# Anticipation as the Source of Synchronization in Discrete Agents

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**Abstract**
Present paper address the synchronization classification of discrete agents by the means of z-transform. The developed agent based anticipative model enables us to change the future as well as the past chain of events. Emergent synchronization patterns determined by the application of z-transform provide the base for determination of stability regions in systems of higher complexities. One of the important results provided is that the proposed agent-based system is apparently controllable by considering the frequency response of the system. The proof of the stability for the arbitrary set of proposed anticipatory agents is proposed.
**Keywords** : agent, synchronization, discrete dynamics

## 1 Introduction

In the present paper, the anticipative formulation of the Kaldor's cobweb system will be developed according to the hyperincursivity paradigm [1, 2] which is applied to the explanation of the emergent holonomic properties of natural and artificial systems. Kaldor's cobweb model is particularly suitable for the examination of discrete dynamics systems since the model is well-known and examined. Extension by the incursivity paradigm should be considered as the approach to the examination of the system structure and its relation to the *time component*. The interaction of several modified cobweb systems will be analyzed in one of the next sections. The developed agent-based model addresses the interaction of $n$ agents incorporating the feedback-anticipative principle. Several such systems could be interconnected in order to form an agent-based model. Interaction between agents will be determined by the discrete rule, representing interactions between several economic systems which could be of great importance in cases of global systems instability. We will examine, how the anticipative component influences system response.

Dubois [1] characterized an incursive system from the contraction of "inclusive" or "implicit" recursion [3, 4].

**Definition 1.1** *Incursive system is defined by:*

$$x(t+1) = F[\ldots, x(t-1), x(t), x(t+1), \ldots] \tag{1}$$

*where the value of a variable $x(t+1)$ at time $t+1$ is a function of this variable at past, present and future times.*

Let us consider the Kaldor's model in a separated form written with difference operator $\Delta$ [5]:

$$
\begin{aligned}
P_{k+1} &= P_k + \Delta P_k &(2)\\
\Delta P_k &= \frac{d}{b}\Big(P_k - \frac{b\,P_k - c + a}{d}\Big) &(3)\\
Q^s_{k+1} &= Q^s_k + \Delta Q^s_k &(4)\\
\Delta Q^s_k &= \frac{d}{b}\Big(Q^s_k - \big(a + \frac{b}{d}(Q^s_k - c)\big)\Big) &(5)
\end{aligned}
$$

Application incursivity idea defined by Def. 1.1 to Kaldor's classic cobweb model yields the following set of equations for the anticipative cobweb model as one of *16* possibilities:

$$
\begin{aligned}
P(k+2) &= \frac{d}{b}\Big(P(k+1) - \big(\frac{b\,P(k) - c + a}{d}\big)\Big) &(6)\\
Q_s(k+2) &= \frac{d}{b}\Big(Q_s(k+1) - \big(a + \frac{b}{d}(Q_s(k) - c)\big)\Big) &(7)
\end{aligned}
$$

In Eq. (6) and Eq. (7) the $\Delta P_k$ from Eq. (3) is a function of present and past time, which has a physical meaning [6]. If we reformulate Eq. (6) and Eq. (7) the dependency of the future-present-past events could be observed:

$$
\begin{aligned}
P(k) &= \frac{bP(k-1) + a - c}{d} + \frac{b}{d}P(k+1) &(8)\\
Q_s(k) &= \frac{b}{d}Q_s(k+1) + \frac{b}{d}Q_s(k-1) + a - \frac{bc}{d} &(9)
\end{aligned}
$$

Eq. (8) and Eq. (9) state that the value of the present is dependent on the past as well as on the future, which in the linear case could be satisfied. The model stated in the *anticipative* form considers that the state value depends not only on the state value in the time $k-1$ but also on the state value in time $k+1$. The equation for the difference operator $\Delta$ has been transformed to the state equation while the time arguments that were applied are in the form $\{t-1, t, t+1\}$; here $t$ represents discrete time $k$.

## 2 Periodicity of Anticipative Kaldor Model

Fig. 1 represents an example of the synchronization results of the anticipative cobweb model. Different modes of cyclic behaviour response could be observed when parameter $d$ is varied. Synchronization patterns are named by the shape of the Poincaré first-return map representing the values of $P_k, P_{k+1}$. Fig. 1 represents the system before, in and after hexagon synchronization. The vertices converge to the edge point of the hexagon. Points on the vertices form the line at the periodic condition values for parameter $d$. The system is in transition to the next full polygon synchronization. Graphs have the time step shown on the $x$-axis, $P(k)$ on $y$-axis and $P(k+1)$ on $z$-axis. Part $A$ of Fig. 1 represents the system before synchronization, Part $B$ represents the system in synchronization (example of parameter values $a = 400$, $b = -20$, $c = -50$, $d = -20$ and $p = 160$) and Part $C$ after synchronization. Synchronisation is addressed here as the parameters' values for which the periodic stability of the system is manifested.

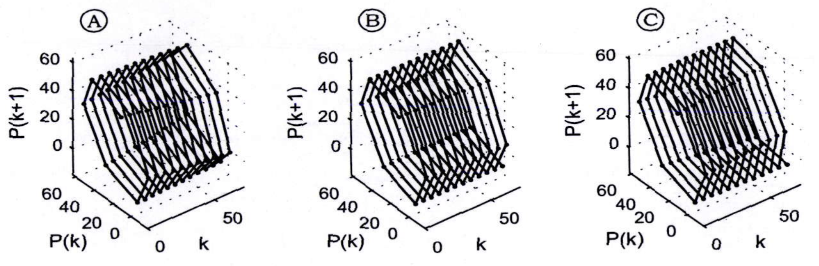


**Fig. 1**: *3-D* mapping of the system before ($A$), in ($B$) and after ($C$) hexagon synchronization; here the variation of the parameter $d$ value is performed.

The application of $z$-transform on Eq. (6) and Eq. (7) with initial conditions gives:

$$Y(z) = \frac{-y_1 z + y_0 d z - y_0 z^2}{-1 + dz - z^2} \tag{10}$$

Inverse $z$-transform yields the following solution:

$$\begin{aligned} Y^{-1}(z) &= 2^{-1-n}\, y_0 \left(d - \sqrt{-4+d^2}\right)^n - \frac{y_1 \left(d - \sqrt{-4+d^2}\right)^n}{2^n \sqrt{-4+d^2}} + \\ &+ \frac{2^{-1-n}\, y_0\, d \left(d - \sqrt{-4+d^2}\right)^n}{\sqrt{-4+d^2}} + 2^{-1-n}\, y_0 \left(d + \sqrt{-4+d^2}\right)^n + \\ &+ \frac{y_1 \left(d + \sqrt{-4+d^2}\right)^n}{2^n \sqrt{-4+d^2}} - \frac{2^{-1-n}\, y_0\, d \left(d + \sqrt{-4+d^2}\right)^n}{\sqrt{-4+d^2}} \end{aligned} \tag{11}$$

In order to gain conditions for the periodic response of the system the following equation should be solved:

$$Y^{-1}(z) \;=\; y_0 \tag{12}$$

Let us compute a numerical example of periodic solution applying the $z$-transform. The period examined will be the period of *9*, i.e. $n = 9$. In Eq. (12) one should put the condition $n = 9$. One of the possible solutions for the initial condition worth examining is the following:

$$d \;=\; \frac{1}{\left(\frac{1}{2}(-1+i\sqrt{3})\right)^{\frac{1}{3}}} + \left(\frac{1}{2}(-1+i\sqrt{3})\right)^{\frac{1}{3}} \tag{13}$$

By inspecting Eq. (13) and considering the equation for the roots of complex numbers [7]:

$$\sqrt[n]{z} = \sqrt[n]{r}\Big(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\Big) \tag{14}$$

the general form of the solution for parameter $d$ could therefore be defined as:

$$d = 2\cos\frac{2\pi m}{n} \tag{15}$$

where $n$ is the period and $m = 1, 2, 3, ..., n-1$. A similar procedure could be performed for the arbitrary period $n$. A more general solution, which applies parameter $b$, which was fixed for the purpose of determining solutions, is:

$$d = 2b\cos\frac{2\pi m}{n} \tag{16}$$

The computation of numerical values is shown in Table 1. The set of values is similar to the gained parameter values for the domain of *2-D* dynamic attraction by Sonis [8]. Asterisk* marks the critical points in calculation at period *2* and *4*.

Stability result corresponds to the polynomial $\lambda^2 = tr J\lambda - det J$ where periodic solutions will be considered. One should consider [8] for details. The discrete map stated according to Eqs. (8) and (9) should be analyzed according to the variation of parameter d and the determinant $\Delta = p^2 - 4q$. As shown in Tab. 1, the following classification of the periodic solution, gained by the *z-transform* could be drawn, shown in Fig. 2. One of the questions that arose at the analysis of similar *2d* systems is the question about the rule that determines the periodicity. In our case the change of parameter $d$ causes the system to switch between different

**Table 1**: Synchronization parameter $d$ values of periodicity conditions up to period 7

| Period $n$ | Shape symbol | Argument $\Omega = 2\pi\frac{m}{n}$ | Algebraic or alternative trigonometric d value representation | Num. value $d = 2\cos\frac{2\pi m}{n}$ |
|---|---|---|---|---|
| 2* | — | $\pi$ | | $-2.00000$ |
| 3 | △ | $\frac{2\pi}{3}$ | | $-1.00000$ |
| 4** | ◇ | $\frac{\pi}{2}$ | | $0.00000$ |
| 5 | ⬠ | $\frac{2\pi}{5}$ | $\frac{1}{2}(-1+\sqrt{5})$ | $0.61803$ |
| | ⛤ | $\frac{4\pi}{5}$ | $\frac{1}{2}(-1-\sqrt{5})$ | $-1.61803$ |
| 6 | ⬡ | $\frac{\pi}{3}$ | | $1.00000$ |
| 7 | | $\frac{2\pi}{7}$ | $\frac{1}{3}(-1+2\sqrt{7}\cos(\frac{1}{3}\arctan(3\sqrt{3})))$ | $1.24698$ |
| | | $\frac{4\pi}{7}$ | $\frac{1}{3}(-1+2\sqrt{7}\cos(\frac{4\pi}{3}+\frac{1}{3}\arctan(3\sqrt{3})))$ | $-0.44504$ |
| | | $\frac{6\pi}{7}$ | $\frac{1}{3}(-1+2\sqrt{7}\cos(\frac{2\pi}{3}+\frac{1}{3}\arctan(3\sqrt{3})))$ | $-1.80194$ |

equilibriums. The ordering of the equilibriums is determined by the general Eq. 16. The rational fraction $\frac{m}{n}$, which is in our case transformed by the Eq. 16 to the value of the parameter $d$, corresponds to the Farey sequence, which could be represented by the Farey tree. Fig. 2 represents the classification of the periodicity values. Aperiodic region is determined by the condition $\Delta > 0$ and the periodicity by the $\Delta < 0$. The vertical classification at $d < 0$ determines the angles which are determined by the three points in the 2-d map in our case, $\alpha_n < \frac{\pi}{2}$; $d > 0$, the angles of the map are $\alpha_n > \frac{\pi}{2}$. The strongest periodicity points are determined by the polygon structures in 2-d mapping. In Fig. 2, the polygons are marked near the hyperbola starting with *digon*, *triangle* etc. Other periodicity is the subset of the main sections that is determined by the $\sum \alpha_n$ and the Farey tree. The emergence of the system periodic stability in the shape of an $n$-sided polygon could be observed not only in economical systems [9]; the $n$-sided polygon and the Farey tree organization of the equilibria could be observed in the technical systems; for example, in laser control as the paradigm of the chaotic system [10].

## 3 Agent Based System

In order to analyze the interactions between several entities modelled as agents the following agent-based model is proposed. In our case, agent interaction represents an alternative control mechanism, which should provide standing oscillations and global equilibrium-seeking behaviour found in real world cases [12]. Consider the following agent-based anticipative system where the dynamics is denoted by the variable $P$ as the function of control parameters $f(a, b, c, d)$:

$$\mathbf{P}_k = \frac{b\mathbf{P}_{k-1} + a - c}{\mathbf{d}_k} + \frac{b}{\mathbf{d}_k}\mathbf{P}_{k+1} \tag{17}$$

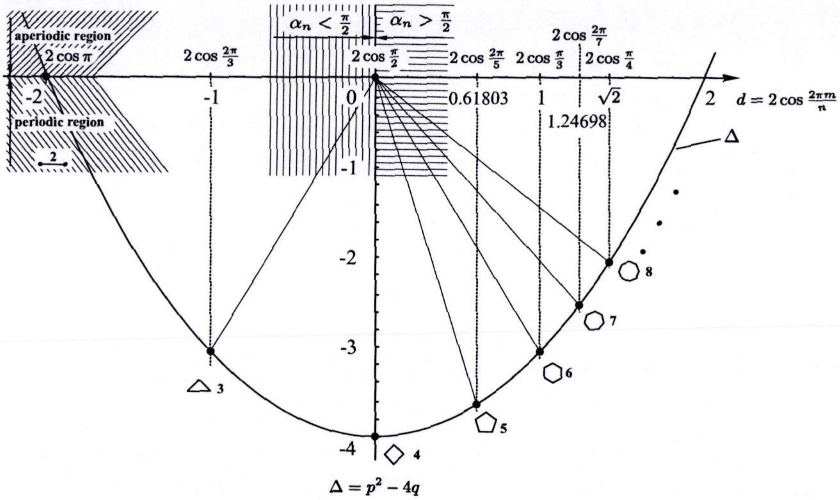


**Fig. 2**: Periodicity of the discrete 2D map — Classification of solutions according to the determinant [11]

Equation 17 captures the general feedback-anticipative mechanism of system control where the present state at time $k$ is dependent on the state at time $k-1$ as well as on the state at time $k+1$. Such model has many possible applications in the field of complex dynamics modelling. In the above equation the matrix annotation represents column vectors, which have the same arbitrary dimension $n$ determined by the number of agents. Initial conditions for Eq. (17) should be stated in matrix form as:

$$\mathbf{P}_{k+1} = \frac{p-a}{b} \tag{18}$$

$$\mathbf{P}_{k} = \frac{bP_{k+1}+a-c}{\mathbf{d}} \tag{19}$$

For the computation of the new values of $P$, shift operator $\rho$ on sequence $P$ is applied, which shifts sequence $p \in P$ one step to the left:

$$\rho(\langle \mathbf{P} \rangle) = \langle \mathbf{p}_{n+1} \rangle \tag{20}$$

providing the forward shifted values for $\mathbf{P}_{k-1}$ and $\mathbf{P}_k$ in Eq. (17). The decision of change in parameter $d$ will be dependent on the sum of two values of variable $P$ at time $k+1$ and time $k-1$. Here, the relative value of $P$ by taking the range of system response in the denominator will be considered:

$$\mathbf{e} = \frac{\xi_{k+1} + \xi_{k-1}}{|\lceil \xi_k \rceil - \lfloor \xi_k \rfloor|} \tag{21}$$

In Eq. (21) $\xi$ represents the estimation chain for $r$ time steps computed in a similar manner to $\mathbf{P}$ in Eq. (17) except for the initial conditions, which are stated in matrix form as in Eqs. (18, 19) for time 0, while for $\xi(0)$ the shift operator $\rho$ is applied forcing the anticipation principle as $\xi_k = f(\mathbf{P}_{k+1})$. Besides the notation for absolute value in the denominator, the *roof* and *floor* operators are applied. In order to perform the control by variation of parameter $d$, where $n$ agents are present, the following state equation with the adaptive rule for $\Delta\mathbf{d}_k$ is introduced:

$$\mathbf{d}_{k+1} \;=\; \mathbf{d}_k + \Delta\mathbf{d}_k \tag{22}$$

where $\Delta \boldsymbol{d}$ determines the change in control parameter $d$:

$$\Delta\mathbf{d}_k = \begin{cases} \beta & \text{if} \quad e = \lceil \mathbf{e} \rceil \\ -\beta & \text{if} \quad e = \lfloor \mathbf{e} \rfloor \end{cases} \tag{23}$$

In the above definition of the agent's rule, the *floor* and *ceiling* functions over a vector of relative values $\mathbf{e}$ reflect only a finite number of lags. One should notice that the mentioned *floor-roof* operators are applied on vector $\mathbf{d}$ rather than on vector $\mathbf{P}$, which would mean the strict, conventional implementation of the *floor-roof* principle [6]. Parameter $\beta$ is the *intensity* of the agent's reaction to the system disequilibrium; $\beta \in (0, 1)$. Initialization of vector $\mathbf{d}$ is determined by random value $r_i \in [-2, 2]$, which falls within the interval of periodic solutions [5, 8, 13, 14] for the anticipative agent-based system. Certainly, one could also assign an arbitrary value for $\mathbf{d}$ as this will also be considered.

The idea captured in the above definition considers a situation where an agent-based system where the state space values in the past and estimated future are at their peak, should be controlled by increasing the value of control parameter $d$, thus changing the frequency response of the system [15]. The case at the lower end of the system response is inverse.

Fig. 3 represents the interaction of eight agents as defined by Eqs. (17)-(23) as an example of the system response. Here the values of the parameters are: $a = 1$, $b = 1$, $c = 1$, $p = \frac{1}{2}$. On the $x$-axis the time step $k$ is represented and on the $y$-axis, the value of parameter $d$ is shown. Each line of the graph represents the variation of parameter $d$ for particular agent $A$. The synchronization plateaux can be observed from Fig. 3, which are marked as $\phi$ and $\Phi$. Synchronization is indicated as the plateaux in the system response, where dynamical equilibrium occurs. One of the main properties of the system of $n$ agents is that equilibrium could occur only as a trivial solution of the system, meaning that the system is not active, i.e. there is no interaction with the environment. This is the property of an agent-based system which should be considered by the proposed agent-based implementation. Therefore, in the case of equilibrium, system $S$ does not exist hence $\nexists\, S$; the system is closed. Another important property which should be considered by the proposed agent-based implementation is the dynamical equilibrium synchronization of agent response. Analyzing real time series of interacting agents, one could observe vivid synchronization plateaux [10, 16].

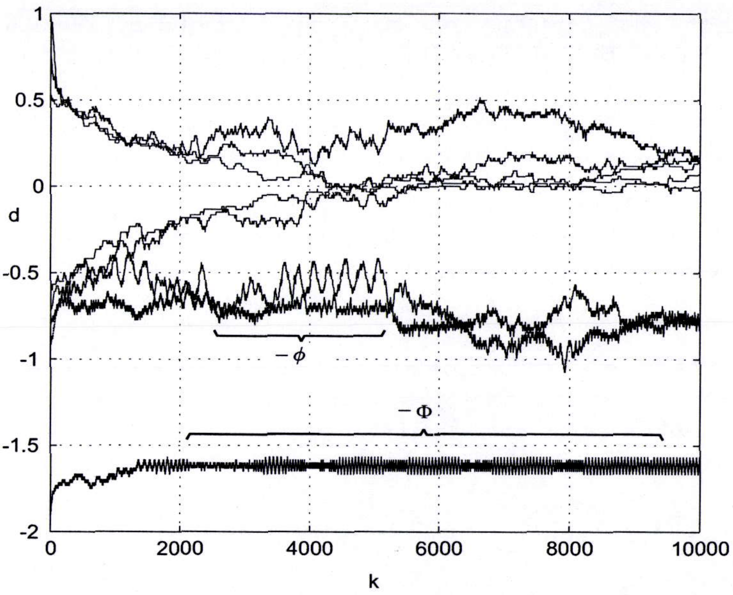


**Fig. 3**: Agents (8) interacting with synchronization plateaux at $\phi$; each curve represents the adjustment dynamics of a particular agent.

## 4 System Stability

In our case, system stability of the anticipative agent-based model is dependent on the value of agent reaction $\beta \in (0, 1)$. Higher values of $\beta$ result in a higher volatility of system response. Another important variable is the number of anticipative agents $n$ considered. Note, that the value for parameter $\mathbf{d}$ is not limited, i.e. by applying $\Delta d$ the values could range from $-\infty$ to $+\infty$, while the standing oscillation response of the agent could only be possible in the interval $d \in [-2, 2]$. Therefore, let us formulate the following proposition:

**Proposition 4.1** *The anticipative agent-based system defined by Equation 17 is stable if $n = 2$, $\Delta d = 1 - \phi$, initial conditions $d_1 = 1$, $d_2 = -1$ when $k \to \infty$; and $\mathbf{d} \in [-1, 1]$.*

*Proof.* Due to the feasibility of proof the interaction of two agents will be analyzed here. One of the agents will be marked with the $[+]$ sign as $A_{[+]}$ and the other with the $[-]$ sign as $A_{[-]}$ due to the initial conditions for the value of parameter $d$. The observed parameters are: $a = 1$, $b = 1$, $c = 1$, $d_{[+]}(0) = 1$, $d_{[-]}(0) = -1$, $p = \frac{1}{2}$. Let us consider the value for $\Delta d = 1 - \phi$. For the first step the value for $\mathbf{d}$ is $d_{[+]}(1) = d_{[+]}(0) - \Delta d$ and $d_{[-]}(1) = d_{[-]}(0) + \Delta d$.

$$\mathbf{d}(1) = \begin{bmatrix} d_{[+]}(0) - \Delta d \\ d_{[-]}(0) + \Delta d \end{bmatrix} \tag{24}$$

By examining the system response, one would expect that further values for $\mathbf{d}$ would vary in the interval $[-1,+1]$. According to this proposition, the following should hold

$$\mathbf{d}_{k+1} = \begin{bmatrix} 1-\Delta d \\ -1+\Delta d \end{bmatrix} \quad if \quad \mathbf{d}_k = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \forall\, \mathbf{P} \in \Re/\{0\} \tag{25}$$

This would limit the value for parameter $\mathbf{d}$ in the prescribed interval $[-1,+1]$. There is an exception at critical point 0. In a further investigation of system stability, one should consider the following condition for parameter $\mathbf{d}$:

$$(|P_{[+]k+1}| = |P_{[-]k+1}|) \wedge (|P_{[+]k}| = |P_{[-]k}|) \wedge (|P_{[+]k-1}| = |P_{[-]k-1}|) \tag{26}$$

Eq. (26) determines the parameter space in which the solution of the system could exist.

For the condition $(P_{[-]k-1} = P_{[+]k-1}) \wedge (P_{[-]k} = -P_{[+]k})$ at $d_{[+]}(0) = 1$ and $d_{[-]}(0) = -1$, the condition $e_{[+]} \leq e_{[-]}$ from Eq. (23) is determined by two planes:

$$\alpha = \frac{P_{k-1}}{|\max(s_1)| + |\min(s_2)|} + \frac{P_k - P_{k-1}}{|\max(s_3)| + |\min(s_4)|} \tag{27}$$

$$\beta = \frac{P_{k-1}}{|\max(s_5)| + |\min(s_6)|} + \frac{P_k - P_{k-1}}{|\max(s_7)| + |\min(s_8)|} \tag{28}$$

In Eq. (27) and Eq. (28) $s_n$ represents the sequence from $\mathbf{P}$ and $\xi$. On account of the periodicity condition, which is met at $d \in \{-1, -\phi, \phi, 1\}$, the minimum number of sequence values are taken for determining the planes. Planes $\alpha(P_{k-1}, P_k)$ and $\beta(P_{k-1}, P_k)$ cross symmetrically with respect to the origin, which could be proven by a reduction of Eq. (27) and Eq. (28):

$$\frac{P_k}{|\max(\varsigma_a)| + |\min(\varsigma_b)|} < \frac{P_k}{|\max(\varsigma_c)| + |\min(\varsigma_d)|} \tag{29}$$

Inequality defined by Eq. (29), where $\varsigma$ represents proper system sequence, holds except for the critical point 0 and limit values as $P \to \pm\infty$. The condition $e_{[+]} \leq e_{[-]}$ for the postitive combination of signs is met, meaning that for such values of $\mathbf{P}$ the direction of parameter $d$ change is correct. The procedure for the negative set of signs is performed respectively.

The above procedure does not provide an answer to what will happen in the limit. The answer is provided by the following four limits:

$$\lim_{P_k \to -\infty} \frac{P_k(\sqrt{5}-1)}{2(|\max(\psi_1)| + |\min(\psi_2)|)} = \frac{1-\sqrt{5}}{2(\sqrt{5}+1)} \tag{30}$$

$$\lim_{P_k \to \infty} \frac{P_k(\sqrt{5}-1)}{2(|\max(\psi_1)| + |\min(\psi_2)|)} = \frac{\sqrt{5}-1}{2(\sqrt{5}+1)} \tag{31}$$

$$\lim_{P_k \to -\infty} \frac{P_k(\sqrt{5}-1)}{2(|\max(\psi_3)| + |\min(\psi_4)|)} = \frac{1}{4}(\sqrt{5}-3) \tag{32}$$

$$\lim_{P_k \to \infty} \frac{P_k(\sqrt{5}-1)}{2(|\max(\psi_3)| + |\min(\psi_4)|)} = \frac{1}{4}(3-\sqrt{5}) \tag{33}$$

The results of limits in Eqs. (30, 31, 32, 33) with fulfilled conditions for $-\infty < P_k < +\infty$ including critical point 0 confirm that the critical stability condition is not met for $P_k \in [-\infty, +\infty]$, thus providing a proper change of parameter $\mathbf{d}$ in a critical step before $\mathbf{d}$ takes the values $d_{[+]k+1}(0) = 1$ and $d_{[-]k+1}(0) = -1$ $\square$.

## 5 Conclusion

The feedback–anticipative principle is an important concept in the modelling of multi-agent systems. The Dubois anticipative paradigm [1, 17] could be further extended to the field of hyperincursive systems. It is important to know that complex systems such as multi-agent systems incorporate two loops: a) a feedback loop and b) an anticipative loop. These two loops inevitably produce oscillatory behaviour of the system which is the main property of real world complex agents. By the proposition of an agent based system stated in the form of linear system with nonlinear rule of interaction the periodic response was determined with a significant $\phi$ value in the example of system response. The gained periodicity results are applicable in further analysis of interacting agent-based systems [11, 18, 19].

The analysis of the agent-based model provides proof of system stability, which is one of the key conditions that should be meet by agent-based models simulating complex systems. The provided proof of system stability for the case of two agents, which also provides promising results for the *n-agent* case, confirms that the model could be set in the global equilibrium mode. All the stated characteristics of the agent-based model as well as the response of the system for eight agents provides a promising methodological platform for the study of the interaction between several agent-based systems that incorporate feedback-anticipative principles. The proposed model provides the means for analyzing interaction, feedback, anticipation, frequency response, synchronization, standing oscillations and system equilibrium. An introduction of *feedback-anticipative* systems interconnection and control by varying the parameter, which influences system frequency response, represents a new perspective for the analysis of complex evolutionary agent-based systems. The findings presented here provide the interaction rules of program agents with potential business applications in the field of informational systems.

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