RANDOMNESS IN THE BIFUZZY SET THEORY

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Abstract

The present paper includes a survey of notions in probability theory, carried over to the ground of the theory of bifuzzy sets. At the same time, it shows a possible combination of randomness and bifuzziness and signals other relationships of both the theories.

Keywords: fuzzy set, probability, probability of a fuzzy event, bifuzzy set.

1 Introduction

When, in 1965, L. A. Zadeh published his first paper concerning fuzzy sets, there appeared opinions of opponents who scented in fuzziness a certain form of probability theory. Their suspicion arose from a false interpretation of a membership function of a fuzzy set as well as from identifying it with a probability function. The difference consists in the fact that randomness concerns phenomena defined precisely, and the indefiniteness connected with them comes from a random character of their appearance. Whereas fuzziness concerns the indefiniteness lying in the very description of phenomena. This, however, does not exclude a simultaneous occurrence of the randomness and fuzziness of experiments.

In 1968 there appeared a paper by L. A. Zadeh initiating the investigations of the problem of the combining of these two theories. The idea described in it still remains a model for the conceptions of fuzzy probability theory which are being created and modified. So, according to that paper, we consider a probability space $(\Omega, \mathcal{B}, P)$ with the σ-field of Borel subsets of a non-fuzzy set Ω and a probability function P. Here, by a fuzzy event in Ω we mean any fuzzy set A whose membership function $\mu_A : \Omega \to \langle 0,1 \rangle$ is Borel measurable. Then the integral

$$P(A) = \int_{\Omega} \mu_A(x) dP(x) \tag{1}$$

defines a probability of the fuzzy event A.

A completely different conception of fuzzy probability theory was described by R. Yager (1979). The unconventional character of his approach consists in the fact that the computed probability is not a number from the interval $\langle 0,1\rangle$, but a fuzzy set treated as a fuzzy number over the space $\langle 0,1\rangle$. R. Yager's approach is based on the notion of an α-level of a fuzzy set as well as on the decomposition theorem and the extension principle (see: Zadeh, 1975).

Adopting the assumptions concerning a fuzzy event as before, R. Yager obtained

$$\underset{\sim}{P}(A) = \bigcup_{\alpha=0}^{1} \alpha * P(A_\alpha) \tag{2}$$

where $P(A_\alpha)$ is a probability (in the ordinary sense) of a non-fuzzy event A_α, while $\underset{\sim}{P}(A)$, as the fuzzy union of the sets $\alpha * P(A_\alpha)$, is the so-called fuzzy probability of a fuzzy event A in $(\Omega, \mathcal{B}, P)$.

Other conceptions of the creation of probability theory for fuzzy events and constructions of fuzzy probability spaces are presented in paper of Gerstenkorn and Mańko (1996).

2 Bifuzzy Sets and Events

Among many papers devoted to fuzzy sets there appear conceptions generalizing the notion of a fuzzy set.

One of such generalizations is K. Atanassov's idea from 1983 (published with S. Stoeva in 1985) of an intuitionistic set. However, in the paper presented here we shall constantly use the name "bifuzzy set" instead of "intuitionistic set" in order not to insinuate any connection of K. Atanassov's idea with the intuitionists' philosophical trend in mathematics (on the other hand, the name "intuitionistic fuzzy set" was used for the first time by Takeuti and Titani (1984), being simultaneously based on intuitionistic fuzzy logic. Basing ourselves on paper of Atanassov (1986), we adopt

DEFINITION 1. By a bifuzzy set A in the space X under consideration we mean an object of the form

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\} \tag{3}$$

where the functions $\mu_A, \nu_A : X \to \langle 0,1\rangle$ satisfy the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \tag{3'}$$

and define, respectively, the degrees of the belonging and of the non-belonging of an element x to the bifuzzy set A. The family of bifuzzy sets in X denoted by BFS(X) (for self-evident reasons, K. Atanassov in his works uses then the symbol IFS(X)).

Then a fuzzy (in the sense of Zadeh) set L is

$$L = \{(x, \mu_L(x), 1 - \mu_L(x)) : x \in X\} \quad (4)$$

and an ordinary non-fuzzy set K, written down as a bifuzzy one, has the form

$$K = \{(x, \chi_K(x), 1 - \chi_K(x)) : x \in X\}. \quad (5)$$

In particular, the empty bifuzzy set is $\phi = \{(x,0,1) : x \in X\}$, whereas a bifuzzy space is the set $U = \{(x,1,0) : x \in X\}$.

DEFINITION 2. For $A, B \in \mathrm{BFS}(X)$, we define the relations of inclusion and equality as well set operations as

(1) $A \subset B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ and $v_A(x) \ge v_B(x)$

(2) $A = B \Leftrightarrow A \subset B$ and $B \subset A$

(3) $A \cup B = \{(x, \mu_{A\cup B}(x), v_{A\cup B}(x) : x \in X\}$, where

$\mu_{A\cup B}(x) = \max\{(\mu_A(x), \mu_B(x)\}$ and $v_{A\cup B}(x) = \min\{v_A(x), v_B(x)\}$

(4) $A \cap B = \{(x, \mu_{A\cap B}(x), v_{A\cap B}(x) : x \in X\}$ where

$\mu_{A\cap B}(x) = \min\{(\mu_A(x), \mu_B(x)\}$ and $v_{A\cap B}(x) = \max\{v_A(x), v_B(x)\}$

(5) $A' = \{(x, \mu_{A'}(x), v_{A'}(x) : x \in X\}$, where $\mu_{A'}(x) = v_A(x)$ and $v_{A'}(x) = \mu_A(x)$

for all $x \in X$.

It is not difficult to justify that the operations $\cup$ and $\cap$ are associative, commutative, mutually distributive and idempotent and satisfy de Morgan's laws, whereas the properties

$A \cap A' = \phi$ and $A \cup A' = U$

are not valid.

An unconventional example of a bifuzzy set and the definitions of other operations can be found in Atanassov (1986). In paper of Gerstenkorn and Mańko (1990/91) there have appeared, for the first time, a conception of combining K. Atanassov's idea with probability and a construction of a probability measure for bifuzzy events. This construction lies in making use of a probability space $(E, \mathcal{F}, P)$ in the ordinary sense, that is, in the sense that $\mathcal{F}$ stands for the σ-field of subsets of a set E, while P- a probability measure on $\mathcal{F}$, and in adapting it to a bifuzzy set. In the family BSF(E) we then consider only those bifuzzy sets for which membership and non-membership functions are measurable. Such particular bifuzzy sets will be called bifuzzy events, and their family denoted by BM(E) (see: Mańko, 1992).

3 Probability of Bifuzzy Events

DEFINITION 3. For $A \in \mathrm{BM}(X)$, the number

$$P(A) = \int_E \frac{\mu_A(x) + 1 - \nu_A(x)}{2} P(dx) \tag{6}$$

is called a probability of the bifuzzy event A (Mańko, 1992).

The function thus defined satisfies the classical probability properties required by A. Kolmogorov, that is,

(1) $P(A) \geq 0$

(2) $P(U) = 1$

(3) $P(A \cup B) = P(A) + P(B)$ for $A \cap B = \emptyset$

(4) $P(\emptyset) = 0$

(5) $P(A) \leq 1$

(6) $P(A') = 1 - P(A)$

(7) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ for $A, B \in \mathrm{BM}(E)$

and, at the same time, reduces to formula (1) in the case when K. Atanassov's bifuzziness reduces to the fuzziness in the sense of L. A. Zadeh.

Gerstenkorn and Mańko (1998a) presented a conception of a bifuzzy probability of bifuzzy events, referring to R. Yager's idea. It is based on the notion of an (α, β)-level of a bifuzzy set and on the extension principle of Stojanova (1990). Therefore we introduce, for any numbers $\alpha, \beta \in \langle 0,1 \rangle$ such that $\alpha + \beta \leq 1$, the notion of a product of the pair (α, β) and the bifuzzy set A as the set $(\alpha, \beta) * A$ in the form

$$(\alpha, \beta) * A = \{(x, \alpha \cdot \mu_A(x), \beta + (1 - \beta) \cdot \nu_A(x)) : x \in E\} \tag{7}$$

and an (α, β)-level of the bifuzzy set A as

$$A_{\alpha,\beta} = \{x \in E : \mu_A(x) \geq \alpha \text{ and } \nu_A(x) \leq \beta\}. \tag{8}$$

With that, the non-fuzzy set

$$N_{\alpha,\beta}(A) = \{(x, 1, 0) : x \in A_{\alpha,\beta}\} \tag{9}$$

is called an analogue of the (α, β)-level of the bifuzzy set A.

In virtue of formula (7), we then have

$$(\alpha,\beta)* N_{\alpha,\beta}(A) = \{(x,\alpha,\beta):\ x \in A_{\alpha,\beta}\}. \tag{10}$$

Stojanova proved the theorem on the decomposition of a set $A \in \mathrm{BFS}(E)$ into its (α,β)-levels as

$$A = \bigcup_{\alpha,\beta} (\alpha,\beta)* N_{\alpha,\beta}(A) \tag{11}$$

where the symbol $\bigcup$ denotes the union in the sense of definition 2 and extends to all $\alpha,\beta \in \langle 0,1\rangle$ such that $\alpha+\beta \le 1$. With that, let $f: G \to H$ be any function in the ordinary sense, G and H - any non-fuzzy sets. Let $A \in \mathrm{BFS}(G)$. The formula

$$f(A) = \{(f(x), \mu_A(x), \nu_A(x)): f(x) \in H\} \tag{12}$$

defines the image of the bifuzzy set A under the mapping f, that is, extends the domain of operation of the function f from sets in the ordinary sense to bifuzzy ones. It is sometimes convenient to write formula (12) down in the form

$$f(A) = \bigcup_{\alpha,\beta} (\alpha,\beta)* f(N_{\alpha,\beta}(A)) \tag{13}$$

being the contents of the so-called bifuzzy extension principle (Stojanova, 1990).

Now, using the probability function P in formula (13) and maintaining the assumptions about the function P and the bifuzzy event A to be as in (6), we get the so-called fuzzy probability $\widetilde{P}(A)$ of a bifuzzy event $A \in \mathrm{BM}(E)$ as

DEFINITION 4

$$\widetilde{P}(A) = \bigcup_{\alpha,\beta} (\alpha,\beta)* P(N_{\alpha,\beta}(A)). \tag{14}$$

The above procedure is a generalization of that proposed by R. Yager (1979) and was discussed more extensively by Gerstenkorn and Mańko (1998a, 1998b).

4 The Classical Probability

Let now $E = \{x_1, x_2, \ldots, x_n\}$ be a non-fuzzy finite set and let $A \in \mathrm{BFS}(E)$ be a bifuzzy set A in E with a membership function $\mu_A(x_i)$ and a non-membership function $\nu_A(x_i)$. Then we accept (Gerstenkorn and Mańko, 2000) to mean by the cardinality of the set A a real number *card* (A) defined by the formula

$$card(A) = \sum_{i=1}^{n} \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} \tag{15}$$

being a natural generalization of the notion of the cardinality of a fuzzy set of de Luca and Termini (1972). Extending formula (13) to a function $f = card$, we then define the so-called bifuzzy cardinality of the set A in the form of a bifuzzy set $card_f(A)$ as

$$card_f(A) = \bigcup_{\alpha,\beta} (\alpha,\beta) * card(N_{\alpha,\beta}(A)) . \tag{16}$$

Let us now suppose that a discrete probability function has been defined in E according to the assumptions of Laplace, i.e. that each x_i is equally possible and appears as a result of some random experiment with a constant probability $\frac{1}{n}$ for each x_i $(i = 1,2,\ldots,n)$. In accordance with the so-called classical Laplace definition of probability, we then have for $A \in \mathrm{BM}(E)$,

$$P(A) = \frac{card(A)}{card(E)} = \sum_{i=1}^{n} \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} p(x_i) \tag{17}$$

which is a generalization of formula (1) carried over to the ground of a finite space of elementary events for (6).

Using now formula (16), we simultaneously have

$$\tilde{P}(A) = \frac{card_f(A)}{card(E)} = \bigcup_{\alpha,\beta} (\alpha,\beta) * P(N_{\alpha,\beta}(A)) \tag{18}$$

which, in turn, generalizes formula (2) and is a special case of (14).

The comparison of both the situations described here and numerical examples can be found in Gerstenkorn and Mańko (2000).

5 Bifuzzy Probabilistic Sets

One of the alternative conceptions of combining randomness and fuzziness are fuzzy and bifuzzy probabilistic sets. It is the theory extending of K. Hirota's paper (1981), given (first for the probabilistic sets) by Gerstenkorn and Mańko (1992).

So, let $(\Omega, \mathcal{B}, P)$ be, as before, a probability space and let $(\langle 0,1\rangle, \mathcal{B}_{\langle 0,1\rangle})$ be the characteristic space associated with it. According to Gerstenkorn and Mańko (1995a), we adopt

DEFINITION 5. By a bifuzzy probabilistic set in a space $X \neq \emptyset$ we mean a bifuzzy set described by defining functions $\mu_A, \nu_A \colon X \times \Omega \to \mathcal{B}_{\langle 0,1\rangle}$ where $\mu_A(x,\cdot)$ and $\nu_A(x,\cdot)$ are $(\mathcal{B}, \mathcal{B}_{\langle 0,1\rangle})$-measurable and

$$0 \leq \mu_A(x,\cdot) + \nu_A(x,\cdot) \leq 1 \tag{19}$$

for almost all $x \in X$.

It turns out that the family of bifuzzy probabilistic sets is a partially ordered set forming a Boolean pseudo-algebra.

The conception of bifuzzy probabilistic sets enables one to define and to analyse parameters characteristic of a distribution of a random variable. So, we have

DEFINITION 6. For a bifuzzy probabilistic set A in $X \times \Omega$, by its mean value $E_A(x)$, variance $V_A(x)$ and n-th moment $m_A^n(x)$ we mean the numbers

$$E_A(x) = \int_\Omega \frac{\mu_A(x,\omega) + 1 - \nu_A(x,\omega)}{2} \cdot P(d\omega) \tag{20}$$

$$V_A(x) = \int_\Omega \left[\frac{\mu_A(x,\omega) + 1 - \nu_A(x,\omega)}{2} - E_A(x) \right]^2 \cdot P(d\omega) \tag{21}$$

$$m_A^n(x) = \int_\Omega \left(\frac{\mu_A(x,\omega) + 1 - \nu_A(x,\omega)}{2} \right)^n \cdot P(d\omega) \tag{22}$$

with a fixed $x \in X$.

6 Concluding Remarks

The ideas of connecting the theory of bifuzzy sets with probability theory, described in Sections 3, 4 and 5, are not the only possibilities of combining them. A completely separate and independent subject matter are the notions of bifuzzy entropy, energy, correlation and independence, being a generalization of the conception of measuring fuzziness and sharpness as well as the dependence of bifuzzy sets both in the deterministic case (Gerstenkorn and Mańko (1991); Mańko 1993) and in the random one (Gerstenkorn and Mańko, 1994, 1995b). The premises for creating such conceptions, coming from various fields of science such as, for instance, the theory of pattern recognition, decision-making, artificial intelligence, and the like, were:

a) the indefiniteness of objects,

b) the variability of the object being examined,

c) the subjectivity of the observers,

d) the evolution of the knowledge and the educational process of the observers.

And, although there function mathematical theories combining separately premises selected from the above ones (e.g. multi-valued logic for a) and b)), modal logic for b) and d)), probability calculus for c)), neither of them embraces all.

The ideas presented in our paper look very promising at this background and can have great possibilities in applications, being, at the same time, quite simple numerically.

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