Polynomial Lattice Equations: the Key to Fuzzy Systems Modelling

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Abstract

Fuzzy relational equations are without doubt the most important inverse problems arising from fuzzy set theory, and in particular from fuzzy relational calculus. Indeed, the calculus of fuzzy relations is a powerful one, with applications in fuzzy control and fuzzy systems modelling in general, approximate reasoning, relational databases, clustering, etc. In this paper, fuzzy relational equations are approached from an order-theoretical point of view. It is shown how all inverse problems can be reduced to systems of polynomial lattice equations. The exposition is limited to the description of exact solutions of systems of sup-$\mathcal{T}$ equations, and analytical ways are presented for obtaining the complete solution set when working in a broad and interesting class of distributive lattices.

Keywords: fuzzy relational equation, inverse problem, polynomial lattice equation, root system, triangular norm.

1 Introduction

Mathematical relations, just as human relations, are invaluable to any scientific researcher. As Goguen (1967) writes, "Science is, in a sense, the discovery of relations between observables. Zadeh has shown the study of relations to be equivalent to the general study of systems (a system is a relation between an "input" and an "output" space)." However, the typical behaviour of a mathematician to describe relationships in a black-or-white manner is not suitable for every-day problems. Fuzzy relations, however, allow for a description of gradual relationships. Depending on the context, fuzzy relations can be interpreted in two ways: a disjunctive way, or a conjunctive way, see e.g. De Baets (1996) and Dubois and Prade (1992).

In the *disjunctive interpretation*, a fuzzy relation is considered as an elastic restriction on the more or less possible values of a couple of variables, each of which takes exactly one value in reality. In this interpretation, fuzzy relations are extensively used for modelling systems of which the input-output behaviour is known only through linguistic descriptions. The basic operations on fuzzy relations are the *direct image* and the *composition*. The generalized modus ponens and the generalized modus tollens, the key operations behind forward and backward chaining inference mechanisms in fuzzy rule-based systems, perfectly fit within this disjunctive interpretation and are immediate applications of the basic operations. The design and deployment of such systems require the solution of inverse problems connected with the basic operations: from known input-output behaviour a system model is built through the solution of fuzzy relational equations.

The notion of *mutually exclusive values* is no longer present in the conjunctive interpretation. In a diagnostic problem, for instance, a single cause may lead to multiple defects. In a decision problem, alternatives are evaluated on a number of criteria, of which, of course, more than one may be satisfied. Although most attention has been directed towards the conjunctive interpretation, this disjunctive interpretation also has an incredible potential. The basic operations from the conjunctive interpretation are still applicable in the disjunctive interpretation. However, Bandler and Kohout (1980a, 1980b) realized that in the conjunctive interpretation additional operations play an important role. The most important such operations are their *triangular compositions*. The contributions of Bandler and Kohout are, without doubt, among the most fundamental in the study of fuzzy relations. However, the solution of inverse problems related to these 'new' operations has received only little attention.

In this paper, we present a uniform framework for solving the inverse problems arising in both interpretations. Remarkably, particular solutions in the disjunctive interpretation, for instance, can be described by means of operations from the conjunctive one. This demonstrates that it is really opportune to treat these problems in one common framework. However, due to space limitations, only the inverse problems arising in the disjunctive interpretation will be discussed. This paper is organized as follows. In Section 2, we briefly introduce the reader to the basic operations of fuzzy relational calculus, followed by an identification of the corresponding inverse problems in Section 3. The most important fuzzy relational equations are related to systems of sup-$\mathcal{T}$ equations. Their solution procedures are explained in Sections 4 and 5.

2 Images and Compositions

2.1 Relational Calculus and Boolean Equations

In essence, there exist two basic types of operations in relational calculus: *images* and *compositions*. Consider a relation R from X to Y. The *afterset* xR of $x \in X$ is the subset of Y defined as $xR = \{y \in Y \mid (x, y) \in R\}$. The *foreset* Ry of $y \in Y$ is the subset of X defined as $Ry = \{x \in X \mid (x, y) \in R\}$. Given a subset A of X, the *direct image* $R(A)$, *subdirect image* $R_{\triangleleft}(A)$ and *superdirect image* $R_{\triangleright}(A)$ of A under R are the subsets of Y defined as:

$$\begin{aligned} R(A) &= \{y \in Y \mid (\exists x \in X)(x \in A \wedge (x, y) \in R)\} \\ &= \{y \in Y \mid A \cap Ry \neq \emptyset\} \end{aligned}$$

$$\begin{aligned} R_{\triangleleft}(A) &= \{y \in Y \mid (\forall x \in X)(x \in A \Rightarrow (x, y) \in R)\} \\ &= \{y \in Y \mid A \subseteq Ry\} \end{aligned}$$

$$\begin{aligned} R_{\triangleright}(A) &= \{y \in Y \mid (\forall x \in X)((x, y) \in R \Rightarrow x \in A)\} \\ &= \{y \in Y \mid Ry \subseteq A\}. \end{aligned}$$

Similarly, given a second relation S from Y to Z, the *(round) composition* $R \circ S$, *subcomposition* $R \triangleleft S$ and *supercomposition* $R \triangleright S$ of R and S are the relations from X to Z defined as:

$$\begin{aligned} R \circ S &= \{(x, z) \in X \times Z \mid (\exists y \in Y)((x, y) \in R \wedge (y, z) \in S)\} \\ &= \{(x, z) \in X \times Z \mid xR \cap Sz \neq \emptyset\} \end{aligned}$$

$$\begin{aligned} R \triangleleft S &= \{(x, z) \in X \times Z \mid (\forall y \in Y)((x, y) \in R \Rightarrow (y, z) \in S)\} \\ &= \{(x, z) \in X \times Z \mid xR \subseteq Sz\} \end{aligned}$$

$$\begin{aligned} R \triangleright S &= \{(x, z) \in X \times Z \mid (\forall y \in Y)((y, z) \in S \Rightarrow (x, y) \in R)\} \\ &= \{(x, z) \in X \times Z \mid Sz \subseteq xR\}. \end{aligned}$$

Strictly speaking, for the sub- and superdirect images, and the sub- and supercompositions, one should also take into account some non-emptiness conditions, see De Baets and Kerre (1993). In the context of relational equations, however, the above formulations are more suitable. Sub- and supercompositions of relations have been studied extensively by Bandler and Kohout (1980a, 1980b) under the name *triangle relational products*.

It is now not difficult to imagine what types of inverse problems one can consider:

(i) *image equations*, i.e. equations of the type $R(A) = B$, $R_{\triangleleft}(A) = B$ or $R_{\triangleright}(A) = B$ in the unknown set A or in the unknown relation R;

(ii) *composition equations*, i.e. equations of the type $R \circ S = T$, $R \triangleleft S = T$ or $R \triangleright S = T$ in the unknown relation R or in the unknown relation S.

Identifying sets and relations with their characteristic mapping, these inverse problems, called relational equations, can be translated into Boolean equations (see e.g. Rudeanu (1974)). Note that the characteristic mapping χ_A of a crisp subset A of X is defined by

$$\chi_A(x) = \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ elsewhere} \end{cases} .$$

2.2 Fuzzy Relational Calculus

The above images and compositions are extended to fuzzy relations in a by now traditional manner: supremum and infimum are playing the role of the existential and universal quantifiers and a conjunctor and implicator are modelling pointwise intersection and inclusion. The most natural mathematical framework for dealing with these operations is without doubt the lattice-theoretic one (see e.g. Birkhoff (1967), Davey and Priestley (1990)). Hence, we will work with so-called L-fuzzy relations, not to artificially increase the level of mathematical abstractness, but because lattice theory provides the proper framework for formulating and solving fuzzy relational equations.

Consider a complete lattice $(L, \leq)$ with smallest element 0 and greatest element 1. Denote the meet and join operation by $\frown$ and $\smile$. An L-fuzzy set A in a universe X is simply an $X \to L$ mapping; an L-fuzzy relation R from X to Y is an $X \times Y \to L$ mapping, see Goguen (1967). From here on, we will omit the prefix L if no confusion can occur. The binary Boolean logical operations $\wedge$ and $\to$, for instance, can be extended in the following obvious way:

(i) a *conjunctor* $\mathcal{C}$ is a binary operation on L with order-preserving partial mappings such that $\mathcal{C}(0, 1) = \mathcal{C}(1, 0) = 0$ and $\mathcal{C}(1, 1) = 1$;

(ii) an *implicator* $\mathcal{I}$ is a binary operation on L with order-reversing first partial mappings and order-preserving second partial mappings such that $\mathcal{I}(0, 1) = \mathcal{I}(1, 0) = 0$ and $\mathcal{I}(1, 1) = 1$.

Obviously, additional properties can be imposed on these operators. For instance, if a conjunctor is also required to be commutative, associative and to have 1 as neutral element, then we are in the setting of t-norms, see De Cooman and Kerre (1994) and Schweizer and Sklar (1983). For the sake of simplicity and in view of their familiarity to the reader, we will mainly restrict to t-norms from here on. Most results, however, can be stated much more generally. The three most important continuous t-norms on the real unit interval are the minimum operator M, the (algebraic) product P and the Łukasiewicz t-norm W defined by $W(x, y) = \max(x + y - 1, 0)$.

We are now ready to introduce the basic operations of fuzzy relational calculus. Consider a fuzzy relation R from X to Y. The *afterset* xR of $x \in X$ is the fuzzy set in Y defined by $xR(y) = R(x, y)$. The *foreset* Ry of $y \in Y$ is the fuzzy set in X defined by $Ry(x) = R(x, y)$. Let $\mathcal{C}$ be a conjunctor and $\mathcal{I}$ be an implicator. Given a fuzzy set A in X, the *direct image* $R(A)$, *subdirect image* $R_{\triangleleft}(A)$ and *superdirect image* $R_{\triangleright}(A)$ of A under R are the fuzzy sets in Y defined by:

$$\begin{aligned} R(A)(y) &= \sup_{x\in X} \mathcal{C}(A(x), R(x,y)) \\ R_{\triangleleft}(A)(y) &= \inf_{x\in X} \mathcal{I}(A(x), R(x,y)) \\ R_{\triangleright}(A)(y) &= \inf_{x\in X} \mathcal{I}(R(x,y), A(x)). \end{aligned}$$

Given a second fuzzy relation S from Y to Z, the *(round) composition* $R \circ S$, *subcomposition* $R \triangleleft S$ and *supercomposition* $R \triangleright S$ of R and S are the fuzzy relations from X to Z defined by:

$$\begin{aligned} R\circ S(x,z) &= \sup_{y\in Y} \mathcal{C}(R(x,y), S(y,z)) \\ R\triangleleft S(x,z) &= \inf_{y\in Y} \mathcal{I}(R(x,y), S(y,z)) \\ R\triangleright S(x,z) &= \inf_{y\in Y} \mathcal{I}(S(y,z), R(x,y)). \end{aligned}$$

For an in-depth study of these operations and their applications, we refer to De Baets and Kerre (1995).

3 Types of Inverse Problems

As in the Boolean case, we can consider the following types of inverse problems:

(i) *image equations*, i.e. equations of the type $R(A) = B$, $R_{\triangleleft}(A) = B$ or $R_{\triangleright}(A) = B$ in the unknown fuzzy set A or in the unknown fuzzy relation R;

(ii) *composition equations*, i.e. equations of the type $R \circ S = T$, $R \triangleleft S = T$ or $R \triangleright S = T$ in the unknown fuzzy relation R or in the unknown fuzzy relation S.

These inverse problems are what one usually calls *fuzzy relation(al) equations*. They can all be formulated as systems of particular lattice equations. Consider a binary operator $\mathcal{O}$ on L with monotone partial mappings, then we can distinguish the following four basic types of equations, given a family $(a_i)_{i\in I}$ in L and b in L:

◇ left sup-$\mathcal{O}$ equation: $\sup_{i\in I} \mathcal{O}(x_i, a_i) = b$;

◇ right sup-$\mathcal{O}$ equation: $\sup_{i\in I} \mathcal{O}(a_i, x_i) = b$;

◇ left inf-$\mathcal{O}$ equation: $\inf_{i\in I} \mathcal{O}(x_i, a_i) = b$;

◇ right inf-$\mathcal{O}$ equation: $\inf_{i\in I} \mathcal{O}(a_i, x_i) = b$.

These equations can be considered as *polynomial lattice equations.* Finite index sets will be denoted I_n, containing the first n integers. A first observation is that, due to the monotonicity of the operator $\mathcal{O}$, the solution set of any of the above equations is *order convex*, i.e. if $(x_i)_{i\in I}$ and $(z_i)_{i\in I}$ are two solutions of one of the above equations, then so is $(y_i)_{i\in I}$, whenever $x_i \leq y_i \leq z_i$, for any $i \in I$. This allows us to focus on extremal solutions, i.e. maximal (greatest) and minimal (smallest) solutions. In case of a commutative operator $\mathcal{O}$, there is, of course, no need to talk about left or right equations.

Let us consider the image equations $R(A) = B$ and composition equations $R \circ S = T$:

- ◇ the equation $R(A) = B$ in the unknown fuzzy relation R, called image equation of *type 1*, is equivalent to a family of independent sup-$\mathcal{C}$ equations in the foresets Ry, $y \in Y$:

$$\sup_{x\in X} \mathcal{C}(A(x), Ry(x)) = B(y);$$

- ◇ the equation $R(A) = B$ in the unknown fuzzy set A, called image equation of *type 2*, is equivalent to a system of sup-$\mathcal{C}$ equations, $y \in Y$:

$$\sup_{x\in X} \mathcal{C}(A(x), R(x,y)) = B(y);$$

- ◇ the composition equation $R \circ S = T$ in the unknown fuzzy relation R or S can be reformulated as a system of image equations of type 1 or as a family of independent image equations of type 2.

Of course, the same observations hold for the other images and compositions.

In this paper we will only deal with sup-$\mathcal{T}$ equations, with $\mathcal{T}$ a t-norm, and the above corresponding fuzzy relational equations. Note that the associativity of the t-norm $\mathcal{T}$ plays no role, while the commutativity is only imposed for reducing the number of residual operators associated with it (see Section 4). It would therefore be sufficient to consider a t-seminorm, see De Cooman and Kerre (1994). Our treatment of fuzzy relational equations is built up in the following way: we will gradually impose conditions on the t-norm involved and restrict the class of lattices considered, in order to go from a description of the greatest solution, and necessary and sufficient solvability conditions, to a full description of the solution set. Conditions imposed on the t-norm are typically continuity conditions, or in the lattice-theoretic framework, *morphism* conditions. Recall that an $L \rightarrow L$ mapping is called a *sup-morphism (inf-morphism)* if for any non-empty subset A of L it holds that $f(\sup A) = \sup f(A)$ ($f(\inf A) = \inf f(A)$). A mapping is called a *homomorphism* if it is both a sup-morphism and an inf-morphism. For instance, in the more familiar case of the unit interval, sup-morphims, resp. inf-morphisms, are nothing else but increasing left-continuous, resp. right-continuous, mappings.

4 Sup-$\mathcal{T}$ Equations

4.1 The Equation $\mathcal{T}(a, x) = b$

The most basic equation we consider is the equation $\mathcal{T}(a, x) = b$ in the unknown x, with $\mathcal{T}$ a t-norm on a complete lattice $(L, \leq)$. In order to describe the solution set of this equation, we associate to a given t-norm $\mathcal{T}$ two binary operators $\mathcal{I}_\mathcal{T}$ and $\mathcal{L}_\mathcal{T}$ on L, called *residual operators*, defined by

$$\mathcal{I}_\mathcal{T}(x, y) = \sup\{z \in L \mid \mathcal{T}(x, z) \leq y\}$$

$$\mathcal{L}_\mathcal{T}(x, y) = \inf\{z \in L \mid \mathcal{T}(x, z) \geq y\}.$$

The first operator is called the *residual implicator* of $\mathcal{T}$ and satisfies $(\forall(x, y) \in L^2)$ $(x \leq y \Rightarrow \mathcal{I}_\mathcal{T}(x, y) = 1)$, the second one has no particular interpretation.

Since the solution set of the equation $\mathcal{T}(a, x) = b$ can be seen as the intersection of the solution sets of the inequalities $\mathcal{T}(a, x) \leq b$ and $\mathcal{T}(a, x) \geq b$, we start by investigating the latter.

On a complete lattice. The following observations hold true:

- ◇ If the partial mapping $\mathcal{T}(a, \cdot)$ is a sup-morphism, then the solution set of the inequality $\mathcal{T}(a, x) \leq b$ is given by $[0, \mathcal{I}_\mathcal{T}(a, b)]$.

- ◇ If the partial mapping $\mathcal{T}(a, \cdot)$ is an inf-morphism, then the solution set of the inequality $\mathcal{T}(a, x) \geq b$ is given by

$$\begin{cases} [\mathcal{L}_\mathcal{T}(a, b), 1] & , \text{ if } b \leq a \\ \emptyset & , \text{ elsewhere} \end{cases}.$$

- ◇ If the partial mapping $\mathcal{T}(a, \cdot)$ is a homomorphism, then the solution set of the equation $\mathcal{T}(a, x) = b$ is given by

$$\begin{cases} [\mathcal{L}_\mathcal{T}(a, b), \mathcal{I}_\mathcal{T}(a, b)] & , \text{ if } b \in \operatorname{rng}(\mathcal{T}(a, \cdot)) \\ \emptyset & , \text{ elsewhere} \end{cases};$$

  if the partial mapping $\mathcal{T}(a, \cdot)$ is a maximally surjective homomorphism (i.e. if $\operatorname{rng}(\mathcal{T}(a, \cdot)) = [0, a]$), then the solution set of the equation $\mathcal{T}(a, x) = b$ is given by

$$\begin{cases} [\mathcal{L}_\mathcal{T}(a, b), \mathcal{I}_\mathcal{T}(a, b)] & , \text{ if } b \leq a \\ \emptyset & , \text{ elsewhere} \end{cases}.$$

On the real unit interval. A t-norm $\mathcal{T}$ on $[0, 1]$ is continuous if and only if all of its partial mappings are homomorphisms. Moreover, the partial mappings of a continuous t-norm $\mathcal{T}$ on $[0, 1]$ are maximally surjective. Hence, if $\mathcal{T}(a, \cdot)$ is continuous, then the solution set of the equation $\mathcal{T}(a, x) = b$ is given by

$$\begin{cases} [\mathcal{L}_\mathcal{T}(a, b), \mathcal{I}_\mathcal{T}(a, b)] & , \text{ if } b \leq a \\ \emptyset & , \text{ elsewhere} \end{cases}.$$

Therefore, a necessary and sufficient solvability condition is given by $b \leq a$.

4.2 Greatest Solution – Solvability Conditions

Let $(a_i)_{i\in I}$ be an arbitrary family in L and $b \in L$, then we want to determine the solution set of the equation

$$\sup_{i\in I} \mathcal{T}(a_i, x_i) = b$$

in the family of unknowns $(x_i)_{i\in I}$ in L. The families $(a_i)_{i\in I}$ and $(x_i)_{i\in I}$ can be seen as L-fuzzy sets in the index set I. The problem can therefore be reformulated as follows. Given $A \in \mathcal{F}_L(I)$ (with $\mathcal{F}_L(I)$ the set of L-fuzzy sets in I) and $b \in L$, determine the solution set of the equation

$$\sup_{i\in I} \mathcal{T}(A(i), X(i)) = b \tag{1}$$

in the unknown L-fuzzy set X in I.

Recall that $(\mathcal{F}_L(I), \subseteq)$ is a complete lattice, with order relation $\subseteq$ (inclusion) defined by

$$A \subseteq B \Leftrightarrow (\forall i \in I)(A(i) \leq B(i)),$$

and as infimum and supremum the intersection and union defined by

$$\left(\bigcap_{j\in J} A_j\right)(i) = \inf_{j\in J} A_j(i) \quad \text{and} \quad \left(\bigcup_{j\in J} A_j\right)(i) = \sup_{j\in J} A_j(i).$$

The solution set of equation (1) then is a subset of the complete lattice $(\mathcal{F}_L(I), \subseteq)$ and can be seen as the intersection of the solution sets of the inequalities

$$\sup_{i\in I} \mathcal{T}(A(i), X(i)) \leq b \tag{2}$$

and

$$\sup_{i\in I} \mathcal{T}(A(i), X(i)) \geq b. \tag{3}$$

Obviously, due to monotonicity considerations, inequality (2) always has a solution, namely $\chi_\emptyset$, while inequality (3) has a solution if and only if χ_I is a solution, i.e. if and only if

$$b \leq \sup_{i\in I} A(i). \tag{4}$$

In general, inequality (3) is a lot harder to solve than inequality (2). For the moment, we are only concerned with the greatest solution of equation (1).

On a complete lattice. The following statements can be verified:

- ◇ If the partial mappings of $\mathcal{T}$ are sup-morphisms, then the solution set of inequality (2) is given by $[\chi_\emptyset, G]$ with G defined by

  $$G(i) = \mathcal{I}_\mathcal{T}(A(i), b). \tag{5}$$

  In particular, this result shows us the solution set for equation (1) in case $b = 0$.

- ◇ If the partial mappings of $\mathcal{T}$ are sup-morphisms, then the solution set of equation (1) is not empty if and only if the fuzzy set G defined by (5) is a solution. Moreover, if G is a solution, then it is the greatest solution.

Necessary and sufficient solvability conditions. Notice that, in general, no simple necessary and sufficient solvability condition for equation (1) can be given: one has to construct the potential greatest solution and verify that it is indeed a solution. In some cases, however, such a simpler condition can be formulated:

- ◇ In any case, condition (4) is a necessary condition for the existence of a solution to equation (1). If L is a complete chain and the partial mappings of $\mathcal{T}$ are maximally surjective homomorphisms, then condition (4) becomes also a sufficient condition. Moreover, in case $\mathcal{T}$ is the meet operation, then L need not be totally ordered (in this case, we are working in a complete Brouwerian lattice), see Zhao (1987).

- ◇ If the partial mappings of $\mathcal{T}$ are maximally surjective homomorphisms, then the condition
$$(\exists k \in I)(b \leq A(k)), \tag{6}$$
a stronger version of condition (4), is a sufficient condition for solvability.

On the real unit interval. For a left-continuous t-norm $\mathcal{T}$ on $[0, 1]$, the solution set of equation (1) is not empty if and only if the fuzzy set G defined by (5) is the greatest solution. Moreover, if $\mathcal{T}$ is continuous, then condition (4) is a necessary and sufficient solvability condition.

4.3 Complete Solution Set

For most practical considerations, it is sufficient to know the greatest solution of equation (1). From a mathematical point of view, one is, of course, highly interested in the complete solution set. Before we can discuss this solution set, we need to develop an appropriate representation for it. Obviously, it would be desirable to find a representation that is closed under (arbitrary) intersections, since that would learn us how to solve systems of equations at the same time. Fortunately, such a representation exists.

Root systems. A subset R of an ordered set $(P, \leq)$ is called a *root system* if there exists an element σ in P and an antichain O in $\downarrow \sigma = \{x \in P \mid x \leq \sigma\}$ such that
$$R = \bigcup_{\omega \in O} [\omega, \sigma].$$

For a root system R, the corresponding element σ and antichain O are unique. The element σ is called the *stem* and the elements of the antichain O are called the *offshoots* of the root system. A root system is called *finitely generated* if the set of offshoots is finite. The stem is the greatest element and the offshoots are the minimal elements of the root system. Hence, a root system clearly is an order convex subset that is completely determined by its greatest element and its minimal elements.

The following results are of extreme importance in the present discussion, see De Baets (1995b, 1998b):

- ◇ Let $(R_i)_{i\in I}$ be a family of finitely generated root systems of a complete lattice $(L, \leq)$ with stem σ_i and set of offshoots O_i. If the intersection $\bigcap_{i\in I} R_i$ is not empty, then it is a root system with stem $\sigma = \inf_{i\in I} \sigma_i$ and as offshoots the minimal elements of the set

$$\{\sup_{i\in I} \omega_i \mid (\forall i \in I)(\omega_i \in O_i \wedge \omega_i \leq \sigma)\}. \tag{7}$$

- ◇ If the intersection of a finite family of finitely generated root systems of a complete lattice is not empty, then it is a finitely generated root system. These offshoots can be determined in an exhaustive way, or following an iterative algorithm, see De Baets (1995b, 1998b). We will call this method the pen-and-paper method.

If the root systems are not finitely generated, then, in general, nothing can be said about their intersection.

On a complete lattice. So far, we know the greatest solution of equation (1), if there exists a solution at all. Under very general conditions, we can always find a root system of $(\mathcal{F}_L(I), \subseteq)$ that is contained in the solution set, see De Baets (1995b). Indeed, if the partial mappings of $\mathcal{T}$ are maximally surjective homomorphisms and condition (6) holds, then the solution set of equation (1) contains the root system with stem G defined by (5) and as set of offshoots the set $\{M_k \mid b \leq A(k)\}$, where M_k is defined by

$$M_k(i) = \begin{cases} \mathcal{L}_\mathcal{T}(A(k), b) & , \text{if } i = k \\ 0 & , \text{elsewhere} \end{cases} . \tag{8}$$

Moreover, these offshoots are minimal solutions. If the index set I is finite, then this root system is, of course, finitely generated. This result shows how particular minimal solutions, that could be considered as Dirac-solutions or point solutions, can be constructed. In general, it is not known how other minimal solutions can be determined. As pointed out further on, in some cases, the above root system coincides with the solution set. For instance, under the same conditions on $\mathcal{T}$ and condition (6), but for a complete chain L and a finite index set I, the solution set coincides with the above root system, which is then finitely generated.

On a distributive, complete lattice with join-irreducible and join-decomposable elements. So far, the above discussion reveals only the complete solution set in case of a complete chain and under some additional conditions. However, there is an interesting class of lattices, in particular from a practical point of view, in which the complete solution set can be determined. Let us first recall some order-theoretic notions, see Birkhoff (1967) and Davey and Priestley (1990):

(i) A lattice $(L, \leq)$ is called *distributive* if the following property holds:

$$(\forall(a,b,c) \in L^3)(a \frown (b \smile c) = (a \frown b) \smile (a \frown c)).$$

For instance, any chain is distributive and also the Cartesian product of distributive lattices is distributive.

(ii) An element a of a lattice $(L, \leq)$ is called *join-irreducible* if

$$(\forall(b,c) \in L^2)(b \smile c = a \Rightarrow (b = a \vee c = a)),$$

and *join-decomposable* if there exists a set A of join-irreducible elements of L, with $\#A \geq 2$, such that $a = \sup A$.

For instance, all elements of a chain are join-irreducible, and all elements of a finite distributive lattice or a Cartesian product of a finite number of chains are join-irreducible or join-decomposable.

In order to describe the complete solution set of equation (1), we now direct our attention towards inequality (3), inequality (2) being solved already on a complete lattice. The following results hold, for a distributive, complete lattice $(L, \leq)$ and a finite index set I, see De Baets (1995a):

◇ **The case of a join-irreducible right-hand side.** If b is join-irreducible, the partial mappings of $\mathcal{T}$ are inf-morphisms and the solution set of inequality (3) is not empty (see condition (4)), then it is a finitely generated root system with stem χ_I and as set of offshoots the set $\{M_k \mid b \leq A(k)\}$, where M_k is defined by (8).

◇ **The case of a join-decomposable right-hand side.** If b has a join-decomposition $b = \sup_{j \in J} b_j$, the partial mappings of $\mathcal{T}$ are inf-morphisms and the solution set of inequality (3) is not empty (see condition (4)), then it is a root system with stem χ_I and as offshoots the *minimal* elements of the set

$$\{\bigcup_{j \in J} M_k^j \mid b_j \leq A(k)\},$$

where M_k^j is defined by

$$M_k^j(i) = \begin{cases} \mathcal{L}_{\mathcal{T}}(A(k), b_j) & , \text{if } i = k \\ 0 & , \text{elsewhere} \end{cases}. \tag{9}$$

If the join-decomposition is finite, then this root system is also finitely generated.

Combining this information with our knowledge of the solution set of inequality (2) leads to a full description of the solution set of equation (1), again for a distributive, complete lattice and a finite index set I, see De Baets (1995a):

◇ **The case of a join-irreducible right-hand side.** If b is join-irreducible, the partial mappings of $\mathcal{T}$ are homomorphisms and the solution set of equation (1) is not empty, then it is a finitely generated root system with stem G defined by (5) and as set of offshoots the set $\{M_k \mid b \leq A(k) \wedge M_k \subseteq G\}$, where M_k is defined by (8); moreover, if the partial mappings of $\mathcal{T}$ are maximally surjective, then the set of offshoots can be written as $\{M_k \mid b \leq A(k)\}$. In the case of a complete chain, the latter result was already mentioned above.

◇ **The case of a join-decomposable right-hand side.** If b has a join-decomposition $b = \sup_{j\in J} b_j$, the partial mappings of $\mathcal{T}$ are homomorphisms and the solution set of equation (1) is not empty, then it is a root system with stem G defined by (5) and as offshoots the *minimal* elements of the set

$$\{\bigcup_{j\in J} M_k^j \mid b_j \leq A(k) \wedge M_k^j \subseteq G\},$$

where M_k^j is defined by (9). If the join-decomposition is finite, then this root system is also finitely generated, and if $\mathcal{L}_\mathcal{T} \leq \mathcal{I}_\mathcal{T}$, then the offshoots are the *minimal* elements of the set

$$\{\bigcup_{j\in J} M_k^j \mid b_j \leq A(k)\}.$$

Note that if L is a complete chain and the partial mappings of $\mathcal{T}$ are maximally surjective, then the inequality $\mathcal{L}_\mathcal{T} \leq \mathcal{I}_\mathcal{T}$ always holds.

These results have been obtained by Zhao (1987) in the case of a complete Brouwerian lattice, i.e. when choosing as t-norm the meet operation, of which all elements have a finite irredundant decomposition in join-irreducible elements. They have been generalized and rephrased in the terminology of root systems by De Baets (1995a).

On the real unit interval. The real unit interval is a distributive, complete lattice of which all elements are join-irreducible. Hence, the above results can be applied. For a continuous t-norm $\mathcal{T}$ and a finite index set I, the solution set of equation (1) is not empty if and only if condition (6) holds. This also follows from the previous subsection, since for finite I conditions (4) and (6) are equivalent. If the solution set is not empty, then it is the finitely generated root system with stem G defined by (5) and as set of offshoots the set $\{M_k \mid b \leq A(k)\}$, where M_k is defined by (8).

On the real unit hypercube. A very attractive lattice satisfying the above requirements is the unit hypercube $([0,1]^m, \leq)$. From an implementational point of view, the real unit hypercube is the ultimate ordinal and numerical working environment; it allows for incomparability and offers at the same time the possibility to fall back on the underlying real unit interval. Moreover, it is a distributive, complete lattice and all elements of it are either join-irreducible or have a finite join-decomposition. Consider $a \in [0,1]^m$, $a \neq (0,\ldots,0) = 0_m$, then a can be decomposed as

$$a = \sup_{a(i)\neq 0} \underline{a}_i$$

with $\underline{a}_i = (0,\ldots,a(i),\ldots,0)$.

From the above discussion, it is clear that we should consider a t-norm $\mathcal{T}$ on $[0,1]^m$ with partial mappings that are homomorphisms. However, this implies that the t-norm $\mathcal{T}$ is the direct product of m t-norms $\mathcal{T}_i$, $i = 1,\ldots,m$, on $[0,1]$, i.e. for any x and y in $[0,1]^m$ it holds that

$$\mathcal{T}(x,y) = (\mathcal{T}_1(x(1),y(1)),\ldots,\mathcal{T}_m(x(m),y(m))).$$

The latter follows even under weaker conditions, see De Baets and Mesiar (1998). For the sake of simplicity, let us consider a t-norm $\mathcal{T}$ on $[0,1]$ and the t-norm $\mathcal{T}^{(m)}$ on $[0,1]^m$ defined by:

$$\mathcal{T}^{(m)}(x,y) = (\mathcal{T}(x(1),y(1)),\ldots,\mathcal{T}(x(m),y(m))).$$

The corresponding operators $\mathcal{I}_{\mathcal{T}^{(m)}}$ and $\mathcal{L}_{\mathcal{T}^{(m)}}$ are then given by, with $1_m = (1,\ldots,1)$, see De Baets (1995a):

$$\mathcal{I}_{\mathcal{T}^{(m)}}(x,y) = (\mathcal{I}_{\mathcal{T}}(x(1),y(1)),\ldots,\mathcal{I}_{\mathcal{T}}(x(m),y(m)))$$

and

$$\mathcal{L}_{\mathcal{T}^{(m)}}(x,y) = (\mathcal{L}_{\mathcal{T}}(x(1),y(1)),\ldots,\mathcal{L}_{\mathcal{T}}(x(m),y(m)))$$

if $(\forall i \in I_m)(y(i) \leq x(i))$, and $\mathcal{L}_{\mathcal{T}^{(m)}}(x,y) = 1_m$ elsewhere.

Combining the above results on join-decomposable and join-irreducible right-hand sides, we can state the next result. The most important difference is that now the offshoots can be written down *immediately*, and do not have to be determined as minimal elements of an auxiliary set. If $\mathcal{T}$ is continuous, I is finite and the solution set of the equation

$$\sup_{i\in I} \mathcal{T}^{(m)}(A(i),X(i)) = b \tag{10}$$

is not empty, then it is a finitely generated root system with stem G defined by

$$G(i) = \mathcal{I}_{\mathcal{T}^{(m)}}(A(i),b)$$

and as set of offshoots the set

$$\{\bigcup_{j\in J\wedge b(j)\neq 0} M_k^j \mid b(j) \leq A(k)(j)\}$$

with M_k^j defined by

$$M_k^j(i) = \begin{cases} \mathcal{L}_{\mathcal{T}^{(m)}}(A(k), \underline{b}_j) & \text{, if } i = k \\ 0_m & \text{, elsewhere} \end{cases}.$$

Note that $\mathcal{L}_{\mathcal{T}^{(m)}}(A(k), \underline{b}_j) = (0, \ldots, \mathcal{L}_{\mathcal{T}}(A(k)(j), b(j)), \ldots, 0)$.

4.4 Systems of Sup-$\mathcal{T}$ Equations

It should be clear from the above that in many important cases the solution set of a sup-$\mathcal{T}$ equation, if not empty, is a finitely generated root system. Recalling that the intersection of a family of finitely generated root systems, if not empty, is again a root system, leads immediately to the description of solution procedures for systems of sup-$\mathcal{T}$ equations.

The problem we are concerned with is the following: given a complete lattice $(L, \leq)$ and a family $(A_s)_{s\in S}$ in $\mathcal{F}_L(I)$ and a family $(b_s)_{s\in S}$ in L, determine the solution set of the system of equations $(E_s)_{s\in S}$

$$\sup_{i\in I} \mathcal{T}(A_s(i), X(i)) = b_s \tag{11}$$

in the unknown L-fuzzy set X in I.

Greatest solution. If the partial mappings of $\mathcal{T}$ are sup-morphisms, then the solution set of system (11) is not empty if and only if the fuzzy set $G = \bigcap_{s\in S} G_s$, with G_s the potential greatest solution of equation E_s, i.e.

$$G(i) = \inf_{s\in S} \mathcal{I}_{\mathcal{T}}(A_s(i), b_s), \tag{12}$$

is a solution. Moreover, if G is a solution, then it is the greatest solution.

In general, no simpler necessary and sufficient solvability conditions are known. Let us therefore turn our attention to the description of the complete solution set.

On a distributive, complete lattice with join-irreducible or finitely join-decomposable elements. Consider a finite index set I and assume that the right-hand sides of system (11) are join-irreducible or finitely join-decomposable. If the partial mappings of $\mathcal{T}$ are homomorphisms and the solution set of system (11) is not empty, then it is a root system; if the system is finite, then this root system is finitely generated. Suppose that the solution is not empty, and that the solution sets of the equations E_s are the root systems with stem G_s and set of offshoots O_s, then the solution set of the system $(E_s)_{s\in S}$ is the root system with stem $G = \bigcap_{s\in S} G_s$ and as set of offshoots the set

$$\{\bigcup_{s\in s} N_s \mid N_s \in O_s \wedge N_s \subseteq G\}.$$

A necessary and sufficient solvability condition therefore is that G is a solution. The case of a complete Brouwerian lattice was again discussed by Zhao (1987), the above more general case by De Baets (1998a).

5 Fuzzy Relational Equations

We are now in a position to tackle what are called fuzzy relational equations. For the sake of brevity, we will restrict ourselves to the real unit interval; the reader can easily generalize the results to the more general setting of distributive, complete lattices with join-irreducible or finitely join-decomposable elements.

Image equations of type 1. Consider a fuzzy set A in X and a fuzzy set B in Y. As mentioned before, the image equation $R(A) = B$ in the unknown fuzzy relation R is equivalent to a family of independent sup-$\mathcal{T}$ equations in the foresets Ry, $y \in Y$. This equation is an important identification problem in fuzzy rule modelling: determine a relational correspondence given the input A and the output B. Clearly, the solution set of a family of independent equations is isomorphic to the Cartesian product of the solution sets of the individual equations. Since the Cartesian product of root systems obviously also is a root system, our knowledge of sup-$\mathcal{T}$ equations is sufficient for solving this type of image equation. Note that the set of offshoots of a Cartesian product of root systems is nothing else but the Cartesian product of the corresponding sets of offshoots. For instance, the solution set is not empty if and only if the solution sets of all individual equations are not empty.

We then have that for a left-continuous t-norm $\mathcal{T}$, the solution set of the above image equation is not empty if and only if the fuzzy relation G defined by

$$G(x, y) = \mathcal{I}_\mathcal{T}(A(x), B(y)) \tag{13}$$

is the greatest solution. In case of a complete Brouwerian lattice and the meet operation, this result is known as one of the very first results on fuzzy relational equations, see Sanchez (1977). The case of the real unit interval was discussed by Di Nola, Sessa, Pedrycz and Sanchez (1989). For a continuous t-norm $\mathcal{T}$, a necessary and sufficient solvability condition is given by

$$\sup_{y \in Y} B(y) \leq \sup_{x \in X} A(x), \tag{14}$$

or, in words, the height of A is not smaller than the height of B. In the case of the real unit interval, this condition can also be found in the work of Gottwald and Pedrycz (1988) and Gottwald (1994).

Using the above framework of sup-$\mathcal{T}$ equations and root systems, the complete solution set can be written down immediately. If the universe X is finite, $\mathcal{T}$ is continuous and the solution set of the equation $R(A) = B$ is not empty, then it is a root system with stem G defined by (13) and as set of offshoots the set

$$O = \{M \mid (\forall y \in Y)(My \in O_y)\},$$

where

$$O_y = \{M_u^y \mid B(y) \leq A(u)\}$$

with M_u^y defined by

$$M_u^y(x) = \begin{cases} \mathcal{L}_\mathcal{T}(A(u), B(y)) & \text{, if } x = u \\ 0 & \text{, elsewhere} \end{cases} .$$

If the universe Y is also finite, then this root system is finitely generated.

The complete solution set was described first by Sanchez (1977) for the case of the real unit interval, finite underlying universes and the minimum operator as t-norm. The generalization to continuous t-norms has been done by Di Nola, Sessa, Pedrycz and Sanchez (1989). In both sources, explicit proofs are given.

Image equations of type 2. Consider a fuzzy relation R from X to Y and a fuzzy set B in Y. The image equation $R(A) = B$ in the unknown fuzzy set A is equivalent to a system of sup-$\mathcal{T}$ equations. This inverse problem plays a crucial role in backward chaining in fuzzy rule-based systems. Here, we can immediately apply the results from the previous section.

For a left-continuous t-norm $\mathcal{T}$, the solution set of this equation is not empty if and only if the fuzzy set G defined by

$$G(x) = \inf_{y \in Y} \mathcal{I}_\mathcal{T}(R(x, y), B(y)) \tag{15}$$

is the greatest solution. Denoting the converse (or transpose) of a fuzzy relation R by R^t, i.e. $R^t(y, x) = R(x, y)$, we can rewrite G as follows:

$$G = (R^t)_\triangleright(B),$$

using as implicator the residual implicator $\mathcal{I}_\mathcal{T}$. In view of the importance of this equation, we also mention the complete solution set. If the universe X is finite, $\mathcal{T}$ is continuous and the solution set of the equation $R(A) = B$ is not empty, then it is a root system with stem G defined by (15) and as offshoots the minimal elements of the set

$$\{\bigcup_{y \in Y} M_u^y \mid B(y) \le R(u, y) \wedge M_u^y \subseteq G\}$$

where M_u^y is defined by

$$M_u^y(x) = \begin{cases} \mathcal{L}_\mathcal{T}(R(u, y), B(y)) & \text{, if } x = u \\ 0 & \text{, elsewhere} \end{cases} . \tag{16}$$

This type of image equation has received a lot of attention in the literature. Most attempts concern finite universes, the real unit interval (or a complete chain) and mostly the minimum operator. The case of a (left-)continuous t-norm is also discussed in Pedrycz (1985), Di Nola, Pedrycz and Sessa (1987) and Di Nola, Sessa, Pedrycz and Sanchez (1989). However, using our knowledge about the intersection of finitely generated root systems, the above results can be been derived easily and more generally.

Composition equations. Consider a fuzzy relation S from Y to Z and a fuzzy relation T from X to Z. The composition equation $R \circ S = T$ in the unknown fuzzy relation R is equivalent to a family of independent image equations $S(xR) = xT$ in the unknown aftersets xR, for all $x \in X$. The solution set therefore is isomorphic to the Cartesian product of the solution sets of these image equations, each of which is nothing else but a system of sup-$\mathcal{T}$ equations. Of course, the equation $R \circ S = T$ in the unknown fuzzy relation S is solved at the same time, since it can be reformulated as follows: $S^t \circ R^t = T^t$.

Therefore, for a left-continuous t-norm $\mathcal{T}$, the solution set of this equation is not empty if and only if the fuzzy relation G defined by

$$G(x,y) = \inf_{z \in Z} \mathcal{I}_{\mathcal{T}}(S(y,z), T(x,z)) \tag{17}$$

is the greatest solution. We can rewrite G as follows:

$$G = T \triangleright S^t,$$

using as implicator the residual implicator $\mathcal{I}_{\mathcal{T}}$. In the case of a complete Brouwerian lattice and the meet operation as t-norm, this is again one the first results of Sanchez (1977); in this case, there also exists an alternative necessary and sufficient solvability condition by Di Nola (1990). In the case of finite universes, a completely distributive complete lattice $(L, \leq)$ and the meet operation as t-norm, the solution set is also a root system. Indeed, Di Nola (1987, 1990) has shown that for any solution at least one underlying minimal solution can be constructed; however, not all minimal solutions can be constructed explicitly. This result complements the results obtained in our general framework. It is clear that for specific operators and in specific situations, additional results can be obtained. Of course, in case of join-irreducibility and join-decomposability, we are again in our framework.

However, there is another way of dealing with the equation $R \circ S = T$. Indeed, it can be seen as a system of image equations $S(xR) = xT$ in the unknown fuzzy relation S, for all $x \in X$. At first sight, the latter approach seems to be the most practical one, in particular in view of determining the minimal solutions. Indeed, since each of the image equations $S(xR) = xT$ in the unknown fuzzy relation S stands for a set of independent sup-$\mathcal{T}$ equations, their minimal solutions can be written down immediately and the pen-and-paper method has to be applied only *once* to find the minimal solutions of the composition equation $R \circ S = T$. In the first approach, the minimal solutions of each of the independent image equations $S(xR) = xT$ in the unknown aftersets xR have to be found using the pen-and-paper method. The minimal solutions of the composition equation $R \circ S = T$ are then found immediately. However, in practice it is difficult to say which method is the most efficient one. In the second approach, the pen-and-paper method has to be applied only once, but then possibly on a huge number of *fuzzy relations*, while for each of the independent image equations the minimal solutions have to be found among a smaller number of fuzzy sets. The second approach is the one usually followed in the literature, see Di Nola, Sessa, Pedrycz and Sanchez (1989), while we favour the first one, in particular from an implementational point of view.

6 Conclusion

When the input-output behaviour of a system is (only) described linguistically by an expert, the fuzzy systems modelling approach can be called on. A system model is then built by solving fuzzy relational equations. In this paper, we have explained how these inverse problems can be solved in the framework of polynomial lattice equations and have presented a uniform framework in which various types of image and composition equations can be dealt with. Future work will consist of providing a similar framework for approximate solution methods in case of voidness of the solution set. Also the incorporation of the fuzzy systems modelling approach in other systems modelling approaches, such as Rosen's (1985) anticipatory systems, will be envisaged.

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