Ergodic Properties of the Relaxation Phase in Nonchaotic Unimodal Maps

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Abstract The convergence to the mean values of observables is studied for nonlinear dynamical systems in the period-doubling bifurcation regime. The phase space convergence to the mean values is studied numerically; it reveals a characteristic behaviour induced by several special points in phase space. The convergence to the mean value for these points is exponential as opposed to the power-law convergence of the majority of the phase space. The issue of universality of these results which characterize the period doubling bifurcation behaviour is discussed.

Keywords: Period-Doubling, Mean Values, Relaxation, Unimodal Maps, Phase Space

1 Introduction

During the last decades much effort has been devoted in order to understand in detail the dynamics of nonlinear systems [1, 2, 3, 4]. A typical behaviour of these dynamics is the approach to an attracting subspace of the phase space as time evolves. The complexity of the asymptotic state (attractor) varies, depending on the parameters involved in the dynamics, from a finite set to a multifractal set of points embedded in the phase space of the system. Although many works deal with the properties of the attractor [1, 2, 3, 4] much less is done in order to understand the transient dynamics leading to this asymptotic state. The simplest example of a system possesing such complex dynamical behaviour are nonlinear single humped maps on the interval. In this case the most common regime is the period doubling bifurcation scenario where the attractors of the dynamical system are cycles of period 2^p with increasing p as the corresponding control parameter increases. Recently in [14] a detailed study of the approach to the period 2^p cycle for the logistic map has been been performed.

In [14] the time required for the system to approach the fixed point or the periodic cycle to within a distance ε is studied. Several complicated structures connected with the fixed point or the preimages of the fixed point, or finally with the periodic cycle or the preimages of the periodic cycle, are recognized in the phase space. These structures in the period doubling regime scale with the Feigenbaum constant α.

In the present work we will adopt a different viewpoint in order to explore transient dynamics. Instead of calculating the time needed to approach the attracting set to within an euclidian distance ε, we will focus on the corresponding statistical issue of the dynamics and examine the convergence of a system's trajectory towards the mean value of a prescribed observable. The interest in this case is to classify the phase space points according to their convergence properties to the mean value of the evolving variable.

In order to study the dependence of the mean value from the initial point in the phase space, we used as a tool the numerical simulation. For methodological reasons we have separated the case where the system presents a stable fixed point from the region of multistability. The corresponding properties of convergence and some correlation functions have also been studied in some detail.

The paper is articulated as follows. In Section 2 we present the phase space relaxation modes for the example of the logistic map, before and after the period doubling regime. In Section 3, we focus on the period-doubling regime and we examine the modes of convergence to the mean value, which is undoubtedly the central result of our paper. In Section 4, we study the correlation functions of the phase space above a control parameter value entailing a stable fixed point. The paper ends with a Section of conclusions, where we also discuss plans for future work.

2 Rate of Convergence to the Mean Value in the Phase Space

2.1 Study Before the Period-Doubling Regime

The objective of the present work is the study of the relaxation time to the mean value, i.e. the time required for the convergence of the trajectory to the mean value.

We consider a well-studied system, that is the logistic map in the nonchaotic region [11, 12]. The logistic map is defined in general as the quadratic recurrence

$$x_{n+1} = rx_n(1 - x_n), \tag{1}$$

where $1 < r \leq 4$, $0 \leq x_n \leq 1$.

To start our study we plot the number of iterations required to find a mean value with a difference ε from the asymptotic mean value. Below (Fig. 1 and Fig. 2) we plot this diagram for $r = 1.7$, and $r = 2.5$. We remind the reader that the first bifurcation point of the logistic map is at $r = 3.0$.

As is depicted in the figures below, for $1 < r < 3$ the graph presents two minima and a local maximum, as well as two maxima at the ends of the interval. One of the two minima corresponds to the stable fixed point x_{st} (which is the global minimum), and we shall speak about the other in the sequel.

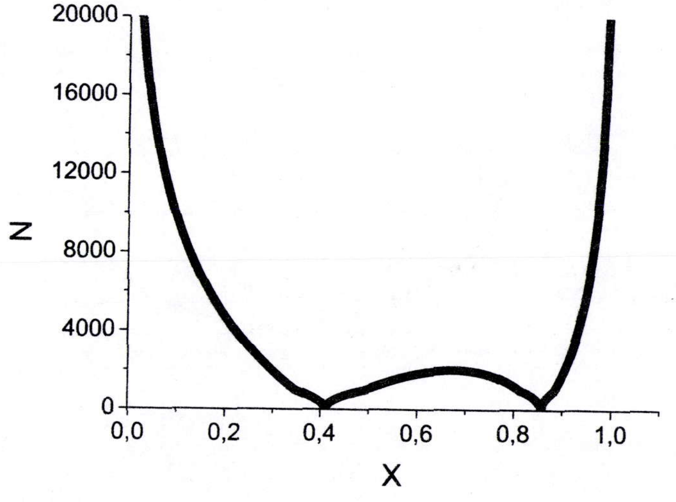

Fig. 1: Number of iterations to arrive to the mean value with a precision $\varepsilon = 10^{-4}$ as a function of the initial position in the phase space for the logistic map. ($r = 1.7$).

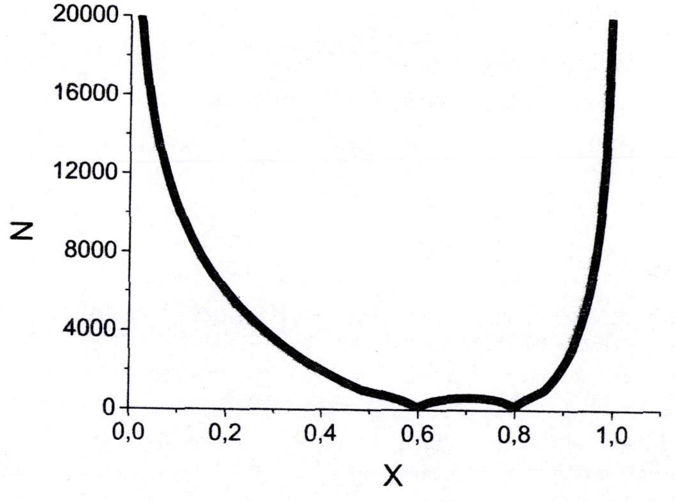

Fig. 2: Number of iterations to arrive to the mean value with a precision $\varepsilon = 10^{-4}$ as a function of the initial position in the phase space for the logistic map. ($r = 2.5$).

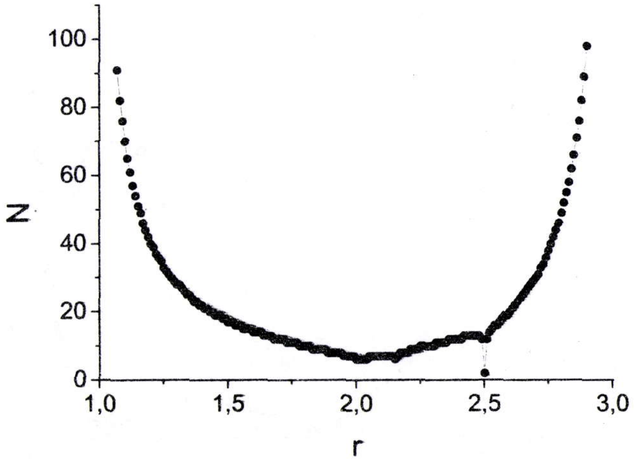

Fig. 3: Number of iterations N for which the equation has converged to the minimum with a precision of $1.0E^{-10}$, as a function of r.

It should be noted that in the case of the convergence to the mean value, the graph is asymmetric. As the value of r increases, the distance between the two minima decreases and the two minima coincide for $r \to 3$.

The second minimum is the second solution of the equation

$$\lim_{N\to\infty} (\sum_{i=1}^{N} x_i - N x_{st}) = 0, \tag{2}$$

where the first solution of this equation is evidently the value $x_1 = x_{st}$. (where x_{st} is the stable fixed point and x_{min} the second solution of the Eq(2)). We may notice, that the existence and uniqueness of the second solution is a numerical statement.

The solutions of the above equation converge to the second minimum when N bypasses a cut-off value which depends on r. In the diagram below (Fig.3) we plot the dependence of N on r (N is the iteration for which the equation has converged to the minimum with a precision of $1.0E^{-10}$).

It should be noted that our numerical analysis leads to the existence of a special point for $r = 2.5$. This special point converges to the mean value extremely rapidly.

Notice that the solutions of eq(2) when N is increased, converge exponentially to the minimum, that is

$$\frac{\rho_{N+1} - x_{min}}{\rho_N - x_{min}} = \lambda \tag{3}$$

(where ρ_N is the root of the equation for the first N iterations) and λ is the value of the derivative in the linear regime. This is depicted in the following figure.

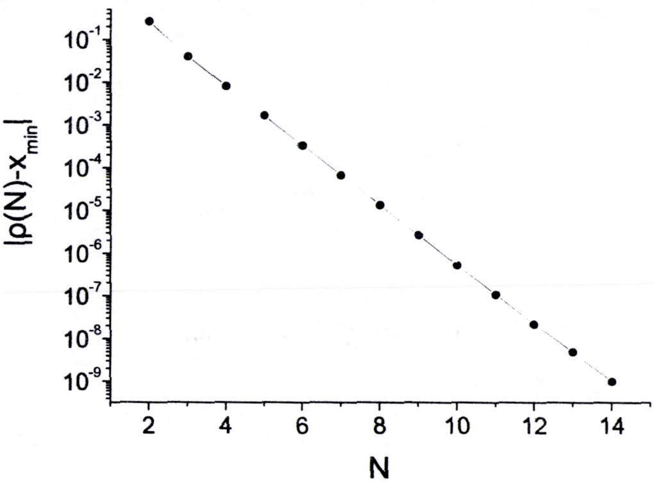

Fig. 4: (ρ_N - x_{min}) as a function of the number of iterations N. This holds for $r = 1.8$.

As we have already mentioned the two minima tend to coincide as $r \to 3$ (this is a numerical statement). In the following figure the dependence of the second minimum on r is depicted. As we can see from the figure, for $r \to 1$ the minimum tends to the unity, whereas for $r \to 3$ the minimum tends to the fixed point.

2.2 Study in the Period-Doubling Regime

As the value of r bypasses 3, the stable fixed point becomes unstable and the cycle-2 is born. The points of the cycle are given by well-known relations [2]. Consequently the asymptotic mean value is given by the relation

$$E[x] = \frac{r+1}{2r}. \tag{4}$$

To bypass the problem of the oscillation for the cycle-2, we study the convergence to the mean value for the quantity

$$y_1 = \frac{x_1 + x_2}{2}, \quad y_2 = \frac{x_3 + x_4}{2}... \tag{5}$$

In the following figure, we present the diagram of the time (number of iterations) required to converge to the mean value as a function of the initial position x_1, for the specific control parameter value $r = 3.1$.

As it is shown in Fig.6, in the centre of the figure in between the three δ-functions which are marked with arrows, there emerge structures similar to those that appear

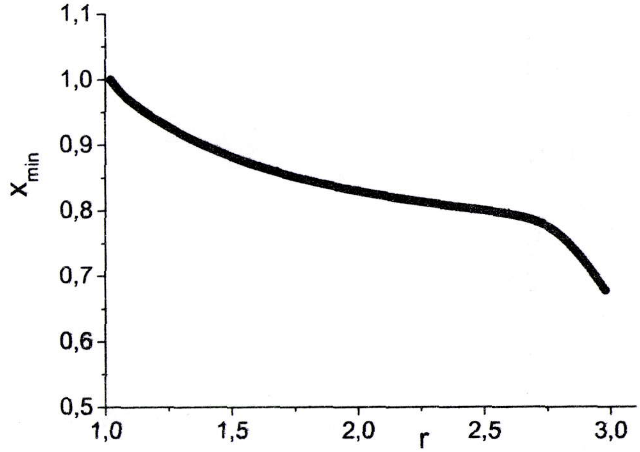

Fig. 5: The special point x_{min} as a function of the control parameter r.

when $r < 3$. The two minima in between $x_a = 0.32$ and $x_b = 0.676$ correspond the first one to one point of the cycle 2 and the other to a special point with similar properties of convergence to the mean value, as those of x_{min} (for $r < 3$).

Similar features are exhibited also for the region in between the second and the third δ-functions (between $x_b = 0.676$ and $x_c = 0.8825$). There are some basic differences in the behaviour of the diagram of the relaxation time in respect to the mean value as a function of the initial position. One important difference is the presence of infinitely many maxima that appear under the form of delta functions. The delta functions correspond to the unstable fixed point (x_b) and to the infinite in number preimages of this unstable fixed point. A second difference is that on each side of the delta functions there exist two minima (one on each side), to which we refer in the sequel. It is worth mentioning that these minima (except from the first four in the centre of the figure), are points that appear in the linear region of the pre-images of the unstable fixed point and are consequently also infinitely many. It is interesting to note that these points are local minima (and not maxima), although they lie in the region of the pre-images of the unstable fixed points.

As the value of the control parameter bypasses the value $r = 1 + \sqrt{6}$, the cycle 4 is born. In the next diagram which corresponds to $r = 3.46$, it is shown that as the control parameter value increases, the diagram of the relaxation time as a function of the initial position becomes more and more complex. In any case we note the presence of many maxima and minima interconnected with the same reasons as in Fig.6. We expect similar behaviour with many maxima and minima until $r = r_\infty$ (the Feigenbaum point).

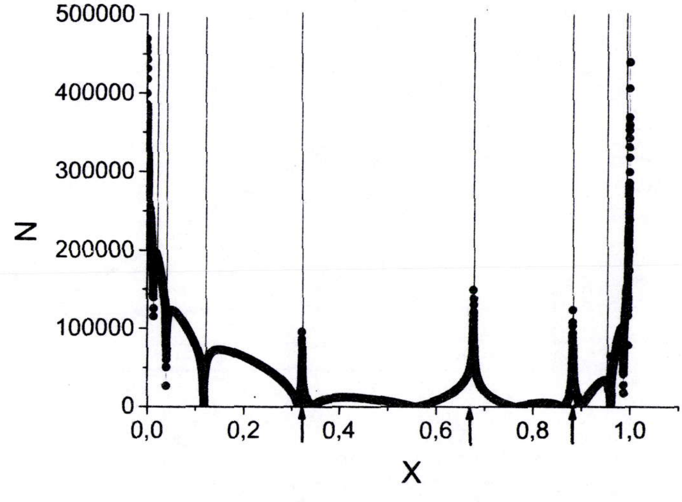

Fig. 6: Number of iterations required to arrive to the mean value with a precision $\varepsilon = 10^{-5}$ as a function of the initial position in the phase space for the logistic map. $(r = 3.1)$(cycle-2).

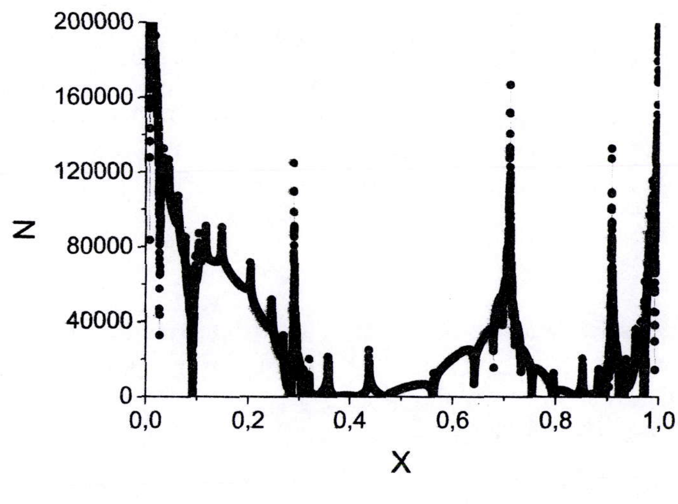

Fig. 7: Number of iterations to arrive to the mean value with a precision $\varepsilon = 10^{-5}$ as a function of the initial position in the phase space for the logistic map. $(r = 3.46)$(cycle-4).

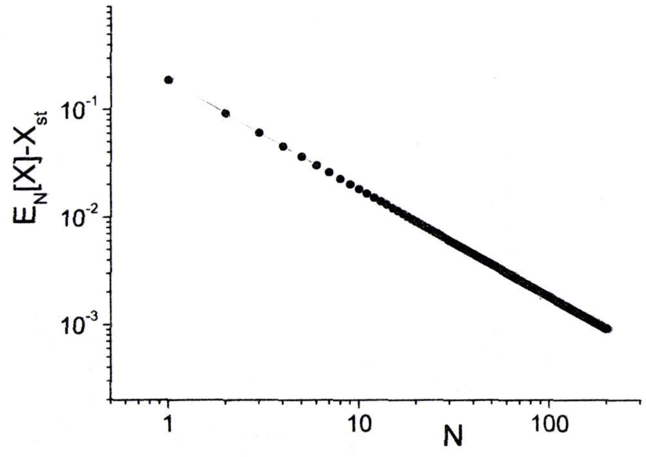

Fig. 8: The difference of the running mean value from the mean value $E_N[x] - x_{st}$ as a function of the number of iterations N. This holds for the control parameter value $r = 1.7$, and with initial point $x_1 = 0.6$ different from x_{min}. We observe a power-law convergence.

3 Modes of Convergence to the Mean Values

3.1 Study Before the Period-Doubling Regime

In the present work we have also studied the mode of convergence to the final mean value of the system as a function of the initial point. In the region $1 < r < 3$, we have found that essentially there are two ways. The first way is followed by the majority of the initial points. Here, the approach of the mean value follows the equation

$$E_N[x] - x_{st} = a/N, \tag{6}$$

where a is a constant and we denote as $E_N[x]$ the "running mean value" of the position x. The graph of $E_N[x] - x_{st}$ as a function of N is depicted in Fig. 8.

The second way is followed only if $x_1 = x_{min}$. In this case the approach to the mean value follows the relation

$$E_N[x] - x_{st} = exp(-\lambda N). \tag{7}$$

The graph of $E_N[x] - x_{st}$ as a function of N is depicted in Fig. 9.

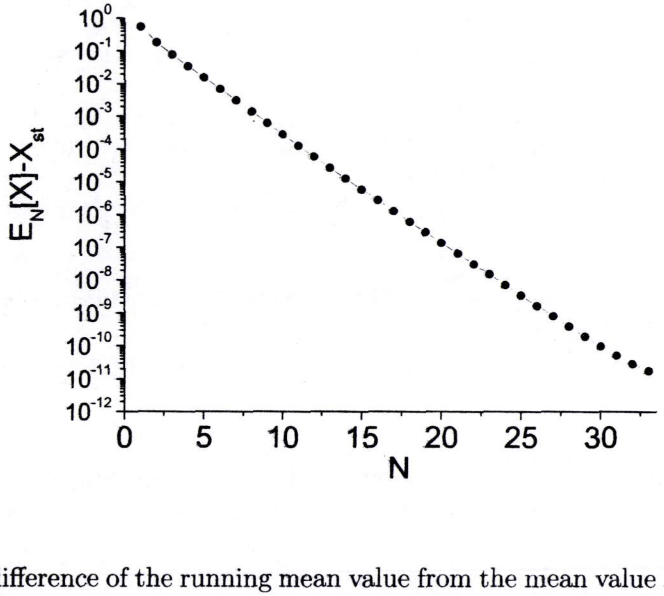

Fig. 9: The difference of the running mean value from the mean value $E_N[x] - x_{st}$ as a function of the number of iterations N. This holds for the control parameter value $r = 1.5$, and with initial point $x_1 = x_{min} = 0.88209$. We observe an exponential convergence.

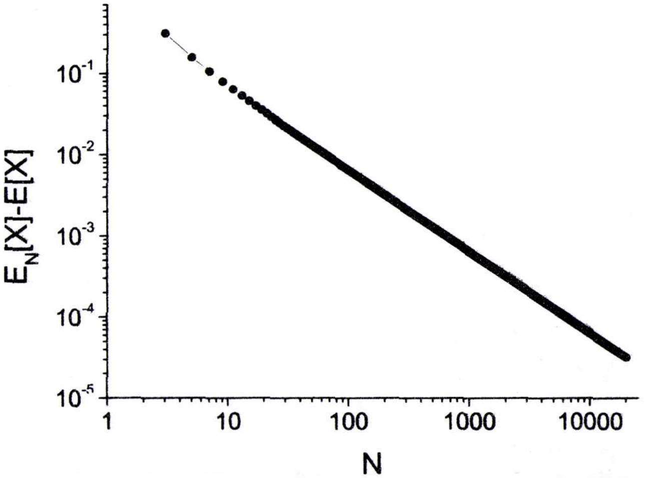

Fig. 10: The difference of the running mean value from the mean value $E_N[x] - E[x]$ as a function of the number of iterations N. This holds for the control parameter value $r = 3.1$, and with initial point $x_1 = 0.2$, different from x_{min}. We observe a power-law convergence.

3.2 Modes of Convergence to the Mean Value in the Period-Doubling Region

In this subsection, we study the case $r > 3$, and in particular for $r = 3.1$. However, our conclusions are quite general.

In Fig.10. we show the evolution of the quantity $E_N[x] - E[x]$ as a function of N, for a randomly chosen point $x_1 = 0.2$. In this case, we observe a power law convergence over many orders of magnitude.

In Fig.11. we show the evolution of the quantity $E_N[x] - E[x]$ as a function of N, for $x_1 = x_{min} = 0.3435$. In this case, the special point x_{min} has similar properties to the corresponding x_{min} when $1 < r < 3$ and we observe an exponential convergence over 10 orders of magnitude.

In Fig.12, we show the evolution of the quantity $E_N[x] - E[x]$ as a function of N, for $x_1 = x_{min} = 0.03962....$ This point belongs to the linearized region of the unstable fixed point. In this case, we first observe initially a power law convergence, and then an exponential convergence.

The universality of these results is an open problem.

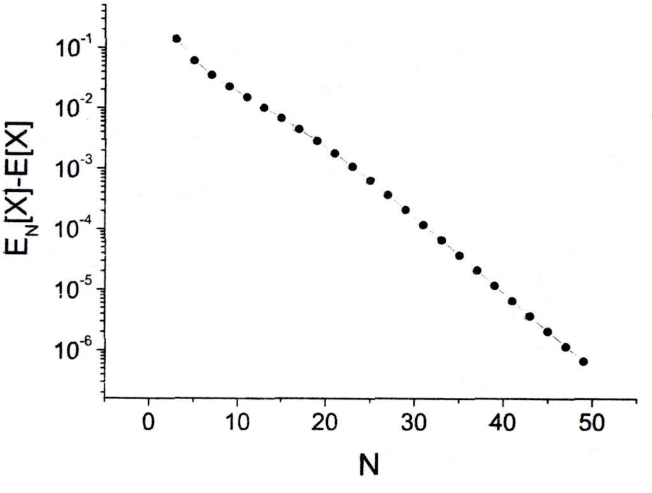

Fig. 11: The difference of the running mean value from the mean value $E_N[x]-E[x]$ as a function of the number of iterations N. This holds for the control parameter value $r = 3.1$, and with initial point $x_1 = x_{min} = 0.3435$. We observe an exponential convergence.

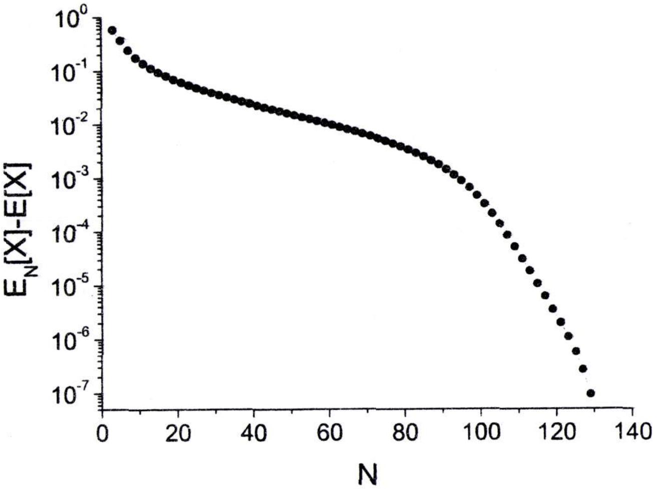

Fig. 12: The difference of the running mean value from the mean value $E_N[x] - E[x]$ as a function of the number of iterations N. This holds for the control parameter value $r = 3.1$, and with initial point $x_1 = x_{min} = 0.03962$. We first observe a power-law convergence, and then an exponential convergence.

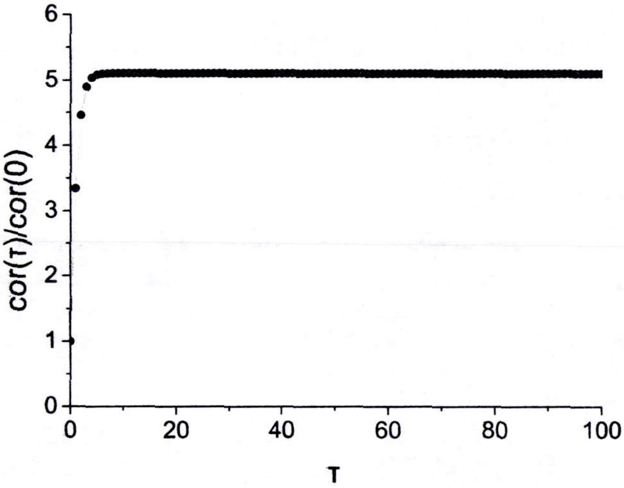

Fig. 13: The correlation function of the logistic map for $r = 1.7$ and with initial point $x_1 = 0.6$, different from x_{min}. The correlation function tends to a non-zero value.

4 Study of the Correlation Functions Below $r = 3$

In our study for control parameter values in between $1 < r < 3$, we have also studied the correlation function, too. The correlation function is defined by the relation

$$C(\tau) = \lim_{N\to\infty} (\frac{1}{N}\sum_{i=1}^{N} x_i x_{i+\tau} - (\frac{1}{N}\sum_{i=1}^{N}(x_i))^2). \quad (8)$$

In this regime the system possesses a stable fixed point and it is obviously non-mixing. This entails that the correlation function converges to a value different from zero, as $\tau \to \infty$. This is depicted, for typical values of the control parameter r and the initial point x_1 in Fig. 13.

As an exception to this basic rule, the correlation function of the special point x_{min} converges to zero, as $\tau \to \infty$. This is presented in Fig.14. In this case the system is mixing. This is related to the particular role of the special point x_{min}.

5 Conclusions

In this work, the phase space convergence to the mean values has been studied for the period doubling scenario on the logistic map. Many special points have

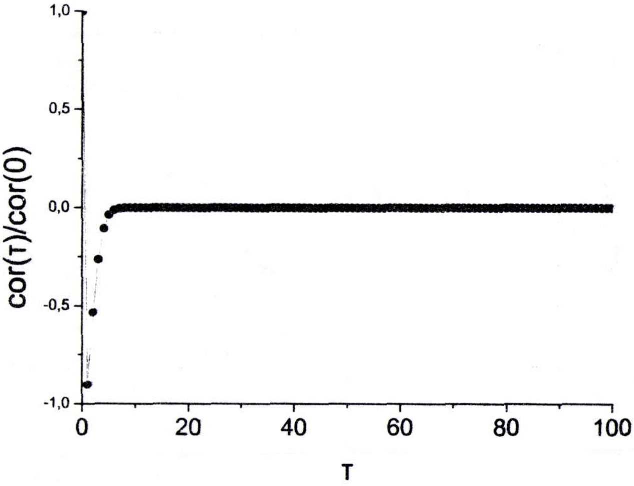

Fig. 14: The correlation function of the logistic map for $r = 1.7$ and with initial point $x_1 = x_{min} = 0.85649$. The correlation function tends to zero.

been recognized and their role examined. The correlation functions in the region $1 < r < 3$ are also considered.

In the general case, a power-law convergence to the mean value is observed for the majority of the phase space points. An exponential convergence is however observed for the special points. The correlation function of the special points corresponds to that of a mixing system unlike the correlation functions of the majority of the phase space. This is quite unanticipated and somehow surprising. All these special properties are generated by the property of the special points as solutions of the equation defining the mean value.

The issue of the universality of these results for unimodal maps, as well as the the rigorous proofs and the extension of these results for continuous time dynamical systems is an open question that could be addressed in the future.

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