# Relativistic Quantum Mechanics from a Single Operator

Peter Rowlands
Department of Physics, University of Liverpool, Oliver Lodge Laboratory, Oxford St, Liverpool, L69 7ZE, UK.
email p.rowlands@liverpool.ac.uk

**Abstract**
Relativistic quantum mechanics can now be constructed minimally from a single creation operator with explicit energy, momentum and mass terms. The phase factor, amplitude, spinor structure and vacuum states are all automatic consequences of the initial definition. As separately-defined entities they are completely redundant. The operator can even be reduced to two terms (energy and momentum) if differentiation is defined in a discrete sense. This version of quantum mechanics is also a full quantum field theory, with an automatic incorporation of vacuum and second quantization. Renormalization is, in principle, eliminated by the intrinsic (vacuum) supersymmetry of the fermion and boson structures, while the fundamental interactions of particle physics are consequences of the mathematical structure alone, and do not require any additional 'physical' assumptions.
**Keywords:** relativistic quantum mechanics, creation operator, quantum field theory, particle physics, universal rewrite system.

## 1 Introduction

Quantum mechanics and quantum field theory have been expressed using many different mathematical formalisms. Formalisms, however, are only as good as the physics they are able to encompass or generate – they have no intrinsic validity based on mathematical characteristics. Even the use of particular algebras does not determine the formalism which produces the most useful physical information. This is especially true of formalisms based on Clifford or geometrical algebra (for example, Hestenes, 1966, 1967, 1975). Thus, while it is mathematically convenient to define wavefunctions as elements of minimal left- or right-ideals (so giving them idempotent characteristics) as an explanation for the need of a spinor structure, this turns out to be a relatively inconvenient description for physical purposes. However, a much more powerful *nilpotent* formalism can be developed which can be easily related to the idempotent description, but which creates a version of quantum mechanics with an immediately holistic structure not available using any other formalism. Idempotents still have a significant role in this representation (as discrete vacuum operators), but nilpotents, which seemingly give a less general picture in mathematical terms, are unique in providing the full range of physical information required, and in generating the most significant physical results while simultaneously reducing the formal apparatus to a minimum.

## 2 The Nilpotent Version of Quantum Mechanics

The nilpotent Dirac equation (Rowlands 2004, 2005, 2006, 2007) can be derived relatively simply by first taking the classical relativistic energy-momentum-mass equation in the form:

$$E^2 - p^2 - m^2 = 0, \tag{1}$$

then factorizing using noncommuting algebraic operators (multivariate 4-vector quaternions or complex double quaternions):

$$(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m), \tag{2}$$

before, finally applying a canonical quantization to the first bracket, with $E$ and $\mathbf{p}$ in the first bracket becoming canonical quantum operators, say $i\,\partial / \partial t$ and $-i\nabla$, with $\hbar = 1$, rather than numerical variables. The first bracket can then be seen as an operator, operating on a phase term of some kind, say $e^{-i(Et - \mathbf{p.r})}$ for a free particle plane wave, and the second bracket the amplitude which results from this operation.

$$\begin{aligned}&(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\\ &= (\mp\, \boldsymbol{k}\partial / \partial t \mp\, i\boldsymbol{i}\nabla + \boldsymbol{j}m)(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\, e^{-i(Et - \mathbf{p.r})} = 0.\end{aligned} \tag{3}$$

This now becomes equivalent to the Dirac equation for a free fermion, and the amplitude has the property, displayed in equation (2), of being a nilpotent, or square root of zero.

The key transition, however, is to assume that, even when the fermion is not a free state and the operators $E$ and $\mathbf{p}$ are not equivalent simply to $\partial / \partial t$ and $\nabla$, but, say, to covariant derivatives or to operators involving field terms, that the amplitude remains nilpotent and that this is the defining characteristic of the fermion state. As soon as this step is taken, we depart from the conventional emphasis on the Dirac equation as the basis of relativistic quantum mechanics and privilege instead the operator represented by $(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, irrespective of the composition of the terms $E$ and $\mathbf{p}$. Of course, if the fermion is not a free state, then the phase term will no longer be $e^{-i(Et - \mathbf{p.r})}$, but some expression which, when operated by $(\pm\, i\boldsymbol{k}E \pm \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$, produces an amplitude which is a square root of zero. In this case, both the phase factor and the amplitude will be uniquely determined once the operator is defined, and hence become redundant as independent information. The same also applies to the quantum mechanical equation. Although an equation can be constructed from the operator, it does not exist independently of it, and the derivation of phase factor and amplitude directly from the operator is actually more true than their derivation from any quantum mechanical equation because it does not depend on the (often incorrect) assumption that the amplitude is a constant.

However, even the operator contains redundant information. The expression which is abbreviated as $(\pm ikE \pm \boldsymbol{i}\mathbf{p} + jm)$ is really a row vector containing four terms, expressing the relative sign variations in $E$ and $\mathbf{p}$:

$$(ikE + \boldsymbol{i}\mathbf{p} + jm) \quad (ikE - \boldsymbol{i}\mathbf{p} + jm) \quad (-ikE + \boldsymbol{i}\mathbf{p} + jm) \quad (-ikE - \boldsymbol{i}\mathbf{p} + jm), \tag{4}$$

It is a 4-component *spinor*, which incorporates the variations representing fermion / antifermion ($\pm E$) and spin up / spin down ($\pm \mathbf{p}$). However, in the nilpotent structure, the sign variation is identical for all fermion states, and so only the first or lead term represents information. In other words, the nilpotent operator entirely removes the need for using such mysterious objects as wavefunctions and spinors. They are strictly redundant, along with phase factors, amplitudes, and quantum mechanical equations. Of course, it will often be convenient to use such terms, but, in every case, they will be constructible uniquely by a completely standard procedure as soon as the first term of the operator is defined.

## 3 Consequences of Quantum Mechanics

The nilpotent contains four creation operators, which, as represented in (4), are respectively for fermion spin up, fermion spin down, antifermion spin down, antifermion spin up. The first or ('lead') term decides the nature of the real particle state; the other three are the vacuum 'reflections', representing the states that the particle could transform into. They are produced by respective $P$, $T$ and $C$ transformations, or by respective pre- and post-multiplication of the lead term by the quaternion operators $\boldsymbol{i}$, $\boldsymbol{i}$; $\boldsymbol{k}$, $\boldsymbol{k}$; $-\boldsymbol{j}$, $\boldsymbol{j}$. Pre-multiplication of the lead term by $\boldsymbol{i}$, $\boldsymbol{k}$, $\boldsymbol{j}$ produces idempotent vacuum states, which can be described respectively as 'strong', 'weak' and 'electric'. The combination of the lead term with the real state equivalents of the three 'vacuum' terms leads to the production of boson-type states, which are, respectively, the fermion-fermion pairing observed in Cooper pairs and other applications of the nonzero Berry phase; spin 1 boson; and spin 0 boson (a state that cannot be massless in nilpotent theory because $(\boldsymbol{i}kE + \boldsymbol{i}\mathbf{p})\ (-\boldsymbol{i}kE - \boldsymbol{i}\mathbf{p}) = 0$). The four states in the nilpotent representation can, of course, be rotated to make any of the three vacuum states into the lead term or real state.

We may ask: what is the physical meaning of defining the fermion as an operator? What is it operating on? The indications are that it is *vacuum*, meaning the rest of the universe. For a 'free' fermion, the phase factor $(exp\,(-i(Et - \mathbf{p}.\mathbf{r}))$ provides the complete range of space and time translations and rotations, but if the $E$ and $\mathbf{p}$ terms represent covariant derivatives or incorporate field terms, then the phase term is determined by whatever expression is needed to make the amplitude nilpotent. Annihilation, of course, requires the opposite signs for $E$ and $\mathbf{p}$ to creation, and the four terms in (2) could be equally taken as four creation operators, four annihilation operators, or as two creation operators and two annihilation operators, together producing a totality zero.

The $\mathbf{p}$ term in the nilpotent is a multivariate vector and already contains the concept of spin, as a result of the full product of multivariate vectors containing both scalar and vector product terms (Hestenes, 1966, 1967, 1975; Gough, 1990). So the nilpotent can also be represented in a more conventional notation as $(ikE + i\boldsymbol{\sigma}.\mathbf{p} + jm)$ or $(ikE - i\mathbf{1}.\mathbf{p} + jm)$, or even $(ikE - i\mathbf{1}\mathbf{p} + jm)$, since $(\boldsymbol{\sigma}.\mathbf{p})(\boldsymbol{\sigma}.\mathbf{p}) = \mathbf{pp} = p^2$. However, because there are four spin states coexisting in the fermionic nilpotent, $\boldsymbol{\sigma}$ can be treated as a dynamical variable. The conventional derivation of half-integral spin follows immediately from

$$[\hat{\boldsymbol{\sigma}}, \mathcal{H}] = [-\mathbf{1}, -j(\mathbf{i}p_1 + \mathbf{j}p_2 + \mathbf{k}p_3) + ikm] = 2ij\,\mathbf{1} \times \mathbf{p}$$

$$[\mathbf{L}, \mathcal{H}] = -ki\,[\mathbf{r}, \mathbf{1}.\mathbf{p}] \times \mathbf{p} = -j\,[\mathbf{r}, \mathbf{1}.\mathbf{p}] \times \mathbf{p} = -ij\,\mathbf{1} \times \mathbf{p},$$

$$[\mathbf{L} + \hat{\boldsymbol{\sigma}} / 2, \mathcal{H}] = 0, \tag{5}$$

which makes $\mathbf{L} + \hat{\boldsymbol{\sigma}} / 2$ a constant of the motion. The spin term here derives specifically from using a multivariate vector p, but, of course, if we take $\mathbf{p}$ or $\nabla$ as a conventional vector, as for example, when we are using polar coordinates, we will necessarily need to include an explicit spin term.

Because of the way they are defined, nilpotent operators are specified with respect to the entire quantum field, they are already second quantized, and a formal second quantization process becomes unnecessary. In effect, the nilpotency condition can be taken as defining the interaction between a localized fermionic state and the unlocalized vacuum or 'rest of the universe', with which it is uniquely self-dual, and the phase becomes the mechanism through which this is accomplished. Defining a fermion, therefore, implies simultaneous definition of vacuum as 'the rest of the universe' with which it interacts. (In terms of the universal rewrite system previously defined, the fermion and the rest of the universe total zero as $(ikE + i\mathbf{p} + jm)$ and $-(ikE + i\mathbf{p} + jm)$.) The nilpotent structure then implies energy-momentum conservation without requiring the system to be closed. The nilpotent structure is thus naturally *thermodynamic*, and provides a mathematical route to defining nonequilibrium thermodynamics. The nilpotent condition (2), thus, appears to have at least *five* independent meanings:

| | |
|---|---|
| classical | special relativity |
| operator × operator | Klein-Gordon equation |
| operator × wavefunction | Dirac equation |
| wavefunction × wavefunction | Pauli exclusion |
| fermion × vacuum | thermodynamics |

Nilpotent operators are also *intrinsically supersymmetric*. The conversion from fermion to boson is by multiplication by an antifermionic operator; the conversion of boson to fermion is by multiplication by a fermionic operator. If we repeatedly post-multiply a fermion operator by any of the discrete idempotent vacuum operators, we will create an alternate series of antifermion and fermion vacuum states, or, equivalently, an alternate series of boson and fermion states without changing the character of the real state. We can interpret this immediately as the series of boson and fermion loops, of the same energy and momentum, required in an exact supersymmetry. Fermions and bosons become their own supersymmetric partners through the creation of these vacuum states. The mutual cancellation of the boson and fermion loops then eliminates the need for renormalization and removes the hierarchy problem altogether.

## 4 Idempotents and Nilpotents

Conventional relativistic quantum mechanics has been assumed to be idempotent (**AA** = **A**), rather than nilpotent (**AA** = 0), but the vacuum operators in the nilpotent theory show that idempotents are also important there. However, the nilpotent theory is a much more significant development than one based on idempotents, because it is the nilpotent nature of the theory that allows us to use constraints, based on zeroing, to remove redundant information. But there is no fundamental conflict, for we can see that the nilpotent equation actually incorporates an idempotent equation. The equations are precisely the same – the difference is purely one of interpretation. There isn't even a transformation required, just a redistribution of algebraic operators between differential operator and amplitude. So the alternative interpretations are:

IDEMPOTENT

$$[(\boldsymbol{ik}\partial / \partial t + \boldsymbol{i}\nabla + \boldsymbol{j}m)\,\boldsymbol{j}]\,[\boldsymbol{j}\,(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,e^{-i(Et - \mathbf{p.r})}] = 0. \qquad (6)$$

*operator* *wavefunction*

NILPOTENT

$$[(\boldsymbol{ik}\partial / \partial t + \boldsymbol{i}\nabla + \boldsymbol{j}m)\,\boldsymbol{j}\,\boldsymbol{j}]\,[(\boldsymbol{ik}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)\,e^{-i(Et - \mathbf{p.r})}] = 0. \qquad (7)$$

*operator* *wavefunction*

## 5 A Discrete Version of Nilpotent Quantum Mechanics

The nilpotent operator has three terms, compartmentalised using the quaternions ***k***, ***i***, ***j***, in a similar way to real and imaginary parts. If we use discrete differentiation, we can even reduce it to two. In discrete differentiation, as defined by Kauffman (2004), to preserve the Leibniz rule, we take

$$\frac{dF}{dt} = [F,H] = [F,E] \quad \text{and} \quad \frac{\partial F}{\partial X_i} = [F,P_i] \qquad (9)$$

The mass term disappears in the operator (though it has to be introduced in the amplitude). Suppose we define a nilpotent amplitude

$$\psi = \boldsymbol{ik}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m \qquad (10)$$

and an operator

$$\partial\!\!\!/ = i\boldsymbol{k}\frac{\partial}{\partial t} - \boldsymbol{i}\mathbf{i}\frac{\partial}{\partial X_1} - \boldsymbol{i}\mathbf{j}\frac{\partial}{\partial X_2} - \boldsymbol{i}\mathbf{k}\frac{\partial}{\partial X_3}, \tag{11}$$

with

$$\frac{\partial \psi}{\partial t} = [\psi, H] = [\psi, E] \quad \text{and} \quad \frac{\partial \psi}{\partial X_i} = [\psi, P_i]. \tag{12}$$

The + and – signs for the differentials are, of course, arbitrary, but are chose to match up with those of the nilpotent Dirac equation (3). In addition, we can use $\partial\psi / \partial t$ rather than $d\psi / dt$, here, because we are making no explicit use of a velocity variable. This means that

$$\begin{aligned} i\boldsymbol{k}\frac{\partial \psi}{\partial t} &= -\boldsymbol{k}[\psi, E] = \boldsymbol{k}\,\psi E + \boldsymbol{k}E\,\psi = \boldsymbol{k}\,\psi i\boldsymbol{k}i\boldsymbol{k}E - \boldsymbol{k}E\,\psi \\ &= i\,\psi i\boldsymbol{k}E - \boldsymbol{k}E\,\psi - 2iE^2 \end{aligned} \tag{13}$$

and

$$\begin{aligned} \boldsymbol{i}\mathbf{i}\frac{\partial \psi}{\partial X_i} &= -i\boldsymbol{i}\mathbf{i}[\psi, P_i] = -i\boldsymbol{i}\mathbf{i}\,\psi P_i + i\boldsymbol{i}\mathbf{i}P_i\psi = i\boldsymbol{i}\mathbf{i}\,\psi\boldsymbol{i}\mathbf{i}\boldsymbol{i}\mathbf{i}P_i + i\boldsymbol{i}\mathbf{i}P_i\psi \\ &= i\,\psi\boldsymbol{i}\mathbf{i}P_i + i\boldsymbol{i}\mathbf{i}P_i\psi - 2i\boldsymbol{i}\mathbf{i}P_i\boldsymbol{i}\mathbf{i}P_i = i\,\psi\boldsymbol{i}\mathbf{i}P_i + i\boldsymbol{i}\mathbf{i}P_i\psi + 2iP_i^2. \end{aligned} \tag{14}$$

Now, if $m$ is a scalar, we may use the identity

$$0 \equiv \boldsymbol{j}\psi m - \boldsymbol{j}m\psi = -\boldsymbol{j}\,\psi\boldsymbol{j}\boldsymbol{j}m - \boldsymbol{j}m\,\psi = -\psi\boldsymbol{j}m - \boldsymbol{j}m\,\psi - 2\,\boldsymbol{j}m\boldsymbol{j}m = -\psi\boldsymbol{j}m - \boldsymbol{j}m\,\psi - 2m^2. \tag{15}$$

Combining equations (13)-(15), term by term, we obtain

$$\begin{aligned} \partial\!\!\!/\,\psi = {} & i\,\psi(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m) \\ & + i(i\boldsymbol{k}E + \boldsymbol{i}\mathbf{i}P_1 + \boldsymbol{i}\mathbf{j}P_2 + \boldsymbol{i}\mathbf{k}P_3 + \boldsymbol{j}m)\,\psi - 2\,i(E^2 - P_1^2 - P_2^2 - P_3^2 - m^2). \end{aligned} \tag{16}$$

When is $\psi$ nilpotent, then

$$\partial\!\!\!/\,\psi = \left(\boldsymbol{k}\frac{\partial}{\partial t} + i\boldsymbol{i}\nabla\right)\psi = 0 \tag{17}$$

or, in fuller form:

$$\partial\!\!\!/\,\psi = \left(\pm\boldsymbol{k}\frac{\partial}{\partial t} \pm i\boldsymbol{i}\nabla\right)\psi = 0, \tag{18}$$

where

$$\psi = \pm\, ikE \pm \mathbf{i}\mathbf{i}P_1 \pm \mathbf{i}\mathbf{j}P_2 \pm \mathbf{i}\mathbf{k}P_3 + jm. \tag{19}$$

This is the discrete form of the nilpotent Dirac equation. Two significant facts emerge immediately from this derivation, and the elimination of the mass term from the operator. The first is that the derivation did not need the introduction of the canonical $i\hbar$ (or $i$ when $\hbar = 1$). Equation (17) is true whether the operators are $\partial / \partial t$ and $-\nabla$ or $i\hbar\, \partial / \partial t$ and $-i\hbar\, \nabla$. It is both classical and quantum, and the transition from classical to quantum is absolutely smooth. The second remarkable fact is that the annihilation operator for any particular fermion / antifermion is the exact negative of the creation operator.

## 6 *Zitterbewegung*

Making explicit use of the constants $c$ and $\hbar$, and of the symbol $\boldsymbol{\alpha}$ for $-\boldsymbol{ij}\mathbf{1}$, we can write the Hamiltonian for a nilpotent fermion in the form $\mathcal{H} = -\boldsymbol{ij}c\boldsymbol{\sigma}.\mathbf{p} - \boldsymbol{ii}mc^2 = -\boldsymbol{ij}c\mathbf{1p} - \boldsymbol{ii}mc^2 = \boldsymbol{\alpha}c\mathbf{p} - \boldsymbol{ii}mc^2$. It is convenient now to regard $\boldsymbol{\alpha}$ as a dynamical variable, and to define a velocity operator, which, for a free particle, becomes:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{1}{i\hbar}\,[\mathbf{r}, \mathcal{H}] = -\boldsymbol{ij}\mathbf{1}c = c\boldsymbol{\alpha}\,. \tag{20}$$

We can also write an equation of motion for the operator $-\boldsymbol{ij}\mathbf{1} = \boldsymbol{\alpha}$, as a function of $t$:

$$\frac{d\boldsymbol{\alpha}}{dt} = \frac{1}{i\hbar}\,[\boldsymbol{\alpha}, \mathcal{H}] = \frac{2}{i\hbar}(c\mathbf{p} - \mathcal{H}\boldsymbol{\alpha}). \tag{21}$$

Since $\mathcal{H}$ is a constant, this yields the solution:

$$\boldsymbol{\alpha}(t) = \frac{\mathbf{v}(t)}{c} = \frac{\dot{\mathbf{r}}(t)}{c} = c\mathcal{H}^{-1}\mathbf{p} + [\boldsymbol{\alpha}(0) - c\mathcal{H}^{-1}\mathbf{p}]\; exp\;(2i\mathcal{H}t\,/\,h). \tag{22}$$

This, in turn, can be solved, to give the equation of motion for a free fermion:

$$\mathbf{r}(t) = \mathbf{r}(0) + \frac{c^2\mathbf{p}}{\mathcal{H}}\,t + \frac{\hbar\, c}{2i\mathcal{H}}\,[\boldsymbol{\alpha}(0) - c\mathcal{H}^{-1}\mathbf{p}](exp\;(2i\mathcal{H}t\,/\,h) - 1). \tag{23}$$

The first term of this solution represents the initial position vector and the second term represents the displacement at time $t$. The third term, however, has no classical analogue, and represents a violent oscillatory motion or high-frequency vibration (*zitterbewegung*) of the particle at frequency $\approx mc^2/\hbar$, and amplitude $\hbar/mc$, which is the Compton wavelength for the particle. Since the dynamical variable here is $\boldsymbol{\sigma}$, it is apparent that this motion is a representation of the switching between the four spin states in the fermionic nilpotent.

## 7 Antisymmetric Wavefunctions

Pauli exclusion is, of course, automatically defined for a nilpotent wavefunction because:

$$\psi = (\pm ikE \pm \mathbf{ip} + jm)(\pm ikE \pm \mathbf{ip} + jm) = 0. \tag{24}$$

However, the standard definition of nilpotent wavefunctions as antisymmetric, with the antisymmetric nature due to the spin, also still applies since

$$\begin{aligned}
&\psi_1\psi_2 - \psi_2\psi_1 = \\
&(\pm ikE_1 \pm \mathbf{ip}_1 + jm_1)(\pm ikE_2 \pm \mathbf{ip}_2 + jm_2) - (\pm ikE_2 \pm \mathbf{ip}_2 + jm_2)(\pm ikE_1 \pm \mathbf{ip}_1 + jm_1) \\
&= 4\mathbf{p}_2\mathbf{p}_1 - 4\mathbf{p}_1\mathbf{p}_2 = -8i\mathbf{p}_1 \times \mathbf{p}_2 = 8i\mathbf{p}_2 \times \mathbf{p}_1
\end{aligned} \tag{25}$$

before normalization, while

$$\begin{aligned}
&\psi_1\psi_2 + \psi_2\psi_1 = \\
&(\pm ikE_1 \pm \mathbf{ip}_1 + jm_1)(\pm ikE_2 \pm \mathbf{ip}_2 + jm_2) + (\pm ikE_2 \pm \mathbf{ip}_2 + jm_2)(\pm ikE_1 \pm \mathbf{ip}_1 + jm_1) \\
&= 4E_1E_2 + 4E_2E_1 - 4\mathbf{p}_1\mathbf{p}_2 - 4\mathbf{p}_2\mathbf{p}_1 - 4m_1m_2 - 4m_2m_1 = 0\,.
\end{aligned} \tag{26}$$

Equation (25) is a remarkable result. It implies that, instantaneously, any nilpotent wavefunction must have a **p** vector in real space (a spin 'phase') at a different orientation to any other. The wavefunctions of all nilpotent fermions instantaneously correlate because their **p** vector directions must all intersect, and the intersections actually create the *meaning* of Euclidean space, with an intrinsic spherical symmetry generated by the fermions themselves. At the same time, equation (24) could also be interpreted as suggesting that each nilpotent also has a unique direction in a quaternionic phase space, in which $E$, **p** and $m$ values are arranged along orthogonal axes. We may suppose here that the mass shell or real particle condition requires the coincidence between the directions in these two spaces. In addition, the **p** vector, as implied in (25), carries *all the information* available to a fermionic state, its direction also determining its $E$ and **p** values uniquely. Three consequences of this are immediately apparent. To avoid direction duplication, one at least of the three nilpotent terms (the mass term, in fact) must have only one algebraic sign; also, a hypothetical massless fermion and antifermion pair would require opposite helicities (say, $ikE$ + $\mathbf{ip}$ and $-$ $ikE$ + $\mathbf{ip}$) to avoid being on the diagonal; and, finally, such a massless fermion could not exist in practice because, since the magnitudes of $E$ and $p$ would always be equal in such cases, then the resultant angles would always be the same.

## 8 Nilpotent Structure and the Fundamental Interactions

The three fundamental interactions of particle physics – electric, strong and weak – and their characteristic force laws and defining symmetries are direct consequences of nilpotent structure alone, and do not require any additional 'physical' input. First, we assume that spherical symmetry requires that we express the momentum term of the operator in polar coordinates, using the Dirac prescription, with an explicit spin term:

$$\nabla = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right) \pm i\frac{j + ½}{r}. \tag{27}$$

The nilpotent Dirac operator now becomes:

$$\left(ikE + ii\left(\frac{\partial}{\partial r} + \frac{1}{r} \pm i\frac{j + ½}{\mathrm{r}}\right) + jm\right). \tag{28}$$

Now, whatever phase factor we apply this to, we will find that we will not get a nilpotent solution unless the $1 / r$ term with coefficient $\boldsymbol{i}$ is matched by a similar $1 / r$ term with coefficient $\boldsymbol{k}$. So, simply requiring *spherical symmetry* for fermion state, requires a term of the form $A / r$ to be added to $E$. A Coulomb term is generated automatically within any fermion state used to define spherically symmetric Euclidean space. All the fundamental interactions of particle physics have a $U(1)$ Coulomb term, the minimum requirement being when no other consideration is being invoked. In this, purely scalar, case, we describe the interaction as *electric*. All the component terms of the nilpotent contribute to the scalar aspect of the spherical symmetry, because all have scalar values, but the $\boldsymbol{j}m$ term alone has no other function. Application of a Coulomb term, of course, leads to the creation of a phase factor for the nilpotent which generates, within six lines of calculation, the energy levels for the so-called 'hydrogen atom' solution of the conventional Dirac equation (Rowlands, 2004, 2005, 2006, 2007).

Secondly, we invoke the vector nature of $\mathbf{p}$ to write down a lead term of the form

$$(\boldsymbol{ik}E \pm \boldsymbol{i}\,\mathbf{i}p_x + \boldsymbol{j}\,m)\,(\boldsymbol{ik}E \pm \boldsymbol{i}\,\mathbf{j}p_y + \boldsymbol{j}\,m)\,(\boldsymbol{ik}E \pm \boldsymbol{i}\,\mathbf{k}p_z + \boldsymbol{j}\,m) \tag{29}$$

which, according to the nilpotent condition, has six allowed phase, i.e. when $\mathbf{p}$ successively and exclusively takes on the values $\pm\, \mathbf{i}p_x$, $\pm\, \mathbf{j}p_y$, and $\mathbf{k}p_z$. The gauge-invariant transformations between these phases has exactly the $SU(3)$ group structure required for the standard 'coloured' or strongly interacting baryon wavefunction made of $R$, $G$ and $B$ 'quarks' (Ryder, 1996),

$$\psi \sim (RGB - RBG + BRG - GRB + GBR - BGR) \tag{30}$$

with transitions via massless generators, but with components that must be massive via the Higgs mechanism because of the simultaneous existence of positive and negative helicities in (29). Also, since the transitions are absolutely nonlocal, any spatial separation of the components of (29) has no effect on the rate of momentum exchange. This is equivalent to a constant force or a potential energy which is linear with distance. As has been demonstrated previously (Rowlands, 2005, 2006, 2007), the combination of linear and inverse linear (Coulomb) potentials added to the energy term in the nilpotent produces an automatic phase factor which requires the component (quark) confinement of the strong interaction, with asymptotic freedom and infrared slavery.

The third aspect of the nilpotent structure which generates a physical effect is the spinor structure, and the accompanying *zitterbewgung*. Though this is only a vacuum process, it specifically requires the creation of bosonic structures of the form $(\boldsymbol{i}\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{i}\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ and $(\boldsymbol{i}\boldsymbol{k}E + \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)(-\boldsymbol{i}\boldsymbol{k}E - \boldsymbol{i}\mathbf{p} + \boldsymbol{j}m)$ via a harmonic oscillator mechanism, which is the characteristic defining process of the weak interaction, in real as well as vacuum states. In effect, the *zitterbewgung* ensures that a fermion is always a weak dipole in relation to its vacuum states, and the single-handedness of the weak interaction can be regarded as the result of a weak dipole moment connected with fermionic ½-integral spin. Significantly, all weak interactions between real particles require sources that are in some senses dipoles (fermion-antifermion) and so can be expected to require a dipolar potential, in addition to the Coulomb term. Any such potential combination applied to the nilpotent operator produces a series of energy levels characteristic of the harmonic oscillator. In this case, the interaction and its $SU(2)$ symmetry appears to be generated by the duality of the pseudoscalar term $\pm\, \boldsymbol{i}\boldsymbol{k}E$ in generating antifermion, as well as fermion states.

The analysis here has shown that it is the *structure* of the nilpotent alone that produces the three fundamental interactions characteristic of particle physics, and that no external physical input is required. Simply by defining an operator which is a nilpotent 4-component spinor with vector properties, we necessarily imply that it is subject to electric, weak and strong interactions. This structure is a product of the three types of quantity (pseudoscalar, multivariate vector and scalar) which it contains, and, ultimately, these are reflections of the need for a discrete (point) source to preserve spherical symmetry and hence to conserve angular momentum. We saw, in section 6, that the angular momentum term (**p**) incorporates, in some sense, all the information relevant to the three terms in the fermionic nilpotent; and we can, in fact, identify these and their associated symmetries as being connected with the three separately conserved aspects of angular momentum: magnitude (scalar, $U(1)$, spherical symmetry does not depend on the length of the radius vector), direction (vector, $SU(3)$, spherical symmetry does not depend on the choice of axes), and handedness (pseudoscalar, $SU(2)$, spherical symmetry does not depend on whether the rotation is left- or right-handed).

Of course, this analysis does not take account of the fourth fundamental interaction, gravity, except insofar as it can be characterized by a $U(1)$ symmetry like the electromagnetic interaction. In fact, the nilpotent structure suggests that gravity has a unique role in providing the basis for instantaneous nonlocal correlation and the continuous vacuum required by quantum mechanics, and that, unlike the other three forces, it is an intrinsically nonlocal interaction, whose continuous nature reflects that of the Higgs field and the zero-point energy. In principle (Rowlands, 2004, 2007), the three quaternionic operators split the vacuum into three discrete (idempotent) components, $\boldsymbol{k}(\pm\, ikE \pm i\mathbf{p} + jm)$, $\boldsymbol{i}(\pm\, ikE \pm i\mathbf{p} + jm)$ and $\boldsymbol{j}(\pm\, ikE \pm i\mathbf{p} + jm)$, which we can specify as 'weak', 'strong' and 'electric', and which produce virtual spin 1, spin 0 and fermion-fermion 'bosonic' vacuum partners for the fermion. The gravitational equivalent would be $1(\pm\, ikE \pm i\mathbf{p} + jm)$, which specifies the complete or continuous vacuum, or the entire universe as seen by the fermion, and the combination of fermion and gravitational vacuum is a totality zero state, not simply a fermion, as it is with the three discrete components. Ultimately, what we measure in 'gravitational' experiments is a localised inertial reaction produced by discrete components of matter. Unlike gravity itself, this can be quantized, and it is in describing this inertial reaction that we find meaning in the mathematical structure of the general relativistic field equations, and are able to predict the acceleration effect now usually referred to as 'dark energy' (Rowlands, 1994, 2007).

## 9 Nilpotent Structure and the Universal Rewrite System

The nilpotent version of quantum mechanics is not only the most streamlined and minimally constructed version available, it is also the most powerful, because it is already a full quantum field theory, with the nilpotent operator potentially incorporating all the physical information available to the fundamental physical state. Perhaps, even more significantly, it can be derived in a fundamental way from a universal rewrite system, which seems to have much more general applications (Rowlands and Diaz, 2002; Diaz and Rowlands, 2005, 2006; Rowlands, 2007), while the algebraic structure can be shown to be derived from the algebras of the four fundamental parameters, space, time, mass and charge, and their mathematically symmetric relationships. This derivation automatically includes quantization and special relativity as part of the abstract formal structure – it doesn't need to assume them – while the classical transition can be effected smoothly by using the discrete version of differentiation, based on commutators, rather than differentials.

## 10 Conclusion

Using the nilpotent approach, the formal apparatus required for quantum mechanics and quantum field theory has been reduced to a single creation operator with explicit energy, momentum and mass terms. All the other apparatus, including renormalization, has been shown to be redundant. In addition the fundamental interactions of particle physics can be shown to be consequences of the mathematical structure alone, with no additional 'physical' assumptions. The nilpotent structure appears to be the one which truly encodes the physical information available to fermions, in addition to providing the simplest and most powerful mathematical formalism for quantum mechanics.

## References


Diaz B. and Rowlands P. (2005). A computational path to the nilpotent Dirac equation, International Journal of Computing Anticipatory Systems, **16**, 203-18.

Diaz B. and Rowlands P. (2006). The infinite square roots of –1, International Journal of Computing Anticipatory Systems, 19, 229-235.

Gough, W. (1990). Mixing scalars and vectors – an elegant view of physics, Eur. J. Phys., 11, 326-3.

Hestenes D. (1966). Space-time Algebras, Gordon and Breach.

Hestenes D. (1967). Real Spinor Fields, J. Math. Phys. 8, 798-808.

Hestenes D. (1975). Observables, operators and complex numbers in the Dirac theory, J. Math. Phys. 16, 556-572.

Kauffman, L. H. (2004). Non-commutative worlds, New Journal of Physics, 6, 2-46.

Rowlands P. (1994). A Revolution Too Far: The Establishment of General Relativity, PD Publications.

Rowlands P. (2004). Symmetry breaking and the nilpotent Dirac equation, AIP Conference Proceedings, 718, 102-115.

Rowlands P. (2005). Removing Redundancy in relativistic quantum mechanics, arXiv.org:physics/0507188.

Rowlands P. (2006). Fermion interactions and mass generation in the nilpotent formalism, AIP Conference Proceedings, 839, 225-35, 2006.

Rowlands, P. (2007). Zero to Infinity, World Scientific.

Rowlands P. and Diaz B. (2002). A universal alphabet and rewrite system, arXiv:cs.OH/0209026.

Ryder, L. H. (1996). *Quantum Field Theory*, Cambridge University Press, second edition, pp. 18-20.