmetry properties. They are defined accurate to constant addenda b_E, $\mathbf{b}_p$. Parametrical families of such functionals $J_t(u,b_E)$, $J_x(u,\mathbf{b}_p)$, corresponding the symmetry in relation to translation of time and space, will be called invariants of energy and impulse, unlike physical values "energy" E(u) and "impulse" $\mathbf{p}$(u), which have to be defined uniquely.

Dynamic functionals on the sphere $\|a\|^2 = 1$ of the space H_a can be represented as EQF with the spectrum $\lambda_k\|u_k\|^2$ +b, and eigen vectors α_k, independent of b and $r_k = \|u_k\|^2$.

Semi-positive invariant of the energy on this set of feasible states will be represented as:

$$J_t(u,b) = J_t(u,m) = (u,M\,u)+ m = (u,(M+m\mathbf{1})u), \qquad (2)$$

where M is the minimal semi-positive operator with eigen values $\mu_k = \lambda_k - \lambda_0 \geq 0$, and eigen vectors identical to u_k; $m \geq 0$ is any constant.

Statistical part of the QS occurs to be undefined by general postulates: observed values y_k are defined accurate to common addendum, and possibly, to scale multipliers r_k^2 (for relativistic systems), whereas the probabilities p_k are invariant with their variation. These parameters are fixed with special postulates. Thus, quantum system is defined if we have defined its dynamic part: space of configurations, wave equation, special postulates; and statistical part: operations of realization of quantum system in adequate physical conditions corresponding to given WF, set of observed characteristics, procedure of observation, corresponding IFD.

3.3 Non-Relativistic free Quantum Particle.

Let us consider in more detail the corollary of the infinitesimality postulate and of special postulates applied to quantum particle (QP). The WF $\varphi(.)$ is defined in a unit cube of the real Euclidian space E with the coordinates vector $\mathbf{x} = \{x^i\} = (x^1, x^2, x^3)$. Its evolution $\varphi(t,\mathbf{x})$ is defined by the Schrödinger equation. The norm $\|\varphi(t,\mathbf{x})\|$ is its dynamic invariant, and the WF are normalized: $\|\varphi(t,\mathbf{x})\| = 1$. The observation procedure is linked to the fixed moments of time. The natural special postulate of the absence of energy, impulse, etc., when the field is non-available, is inapplicable here, unlike the classical physics, since the value $||u|| = 0 \neq 1$ is inadmissible.

Definition. Given WF is considered to be not distinguishable from moving MT during observation of impulse **p** and energy E, if these characteristics are deterministic, dynamically invariant, and are related by one of the equations: $(\mathbf{p}^2/2m) = E$, $E^2 = (mc^2)^2 + (c\mathbf{p})^2$ for corresponding cases.

Dynamic invariance of impulse and energy is a common property of free wave field. Eigen vectors u_k of the EQF $J_x(u,b)$, $J_t(u,b)$ are deterministic. The undistinguishable from moving MT eigen vectors u_k are defined by these equations.

Special postulate of free wave field: there exist a set G of WF, which is not distinguishable from moving MT while observation of the energy-impulse and has the follow property: the requirement $E(u) = J_t(u)$, $p(u) = J_x(u)$, $\forall\, u \in G$, defines the energy and the components of impulse $E(u)$, $p_i(u)$, $i = 1,2,3$, $\forall\, u \in H$. The minimal set of such type will be denoted G_m.

Non-relativistic QP has the invariants

$J_t(u,b_E) = \int(\hbar^2/2m)\, \nabla\varphi^*\nabla\varphi\, dE + b_E$, $J_x(u,b_p) = -i\hbar\int\varphi^*\nabla\varphi\, dE + \mathbf{b}_p$, $dE = dx^1\, dx^2\, dx^3$, the eigen WFs $u_k = \exp[i(\mathbf{kx} - \omega_k t)]$ and the eigen values $J_t(u_k,b) = \hbar\omega_k + b_E$, $J_x(u_k,b) = \hbar\mathbf{k} + \mathbf{b}_p$, where **k** is a wave vector, ω_k is frequency, $k = \|\mathbf{k}\|$. According to the wave equation $\hbar\omega_k = (\hbar k)^2/2m$, the equalities $(J_x(u_k,b)^2/2m) = J_t(u_k,b)$, $k^1 = 0, 2\pi, 4\pi$, $k^2 = k^3 = 0$ are simultaneous only when $b_E = b_p = 0$. From this it follows that this triplet is actually the G_m: its elements are indistinguishable from the MP and the functionals $E(u) = J_t(u,0)$, $p_i(u) = J_{xi}(u,0)$ are defined, as well as that other u_k are also indistinguishable.

Given such definition of the set G_m, we assumed that the normalization is a part of definition of the WF. Let us abandon this fact. The eigen WFs are then defined accurate to the numerical multipliers r_k: $u_k = r_k\exp[i(\mathbf{kx} - \omega t)]$. Accordingly: $J_t(u_k,b,r_k) = |r_k|^2 (\hbar\omega_k)^2 + b_E$, $J_x(u_k,b,r_k) = |r_k|^2\, \hbar^2\omega_k\mathbf{k} + \mathbf{b}_p$.

The coefficients $|r_k|^2$ now are subject to fixation along with the b_E, $\mathbf{b}_p$. The totality G is a set of all eigen vectors u_k. The condition $(J_x(u_k,b)^2/2m) = J_t(u_k,b)$ holds when $b_E = b_p = 0$, $|r_k| = 1$. Thus, the normalization of the WF follows from the special postulate.

The statistical meaning of the function $|\varphi(t,x)|^2$ as of probability density follows from A) and the Special postulate of the free wave field.

3.4 Quantum-Mechanical Systems.

Let us consider the QP in the potential field. WF: $u = \varphi(t,\mathbf{x})$. Lagrangian: $\Lambda = i\hbar\varphi^*\varphi_t - (\hbar^2/2m)\, \nabla\varphi^*\nabla\varphi - V(\mathbf{x})\varphi^*\varphi$. Energy invariant: $J_t(u) = \int[(\hbar^2/2m)\, \nabla\varphi^*\nabla\varphi\, dx + V(\mathbf{x})\, \varphi^*\varphi]d\mathbf{E} + b_E$. When $b_E = 0$, it coincides with the conventional notion of energy. What additional information about the physical significance of the model corresponds to it? The second addendum here is the average potential energy of the MP $\langle V(\mathbf{x})\rangle$ with the density $\rho(\mathbf{x}) = |\varphi(\mathbf{x})|^2$. The first addendum is the energy of the free QP, which can be also represented as the average kinetic energy of the classical MP. Having assumed that

$b_E = 0$, we are postulating that the energy effect from appearance of the potential field is reduced to addition of average energy of this field to the energy of the QP. This postulate holds for any QS, which have a classical mechanical prototype reducible to the system of MPs in a potential field. From here it follows the correctness of its conventional eigen values, but only for this physical significance. For the QS with the same lagrangian, but of non-mechanic nature, it may occur that $b_E \neq 0$, and, therefore, the eigen values will be shifted. For instance, for the quantum-mechanical linear oscillator with the frequency ω we have: $\mu_k = k\hbar\omega$, $m = \hbar\omega/2$, $k = 0,1,2,\ldots$ However, the energy of the zero state of the quantum oscillator of the same model, but of different nature, is not fixed by the value $m= \hbar\omega/2$.

3.5 New Model of the Relativistic Quantum Particle.

A fundamental subject of relativistic quantum theory is the QF. But its synthesis is based on the model of free QP. Statistical part of the QP model is sufficiently formalized only for the non-relativistic case with the above-mentioned peculiarities. In relativistic theory this elegant logic no longer holds [9], pp. 13-17. Namely, the probability density of particle coordinates cannot be expressed in WF terms with necessary properties; precision of the coordinates observation is limited by the condition of absence of quantum-field effects; precision of impulse observation depends on the duration of observation. A new refinement of this model is considered here: the procedure of observation and the formal description and procedure of the observation. In the framework of this model, despite common opinion [9], there appears a theoretical possibility of observation of the space-time coordinates of boson, the density of probability is also defined, there is no dependence of precision of impulse observation on the duration of observation.

Let us begin by considering a scalar neutral boson with the mass m. Let E be a real pseudo-Euclidian space with coordinates vector $x= \{x^\alpha\}=(x^0,\mathbf{x})$, where $\mathbf{x}=\{x^i\}=(x^1,x^2,x^3)$ is a vector of space coordinates, an element of sub-space $\mathbf{E}$; $x^0 =ct$, t is time, c is light speed; metric tensor $e=\{e_{\alpha\beta}\}$ is diagonal: $e_{00}=-1$, $e_{ii}=1$, $i>0$. Wave function $\psi(x)$ is defined in the space E. All constructions must be relativistically covariant, in particular, WF is a scalar and satisfies a corresponding wave equation. Let us assume that these equations describe the motion of a single free particle. Wave process is considered in a cube $\mathbf{V} \subset \mathbf{E}$ within the time interval (0,T), or in a corresponding domain $V \subset E$. QP is considered as QS of a special kind in terms of above-mentioned definition. The applicability of this model is restricted by the condition of absence of quantum-filed effects.

Let us consider the following detailed version of the stochastic part of the QP.

Towards Observation Procedure. QS realization operations are defined in equal physical conditions corresponding the given WF. For each realization an observation of given set of dynamic characteristics with values $y \in R^n = Y$ is performed, which is not conditioned by a momentum of time t. At that one and only one event is being fixed: the particle has appeared (and possibly, has disappeared) at the point y of the set Y. Such model is corresponded, for instance, by the observation of a single photon (the object disappears after the reaction with the device), or by a single "momentary" measurement of a boson. The following measurements should be, in general, excluded from the consideration, as the QP is observed in them already in a different state, i.e. it is already not the same realization of the QS. Implementation feasibility of such a model is not discussed here. Model's features are studied assuming its applicability. Probabilistic measure $P(Q,\psi)$ corresponds to observation of infinite set of realizations for each fixed WF.

Towards Formal Description. Probabilistic measure $P(Q,\psi)$ is IFD in Hilbert space $L_2(V)$, and components of the dynamic functional $J(\psi)$ are EQF in $L_2(V)$. Eigen functions of the latter are identical, and spectra are linked with $P(Q,\psi)$ by equation $\langle Y\rangle(\psi) = J(\psi)$ and by theorems 1-3. Characteristics Y, and, correspondingly, their averages $J(\psi)$, have relativistic transformational features of tensors or their components. At the same time, $P(Q,\psi)$ is a relativistic invariant. Tensor dimension of probability density $g(y,\psi)$ is defined by the equation $dP = gdY = inv.$, where dY – is elementary volume of the observations space.

3.5.1 Observation of Coordinates. Scalar Boson.

Supplement to observation procedure: vector $y = x$ of the space-time coordinates of a particle is observed; the result of each measurement is fixation of the event: the particle appeared in the point x of space-time E.
Towards Formal Description. As $y = x$, then according to 2.4 $g(x,\psi)= |\psi(x)|^2$ when $\|\psi(x)\|=1$. Since $dY = dE$, density $g(x,\psi)$ is a relativistic scalar.

Therefore, in the framework of the discussed model the observation of boson coordinates becomes theoretically feasible, and relativistically invariant, nonnegative, quadratic probability density of space-time coordinates of a particle appears to be defined. Its expression in terms of WF is formally similar to probability density of coordinates of non-relativistic particle, but it has different content. Different observations conditions, different transformational features, different meaning of the function $|\psi(x)|^2$: probability density of events in space-time, instead of probability density of positions in space. Such a change in the meaning corresponds to SR logic and predicts a new feature of boson, still to be experimentally verified. Proposed detailed version of the relativistic QP model introduces considerable differences in understanding of this phenomenon, comparing to non-relativistic particle. Instead of stochastic dance of the particle, described by the flow of probability, we have probabilistic distribution of random appearance of a particle in space-time E, which can not be represented by the motion in space.

Knowing $g(x,\psi)$ and using known formulas, one can find out the probability density of positions $\mathbf{x}$, time x^0 , and particular coordinates.
Note 1. It may happen that the measurements will give the value $x^0 > cT$ (as well as $\mathbf{x} \notin V$). To avoid this, a sufficiently large value of T should be chosen. More precisely, $T\to\infty$, $l\to\infty$, while there should be defined the integral $\|\psi(x)\|^2$. Otherwise, the

measure $P(Q,\psi)$ does not satisfy the condition $P(E,\psi) = 1$. But relative probabilities $P(Q_1,\psi) / P(Q_2,\psi)$ are defined.

3.5.2 Observation of the Coordinates. Massive Vector Boson.

Theoretical admissibility of the formula $g = |\psi(x)|^2$ generalization for the case of vector particles derives from the fact that the relativistically invariant construction-analog $|\psi(x)|^2$ is non-negative. WF of such particles is a vector $u(x)$. We have: $uu^* = \mathbf{uu}^* - u^0u^{0*}$. In the coordinate system, related to a particle, $u^0 = 0$. Thus, $uu^* \geq 0$. This inequality is relativistically invariant. Therefore, $u(x)u^*(x)$ coincides with density of probability of appearance of related events in space-time E, accurate to a constant multiplier.

3.5.3 Observation of Coordinates. Photon.

For a photon $u(x)$ is electromagnetic vector-potential. It is defined accurate to gradient transformation. Thus, gradiently non-invariant expression $\mathbf{u(x)u^*(x)}$ is not defined and the probability density is not defined too.

3.5.4 Observation of Dynamic Invariants.

Let us return to the scalar neutral boson with the mass m. Observable Y is energy-impulse vector. Dynamic invariants

$$J_t(u,b_E) = \int \Pi^{00}(x^0,\mathbf{x})d\mathbf{E} + b_E = J_t(x^0) = \text{const.};\quad J_x(u,b_p) = \int \mathbf{\Pi}^0(x^0,\mathbf{x})d\mathbf{E} + \mathbf{b}_p = \text{const.};\quad (3)$$

where Π^{00}, $\mathbf{\Pi}^0$ – corresponding components of energy-impulse tensor:
$\Pi^{00} = \hbar^2\, \partial_t\varphi^*\partial_t\varphi + \hbar^2c^2\nabla\varphi^*\nabla\varphi + (mc^2)^2\, \varphi^*\varphi$; $\mathbf{\Pi}^0 = \hbar^2\, (\partial_t\varphi^*\nabla\varphi + \partial_t\varphi\, \nabla\varphi^*)$;
Accounting for eq.3, we can describe dynamic invariants as vector functionals in $L_2(E)$: $J_t(u,b_E) = T^{-1}\int \Pi^{00}(x^0,\mathbf{x})dE + b_E$; $J_x(u,b_p) = T^{-1}\int \mathbf{\Pi}^0(x^0,\mathbf{x})dE + \mathbf{b}_p$. They have equal eigen WF: $u_k = r_k\exp[i(\mathbf{kx} - \omega t)]$, where: $(\hbar\omega_k)^2 = (c\hbar\mathbf{k})^2 + (mc^2)^2$, r_k is any multiplier. Accordingly, $J_t(u_k,b,r_k) = 2r_k^2\, (\hbar\omega_k)^2 + b_E$, $J_x(u_k,b,r_k) = 2r_k^2\, \hbar^2\omega_k\mathbf{k} + \mathbf{b}_p$.

Norm $\|u_k\|^2 = (r_k)^2 = 1$ is incompatible with the requirement that waves u_k and relativistic MPs are indistinguishable. Now, coefficients $(r_k)^2$ are subject to fixation along with b_E, $\mathbf{b}_p$. Totality G is a set of all eigen vectors u_k. Condition $J_t(u_k)^2 = (mc^2)^2 + (c\, J_x(u_k))^2$ holds when $b_E = b_p = 0$, $r_k = (2\hbar\omega_k)^{-1/2}$. Thus, normalization for a particle in a unit volume derives from the Special postulate of a particle.

Let $u = \sum_k a_k u_k$, $||a|| = \sum_k|a_k|^2$. Energy-impulse vector takes values $y_k = (E_k,\mathbf{p}_k)$, $E_k = \hbar\omega_k$, $\mathbf{p}_k = \hbar\mathbf{k}$. In relativistic QM both, material and probabilistic versions of measure, make sense. In the former case the distribution is described in terms of average number of particles $N_k = |a_k|^2$. Such a classic boson field, generally speaking, is not a physical object, as it does describe neither a single particle, nor a system of particles. This is a vivid mathematical semi-finished product, a blank for secondary quantization. In the latter case, after additional normalization $||a|| = 1$, the field becomes a physical object, WF of single boson, with probability distribution $P_k = |a_k|^2$. It can be tested experimentally, should the described conditions of single boson observation be implemented.

In the case of charged field, a charge is added to energy and impulse, and in case of vector filed a spin is added. Let us stress that in each session a single particle should be observed. A new element here is exclusion of the time from the procedure and the results of observation. Instead of evolution of distributions we have the distribution of events: a particle is observed in the given dynamic state Y. The conservation laws by no means correlate with such information. For such correlation the time has to be added in the process of observation, at least in the minimal form: initial and final moments of realization of QS $t = 0$ и $t = T = \infty$, as in matrices of dispersion. At the same time, conservation laws, obviously only hold on average.

3.5.5 Relation of Uncertainty.

In terms of the traditional model of relativistic QP such relations are not topical, since the coordinates are not observable. In non-relativistic QM relations coordinates-impulse and time-energy are derived in a different way and have different meaning. In the model under consideration these relations are topical and possess full algorithmic and semantic symmetry. Special comprehension is required for the relation time-speed-impulse $v\ \Delta t\ \Delta p > \hbar$ (Born), and for sequential relativistic relation between impulse observation accuracy and the duration of observation session $\Delta t\ \Delta p > \hbar/c$, [9]. First of all, does the notion of "QP speed" make sense, and if so, what does it exactly mean?

3.6 Energy of Quantum Fields.

Algorithm of synthesis of quantum field, quantization of quantum mechanics equations, leads to known contradictions with theoretical foundations of quantum theory. The first one is a presence of no-observable energy of vacuum state, in the case of boson fields it is infinite. Usually, attention is paid namely to this contradiction – divergence of the energy of quantum field. But there is another contradiction: theoretical prediction contradicts the practice, since in reality only finite energy of particles can be observed. These contradictions comprise the paradox of the energy of boson fields. Commonly accepted technique to overcome this paradox is removal of energy of vacuum from eigen states of the energy of quantum fields. Such technique allows for identification of quantum field with secondarily quantified system of particles, but there is no the strict whys and wherefores for such approach, and thus, the paradox remain unresolved. We demonstrate that this paradox is a misunderstanding, explained by insufficient attention to the problem of special postulates of quantum systems, and namely, by full identification of free wave field with the system of mechanic oscillators. While such identification is acceptable only accurate to special postulates due to different nature of these systems. Additional definition of QS, and particularly, QF, above general postulates of quantization, should be based not on analogy of equations, nor on desirable results, but rather on precision of physical matter of QS, and for QF – on its genesis, coming from classical field, and further on, from relativistic free motion of the MP We demonstrate that the basis for such additional definition is the formulated above special postulate of free wave field, invariant property of these transformations, inherited for primary and secondary quantization.

3.6.1 Field Quantization and Genesis of Energy Paradox.

Let us come back to scalar neutral boson. Let WF be represented in terms of a series:

$$\varphi(t,\mathbf{x}) = \Sigma_{\mathbf{k}}\, q_{\mathbf{k}}(t)\varphi_{\mathbf{k}}(\mathbf{x}), \tag{4}$$

where $\varphi_k(x) = \exp(i\mathbf{kx})$; $q_{\mathbf{k}}(t)$ – complex functions, while $q_k{}^*(t) = q_{(-\mathbf{k})}(t)$. Then functions of Lagrange and Hamilton, and wave equation, are all expressed only via coefficients $q_{\mathbf{k}}(t)$:

$$L(q,q\cdot) = \Sigma_{\mathbf{k}}L_k(q_{\mathbf{k}},q\cdot_{\mathbf{k}});\;\; H(q,q\cdot) = \Sigma_{\mathbf{k}}H_k(q_{\mathbf{k}},\, q\cdot_{\mathbf{k}});\;\; \delta\!\int\! L(q,q\cdot)dt = 0, \tag{5}$$

where $q(t)=\{q_{\mathbf{k}}(t)\}$; $q\cdot = dq/dt$; $L_k(q_{\mathbf{k}},q_{\mathbf{k}}\cdot)$, $H_k(q_{\mathbf{k}},q_{\mathbf{k}}\cdot)$ are corresponding functions of harmonic oscillator D_k with frequency $\omega_k = (k^2c^2 + m^2c^4/\hbar^2)^{1/2}$.

Let us reproduce the quantization operation of the system eq.5 in the framework of Schrödinger wave formalism, directly related to the description of statistical model of the QS, used herewith. The space Q of generalized coordinates with elements q and the WF $\xi(t,q)$ are introduced. The latter is considered as a trajectory in Hilbert space H_q with the norm $L_2(Q)$, $\|\xi(t,q)\| = 1$. Lagrange function, energy invariant and variational equation for this WF are:

$$L(\xi,\xi^*) = (\xi,L\hat{}\xi);\quad H(\xi,\xi^*) = (\xi,H\hat{}\xi);\quad \delta\!\int\! L(\xi,\xi^*)dt = 0; \tag{6}$$

where L^, H^, are corresponding operators. Assuming the field to be stationary, and the systems D_k to be independent, we obtain: $\xi(t,q) = \exp(i\varepsilon t/\hbar)\eta_1(q_1)\eta_2(q_2)\ldots\eta_k(q_k)\ldots$, where ε is a parameter, $\eta_k(q_k)$ is the WF of k-th oscillator. According to eqs.5, 2:

$$H\hat{} = \Sigma_{\mathbf{k}}H\hat{}_k + m\mathbf{1};\;\; L\hat{} = -\,i\hbar\partial_{\mathbf{t}} + \Sigma_{\mathbf{k}}H\hat{}_k; \tag{7}$$

where $H\hat{}_{\mathbf{k}}$ – minimal semi positive operator of the energy of quantified linear oscillator D_k with coordinate $q_{\mathbf{k}}$, energy levels $n(k) = 0,1,\ldots$, corresponding eigen functions $\eta_{nk}(q_k)$, and energy values in formalization eq.2 $\mu_{nk} = n\hbar\omega_k$. Let us denote by $\{n(k)\}$ their given distribution within overall totality of oscillators. By virtue of eq.7 $\varepsilon =$

($(\xi(t,q),H^{\wedge}\xi(t,q))$. For each $\{n(k)\}$ there corresponds an eigen vector of the operator $H^{\wedge}$: $\xi[\{n(k)\}] = \Pi_k\eta_{n(k)k}(q_k)$ and its eigen value: $\varepsilon(\{n(k)\}) = \Sigma_k n(k)\hbar\omega_k + m$, $m \geq 0$ – a any given constant. Energy level $n(k) = 0,1,2,...$ is identified with the number of particles with energy $\hbar\omega_k$, $\{n(k)\}$ – distribution of particles numbers within these energies; m – energy of QP vacuum state.

For traditional values $\lambda[n(k)] = (n(k)+ 1/2)\hbar\omega_k$ there corresponds a paradoxical series $m = (\hbar/2)\Sigma_k\omega_k$. Since this series is infinite, the average energy and all $\varepsilon(\{n(k)\})$ are infinite. Usually, the emphasis is given to this contradiction: QP energy diversion. But this result is contradicting also in the case when the finite partial sum of the series (4) is recognized as a classical prototype: all $\varepsilon(\{n(k)\})$ are defined and according to superposition principle should be observed in measurements, but in reality only particles energy is observed. The set of these contradictions comprise the paradox of boson field's energy. This difficulty can be surmounted by volitional exclusion of vacuum energy from $\varepsilon(\{n(k)\})$. Such an approach has justified itself in the development of the theory as this energy does not participate in reactions, but the paradox in its foundation remains undisclosed. Assumption that the energy m is hidden from the observation contradicts the superposition principle as it fixes the distribution of namely observable values.

3.6.2 Non-contradictory Model of Quantum Field Energy.

The energy of vacuum state m is fixed by additional information about the QF, over and above what is fixed in its equations. According to section 3.4 for the value $m = (\hbar/2)\Sigma_k\omega_k$ there corresponds a refinement of physical matter of the oscillator model as of mechanical system of material points in the potential field. However, such interpretation is inapplicable to the system of oscillators D_k, which are not mechanical systems but components of the filed. In particular, coordinates q_k do not have proper physical sense. Given value m is not necessary for them. Let us consider theoretical foundation for fixation of m.

The system eq.5 is the result of the transformation of classical relativistic equations of free motion of MP. Once the equations are transformed, the system is getting randomized. But there should exist some sort of invariant of these transformations, which is inherited while both the primary and the secondary quantization. This invariant is the existence of the field states, indistinguishable from MP, more precisely described by the Special postulate of the free wave field (see 3.3). Its requirements are executable and they fix the energy and impulse of both QP and QF. Indeed, the totality G is a triplet $u_m = \xi_m(t,q) \in H_q$, $m = 0,1,2$, with distributions $\{n_k\} = \{\delta_{km}\}$. The absence of vacuum state energy follows from the above. Then all eigen values of energy invariant $\varepsilon(\{n_k\}) = \Sigma_k n(k)\hbar\omega_k$ are eigen values of system energy, and are identical to particles energy, and thus all contradictions are removed.

Thus, two approaches to fixing the energy of QP are discussed: a traditional one, based on the analogy of the equations of field components and mechanic oscillators, and extending this analogy to the values of any given constants; and a new approach, based on fixing of the general properties of free motion of MP and the fields, obtained as a result of primary and secondary quantization. It is demonstrated that the first approach is erroneous and leads to paradoxical results; while the special postulate of the second approach reflects the physical matter of the object and yields non-contradictive results.

Note. Besides opinions like "all will work fine would the theoretical predictions be slightly modified", theoretical ideas had been suggested about acceptability of exclusion of the vacuum energy, however not sufficiently stringent so that they could be considered as a solution to the problem. Let us continue the citation from [12]: «This infinity does not pose serious difficulties. In fact, setting the energy of the system we always can choose any additive constant. Having renormalized the energy scale in the way that the energy of one oscillator would start counting... from the energy of its zero state, we can neglect the infinite zero energy». As we have seen now, - not always. The following issues have to be addressed: when namely, why for the systems described by the same mathematical model, sometimes it is possible and sometimes not, and why in this case it is possible. Additional definition of QF above the general postulates should be based not on the analogy of the equations, but on the analysis of the genesis of the physical matter of the QF. The result of this analysis is Special postulate of free wave field, which fixes in the non-contradictive manner the energy and impulse not only of the QP, but also the QF. The evolution of the discussion on this problem in the textbooks is remarkable: it was intensive in 40-50s, [12], then rather humble in the following 20 years, [9], and was ignored in some modern textbooks, although through all these decades very little has changed in understanding of the problem. The demand in such understanding, apparently, is being successfully substituted by the effect of customization.

3.6.3 Generalizations.

The generalization of the obtained results is obvious for other free boson fields: massive vector fields, electromagnetic fields, and charged fields. It is sufficient to introduce addition over extra degrees of freedom into (4).

4 Conclusions.

Let us conclude the analysis of the foundations of the QM in the context of the obtained results.

4.1 Minimal Description of the Axioms, Related to the Principle of Superposition.

As it was mentioned above, the minimal desription of statistical quantum mechanics axioms, related to the princile of superposition is as follows:
1) Function $|\varphi(.)|^2$ is the probability density in the filed of configurations (Born) (with respective generalizations); 2) The distributions are centralized around the values J(u),

following from the WF equations; 3) Second postulate of von Neumann: let the L_y be an operator of the observed Y, if the function $z = f(y)$ is observed, then $<Z>(u) = (u,f(L_y)u)$. The first postulate of von Neumann (the observed average is EQF) follows from the dynamic properties of the QS (the functionals J(u) are EQF); when the coordinates are observed this postulate follows from 1).

In the light of the obtained results the minimal set of statistical axioms is shaped as follows: 1. The Postulate of infinitesimality of the QM: the results of QS observation are described by the probabilistic IFD with WF as a parameter; 2. See 2).

Postulate 1. is supplemented only by the postulate 2), which is necessary for fixing the operator of the observed value. Postulate 1) (the earliest statistical axiom of QM) becomes a theorem. From this it follows that the application area of the statistical laws of the QM is restricted by the conditions: WF equations are linear, and all functionals describing the system, including the probabilistic measure $P(Q,\varphi)$, are EQF; WF are small in L_2.

4.2. Refinement of the Model of Relativistic Quantum Particles (Bosons).

The elaboration of the model regards the formal description and the observation procedure. In terms of this model, despite the common opinion described in the textbooks, [9], there appears a theoretical possibility for observation of the boson's space-time coordinates, and their probability density occurs to be defined in the form $|\psi(x)|^2$. Formally, it is similar to the density of probability of coordinates of non-relativistic particle, but it has a different meaning: different conditions of observation, different transformation properties, different meaning of the function $|\psi(x)|^2$, and overall, different meaning of QP. Instead of stochastic dance of the latter, described by the flow of probability, we have probabilistic distribution of random occurrence of the particle in space-time, non-reducible. generally, to the motion in space. We also refine the theoretical model of observation and the distribution of energy - impulse of classical boson field applied to a single boson. These results are proposed for experimental verification, which feasibility depends mainly on implementation of correct physical realization of the model of single relativistic QP, and in particular, on the absence of QF effects.

4.3 Special Postulates of QM and Fixing the Energy of the QF.

Let us emphasize that the described general postulates of quantization leave the distributions under-defined. They are fixed by the special postulates, which contain additional information. Unlike general postulates, the special ones differ for different QS. Separately is considered the Special postulate of energy-impulse of the free field, which is completes definition of the distribution for free relativistic QP, and also fixes the energy of boson QF in a non-contradictive manner. On this basis we resolve the known contradiction between its observed distribution and the predictions of the quantum theory, and give theoretical background for excluding non-observable (infinite) energy of the vacuum state from the eigen states of the field energy.

References:

1. von Johann V. Neumann (1932) Matematishe Grundlagen der Quantenmechanik. Berlin, Verlag von Julius Springer.
2. J.M. Jauch (1968) Foundations of Quantum Mechanics. Addison –Wesley.
3. E.B. Davies (1976) Quantum Theory of Open Systems. Academic Press.
4. Ludwig G. (1967, 1968) Attempt of an Axiomatic Foundations of Quantum Mechanics and More General Theories. Communications of Mathematical Physics. 4, p.331-348; 9,p.192-211.
5. Pool J.T.(1968) Baer –Semigroups and the Logic of Quantum Mechanics. Communications of Mathematical Physics, 9, p.118-141.
6. Krotov V.F. (1991) Observation of linear systems and the principle of quantum mechanics. Soviet Physics Doclady, v.36, p.p. 18-20.
7. Krotov V.F. (2003) Quantization Property for Probability Distributions of Dynamic System Characteristics Under Random Disturbances in Observations. Avtomatika i Telemekhanika (translated as Automation and Remote Control), N1, pp. 86-104.
8. Akhieser N.I. and Glazman I.M. (1963) Theory of Operators in Hilbert Space. New York: Frederick Ungar.
9. Berestetsky, V.B., Lifshits, E.M. and Pitayevsky L.P. (1968) Relativistic Quantum Theory. Nauka Publ., Moscow, p. 23, 24. (in Russian)
10. Nelson E. (1966) Phys. Rev., v. 150, N4, p.1079.
11. Kholevo A.S. (1976 (Engl. Transl. 1978)) On general theory of statistical solutions. Working papers of Steklov Institute of mathematics, N 124, Nauka Publ., Moscow, issue 3).
12. Sokolov, A.A.(1958) Introduction to Quantum Electrodynamics. Fizmatgiz, Moscow, p. 36. (in Russian)