

Gompertz and Verhulst Frameworks for Growth AND Decay Description

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Abstract

Verhulst logistic curve either grows OR decays, depending on the *growth rate* parameter sign. A similar situation is found in the Gompertz law about human mortality. It is aimed to encompass into ONE simple differential equation the growth AND decay features of, e.g., population sizes, or numbers. Previous generalizations of Verhulst or Gompertz functions are recalled. It is shown that drastic growth or decay jumps or turnovers can be readily described through drastic changes in values of the growth or decay rate. However smoother descriptions can be found if the growth or decay rate is modified in order to take into account some time or size dependence. Similar arguments can be carried through, but not so easily, for the so called carrying capacity, indeed leading to more elaborate algebraic work.

Keywords : Logistic law; Gompertz equation; time and size dependence of growth and decay rate; time and size dependence of carrying capacity

1 Introduction

Verhulst 1845 seminal paper [1] bore on previous (1798) considerations by Malthus [2] on population *size* evolution. Verhulst model attempts to describe how the growth (or *mutatis mutandis* its decay) of a *single* species towards its equilibrium population is limited, taking into account the reproduction rate r and the nowadays so called *carrying capacity* M , i.e., the maximum value the population can sustain in the long term in its environment., due to limited resources. He obtained what he called the logistic curve (or map). In a subsequent paper, Verhulst [3] attempted a different mathematical form of the blocking growth term, thereby obtaining an exponentially slow approach of the carrying capacity, but not a sigmoid.

There have been many considerations starting from Verhulst approach of population *size* evolution, generalizing his original work. However, it is of interest to recall that among the generalizations, one should distinguish between those which stick to one single differential equation, and those which involve a set of equations, because they consider a competition effect with another population. Among the former ones,

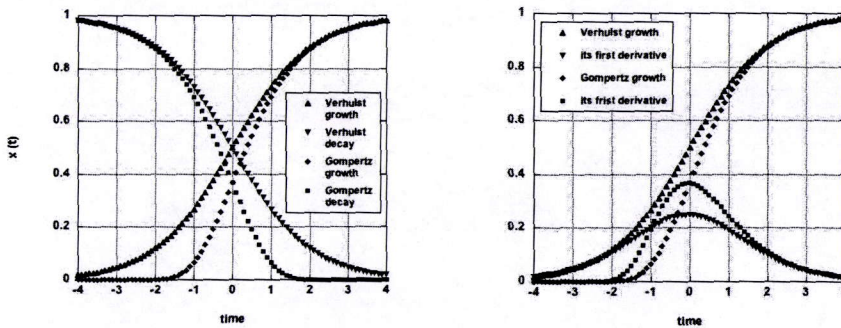


Fig. 1: (l.h.s): comparing Verhulst and a Gompertz growth or decay laws; (r.h.s): comparing the first derivative, - in the case of growth laws; see the asymmetry with respect to its inflection point in each Gompertz law

I am fond of the 1939 work by Avrami [4] on crystal growth, - the so called species changing its thermodynamic phase or state, but there are many others. Among the latter, one has only to go back to the work of Lotka, published [5], in 1925, and of Volterra [6], in 1931, pertaining to so called *prey-predator models*. Not attempting to cite here many subsequent publications., let it be pointed out that Blanco [7] has suggested a trivial relationship between the logistic equation and the Lotka-Volterra (LV) models, indicating that the former is in fact a sort of limit of the LV equations, pending a redefinition of the parameters.

However, growth can neither be infinite nor reach a finite steady state at an infinite asymptotic time, as in the logistic map. Moreover before some decay, some growth must have occurred. In particular, some leveling-off could occur at finite time, though followed either by some growth again or then by some decay, in both cases, with a smooth or an oscillatory variation.

It is of interest in contrast, to recall ideas by Gompertz [8] who in a series of papers near 1825 discussed 'the Nature of the Function Expressive of the Law of Human Mortality, and (...) a New Mode of Determining the Value of Life Contingencies'. The Gompertz model for describing the decreasing number of members of a population can be in fact easily reformulated in terms of a (simple first order differential) *growth* equation of Malthusian form, with an *exponentially decaying birth rate*. This leads to another form of an analytical sigmoid curve, though asymmetric, in contrast to the Verhulst logistic map.

In brief, depending on the sign of the parameters in Verhulst or Gompertz approach, one can describe some growth or some decay see Fig. 1 (l.h.s). However numerous examples show that both Verhulst and Gompertz modelizations are thus too reductive (or restrictive) descriptions of their original purpose. Whence, it is

of interest to aim at finding some simple description combining some less trivial behavior, i.e. obviously, practically inducing some extremum at some time in the function which should be the solution of a, by request, simple first order differential equation. This is the aim of this note, i.e. to encompass into ONE simple first order differential equation *growth AND decay* features of, e.g., population sizes, or numbers, but also of many other measured characteristics found in social and physical science systems.

In Sect. 2, after recalling Malthus and Verhulst equations, a few applications are mentioned in order to substantiate the broad framework in which Verhulst equation can be applied. *Growth*, indeed, is not only found in population studies, through counting the number of individuals in some system, but can be *measured* through other means, like the number of stems, the mass, the length, ... of "something". It is often found, of course, that the Verhulst sigmoid is only a rough representation of the data. Whence one has sometimes invented artificially skewed logistic curves [9]; see Sect. 2, for a general introduction to *growth* and such subsequent considerations.

Subsequent considerations on decay, with examples about languages and religions, are to be found in Sect. 3, before outlining Gompertz law. Discussion about bio-demography applications can be found in an interesting 1997 review, up by Olshansky and Carnes [10]. Nevertheless, *ad hoc*, previous generalizations of Verhulst or Gompertz functions are briefly recalled to conclude Sect. 3.

In Sect. 4, a few examples show that the Verhulst and Gompertz modelizations are indeed too reductive (or restrictive) descriptions of reality. However, drastic growth or decay jumps or evolution turnovers can be readily described through drastic changes in values of the growth or decay rate, as shown in Sect. 5. Sometimes the evolution show drastic jumps or drops indeed, - due to exogenous or endogenous causes. However smoother descriptions must be found, i.e. combining some less trivial behavior. Obviously, one should aim at practically inducing some extremum at some time in a smooth function which should be the solution of a, by request, simple first order differential equation.

It is aimed, in Sect. 6, to encompass into such ONE simple differential equation the growth AND decay features of population sizes, or numbers, but also of many other measured characteristics found in social and physical science systems. In fact one can observe that one could consider that the growth or decay rate has either some time or some size dependence. The algebra is rather simple in order to recover a Verhulst or Gompertz evolution equation. In Sect. 7, the time and/or size dependence of the so called carrying capacity is considered in the same spirit as the growth rate. However this is less trivial, indeed leading to more elaborate algebraic work. Sect. 8 serves as a conclusion.

2 Introduction to Growth

Following the above historical points, let us do some mathematics and data analysis within Verhulst equation framework.

2.1 Growth Equations

First, recall the Malthus equation for describing a (normalized) population *size* x evolution as a function of time t , due to a birth rate r , i.e.

$$\frac{dx}{dt} = r x, \quad (1)$$

which has the (catastrophic) exponential solution $x(t) = e^{rt}$.

The original Verhulst population growth model considers that resources are limited, such that the evolution of the number of members $N(t)$ in the population is described in terms of a carrying capacity M and a growth rate r , according to

$$\frac{dN}{dt} = r N \left[1 - \frac{N}{M} \right]. \quad (2)$$

With the change of variable $x = N/M$, if M is a constant, a dimensionless form equation can be thereby written, i.e.

$$\frac{dx}{dt} = r x [1 - x]. \quad (3)$$

It has for solution, the so called logistic map, a sigmoid curve,

$$x = \frac{e^{rt}}{1 + e^{rt}}, \quad (4)$$

if r is a constant as well. Notice that Verhulst had examined complicated forms of the growth limiting term, like a square, but also a square root [1], both cases without any theoretical justification, but aiming at better fitting available data. Indeed, there is no biological justification for this assumption; Gilpin, Case, & Ayala [11] proposed a growth equation of the form $dx/dt = r x [1 - (x/M)^\theta]$, - a larger than 1 value of θ resulting in a steepening of the growth curve and a higher value for the inflection point, as it is easily calculated.

2.2 Applications of the Logistic Map

Let us point out to a few cases of the "Verhulst population growth model", in order to show some universality value of the logistic map application.

- Population size: Beside those considered by Verhulst, an interesting application on population size is that of Montroll and Badger [12], in 1974, who studied the USA population between 1810 and 1970. An interesting parameter value is found, i.e. $r = 0.02984$, which implies interesting considerations on American sexuality during those times.

- Mass growth: However the logistic map can also be applied to, e.g., mass growth, like in the report of Lee and Werning [13], in 2008, who compared Alligators, Elephants, Shrews and Ostriches, on one hand, but in the other hands Tenontosauruses, Allosauruses, and Tyrannosauruses, using skeletochronology; see reference [13] for the parameter r values, in kg/yr , and subsequent discussions. For completeness, let us mention that the growth curves were compared with an alternative model to that of Verhulst, i.e. the von Bertalanffy [14] curve of mass accumulation.
- Length growth: As far as 1928, Pearl group studied the Cucumis Melo stem length, either under field conditions [15] or in the absence of exogenous food and light [16]. The *population* was considered to be the number of nodes. Notice that r is of the order of 0.7 in their observations.

2.3 Skewed Logistic

In so doing, i.e., studying Canteloup growth, Pearl et al. empirically found that one should generalize the logistic map to better reproduce the data. Rather than the mere logistic, they propose a form like

$$y(t) = \frac{k}{1 + e^{a_0 + a_1 t + a_2 t^2 + \dots}}, \quad (5)$$

where y , in [16], is the number of seedlings of the canteloups. It is *quasi* obvious that a more general form which is discussed below, is

$$y(t) = \frac{k}{1 + e^{f(t)}}. \quad (6)$$

3 Introduction to Decay

Recent applications along Verhulst (limited growth) approach have been concerned with the *decay* of a population size, rather than its growth.

3.1 Religions

The time evolution of several "main" religions was considered to be described, at a so called microscopic level, along the lines of the Avrami-Kolmogorov equation, describing a liquid-solid state formation in a continuous time framework [17], i.e. a peculiar form of Verhulst law [18]. The solution is usually written as

$$c(t) = 1 - e^{-Kt^n} \quad (7)$$

where $c(t)$ is the volume fraction being transformed from one phase to another; K and n are adjustable parameters. However, it is "obvious" that before decaying the number of adepts of any religion had to grow from the founder, $N(0) = 1$.

Nowadays, several "religions" have still a growing number of adepts. Others have reached a maximum, thus decay, or go again into a growing pattern. Population growth-death equations have thus been conjectured for modeling such religion *size* evolution dynamics, in a continuous time framework [19], [20], [21], [22], [23] and [24] attempting to describe the existence of maxima in the number of adepts. An interpretation of the features, after data fitting, can resort to an attachment parameter, as resulting from joining at first one's "mother religion", within exogenous field conditions, as one "joins" at first one's mother tongue.

3.2 Language Death

For completeness, notice that the disappearance of world's languages has also been recently considered through a Verhulst time evolution equation in terms of the number density ρ_i of practitioners of some language i , in [25] and [26], i.e.,

$$\frac{\partial \rho_i}{\partial t} = r_i \rho_i \left(1 - \frac{\rho_i}{C_i} \right) \quad (8)$$

where C_i is the carrying capacity of the environment for the population speaking the i language and r_i is a "negative growth" rate. The r_i values are at this level of no interest. However notice that the speakers do not fully represent a single so called population, since their member evolution is toward a bilingual attitude. In fact, there is a *competition* between two (or more) languages/populations, - which according to the point of view mentioned in the Introduction rather pertains to a prey-predator problem.

4 Gompertz Death Law

In fact, previous to Verhulst, Gompertz [8] discussed human population mortality dynamics from a practicing actuarial point of view. Gompertz used equal long time intervals, i.e. 10 year long intervals, to observe that the differences between the common logarithm of the number of persons living in such successive equal age intervals were almost identical during a significant portion of their life span. Thus the numbers of living individuals in each yearly increase of age are in geometrical progression. Whence mortality progresses geometrically as age increases arithmetically. Therefore Gompertz proposed a 3 parameter expression for the number $N(t)$ of *survivors* of a population at time t :

$$N(t)/N_{inf} = \exp[-b \exp(-rt)], \quad (9)$$

where N_{inf} , b and r are positive constants. This corresponds to an exponentially decaying birth rate r in Malthus equation, i.e. $r = r_0 e^{-\kappa t}$, pending r_0 and κ being positive constants. For completeness, notice that Makeham introduced an age independent mortality rate to be considered in order to better represent available data

[27], i.e. adding a $A + Bt$ term in the right hand side of Eq.(9), thus attempting to take into account extrinsic effects.

The easily derived differential equation describing the exponential rise in death rates between sexual maturity and old age, is commonly referred to as the Gompertz equation, i.e.,

$$\frac{dx}{dt} = r x \log \left[\frac{k}{x} \right] \quad (10)$$

where k has *mutatis mutandis* the same meaning as M , the carrying capacity, in the Verhulst approach. Notice that Gompertz and Makeham wondered whether k can be interpreted in a socio-biological framework, i.e., as a biological limit to life, based on some *vital force*.

4.1 Skewed Gompertz Law

In the original Gompertz model, r and k are supposed to be constant; b , in Eq.(9), and k , in Eq.(10), are related to each other through r and the initial (or final) conditions on $N(t)$. In 1924, Trachtenberg [28] already examined extensions of Gompertz law, sometimes departing from Gompertz geometric progression hypothesis, - extensions which do not necessarily lead to a sigmoid curve, but can be called skewed Gompertz laws. One of them reads

$$N(t) = N_{inf} e^{[a+bt+ct^2+dt^3+et^4]} \quad (11)$$

It can clearly be the solution of a second order differential equation for $\log(N(t))$, as wished by Gompertz, and presents two inflection points at precise t_α and t_β .

In fact, similarly to Eq.(6), the most general extension of a Gompertz form can be hereby proposed to be

$$N(t) = N_{inf} \exp[-b \exp(-g(t))] \quad (12)$$

4.2 Biomedical Considerations

For the purpose of the reader general information, an interesting 1997 review by Olshansky and Carnes [10], on "ever since Gompertz", should be quoted. However, since the discussion is mainly about demography applications and biological causes of mortality, in various species, - like the vital force concept and in particular the influence of radiations, the paper content is quite aside our present considerations. and not further elaborated upon, though of great interest.

Nevertheless, it is of common knowledge that *exogenous* causes can manifest themselves and influence some "decay" in the number of living individuals in a population. Take the trout population data of Lake Ontario measured by Beeton [29], quoted by Meadows et al. [30], for example. Pollution is supposed to be the influential external field. On the other hand, a case of *endogenous*-like cause

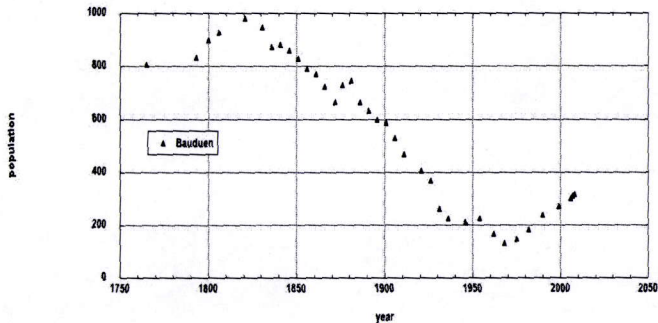


Fig. 2: Evolution of the population in Bauduen, Var, France since *ca.* 1750

was noticed by Kormandy [31], in 1969, who commented on data, which I did not find, and who stated that deer or goats, even though natural enemies are absent, "often overgraze their range and cause erosion or destruction of the vegetation", and consequently die. Finally, in Bauduen, a city in the French department of Var, the city population, increased from 1793 till 1820, thereafter decreased till 1970 and is going back up nowadays, according to the city wikipedia website ([http : //fr.wikipedia.org/wiki/Bauduen](http://fr.wikipedia.org/wiki/Bauduen)). In this case, sharp turnovers are found at the time of war ends; see Fig. 2.

5 Growth and Decay and Growth and Decay and ...

In the case of Lake Ontario trouts, here above mentioned [29], there is obviously a maximum near 1925, followed by a rather sharp drop of three orders of magnitude in 1955 or so. Similar though smoother, features are found for Bauduen population (Fig. 2). In brief, such examples should remind us that, in many measurements, when counting the evolution of a population (of whatever nature), there is a succession of maxima and minima, though the growth and decay regimes can be of different sizes and time spans, - and origins. Remember business cycles as well [32], [33], [34], [35], or [36]. To debate on the evolution laws *per se* is out of question here, but to find an equation containing both types of extrema is the goal of this report, basing our considerations on either Verhulst or Gompertz equation.

5.1 Beyond Verhulst and Gompertz Equations

In view of the above, it is obvious that many multiparameter extensions have been proposed in order to "improve" Verhulst or Gompertz laws [37], see App. A. However, parameter justification is often missing, within our present "agent based

model" requirements. Thus a simple mathematical framework to find several inflection points on a time dependent "growth" curve is still of interest. One may imagine to introduce extra terms, representing endogenous or/and exogenous causes for growth or decay in Verhulst or Gompertz formalisms, like in Bass model [38] of innovative and imitative behavior of consumers, or as in Zwanzig introduction of a (damped exponential) heredity kernel [39], both generalizing Verhulst approach. There is to my knowledge no equivalent extension within Gompertz framework. However both, Bass and Zwanzig considerations, have more physical bases than other purely mathematically oriented extensions.

For completeness, let us recall Bass model [38] of new product acquisition by consumers; it leads to a single "diffusion" equation with constant coefficients, - perfectly suitable to our aims:

$$\frac{dX}{dt} = [N - X(t)] \left[p + \left(\frac{q}{N} \right) X(t) \right]. \quad (13)$$

where $X(t)$ is the number of product users at time t ; N is the number of potential users, like the carrying capacity previously, p is the so called coefficient of innovation, and q is the so called coefficient of imitation. It is a Verhulst-like equation, though the r.h.s has an extra zero order term. The solution reads

$$X(t) = N \frac{1 - e^{-(p+q)t}}{1 + (q/p)e^{-(p+q)t}}. \quad (14)$$

By differentiation, one can easily find the time at which the function has a maximum as well as its value.

On the other hand, Zwanzig [39] writing of Verhulst equation amounts to a change of variable $e^v = N/M$, - somewhat as in Montroll [40] considering that the evolution concerns $\log(N)$ rather than N , and the introduction of a heredity kernel $K(t) = K(0) e^{t/T}$, such that one obtains an integro-differential equation (not written here). $K(0)T$ is in fact like the growth rate r in Eq.(2). For finite T , one obtains an oscillatory but damped evolution, with an overshoot of the carrying capacity value in fact; see Fig. 1 in [39].

Both previous formalisms present at least one hump in the "population" evolution. Nevertheless already within e.g. Verhulst framework, it can be observed that describing either (i) sharp turn-overs or (ii) drop or jump transitions is possible. It is for example, simple to introduce a drastic change in the r value. Similar situations can be illustrated with Gompertz framework, as shown for both cases here below.

5.2 Sharp Turn-overs

The four possible cases of sharp turn-overs forcing a continuous evolution of the population number, in a Verhulst approach, due to a drastic shift in r values, i.e. from some r to some $-r$ value, or the opposite, at some time t , are shown in Fig. 3. Similar curves are shown for the Gompertz evolution law in Fig. 4. Note the pronounced asymmetry in the latter case.

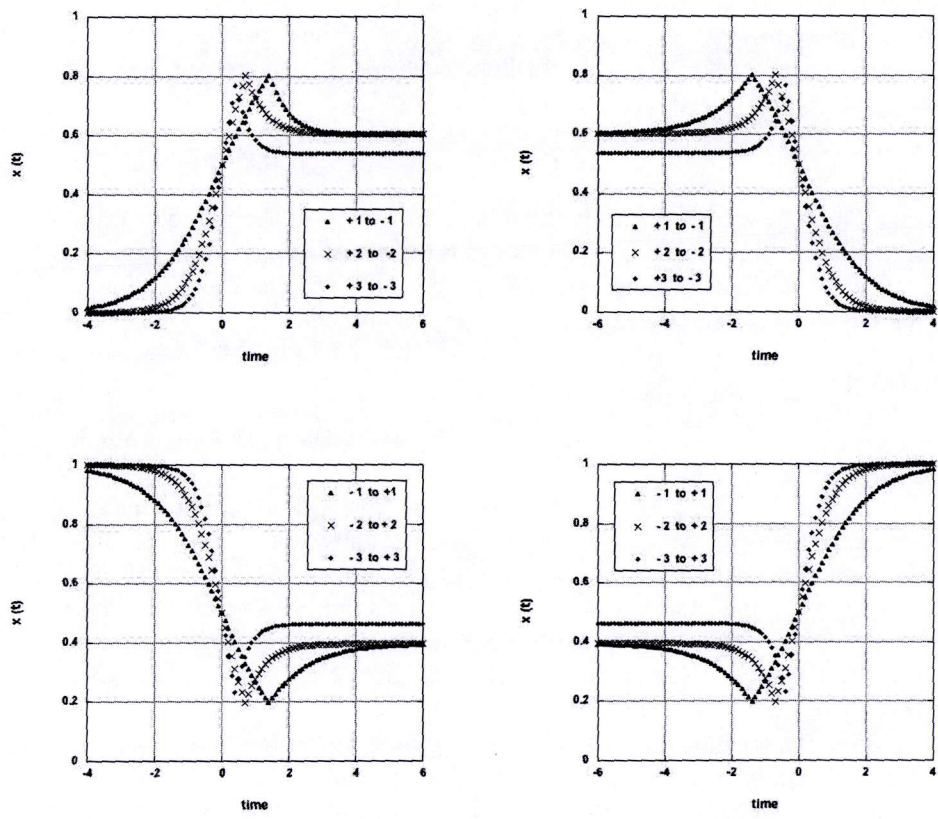


Fig. 3: The four possible cases of sharp turn overs with continuous evolution in a Verhulst approach due to a drastic shift in r values, see insert, at different times t

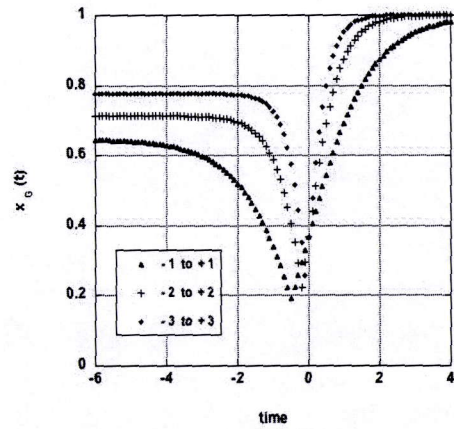
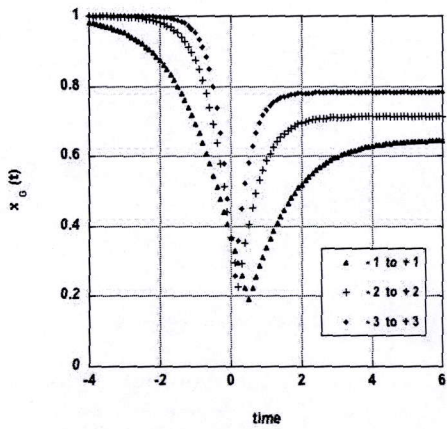
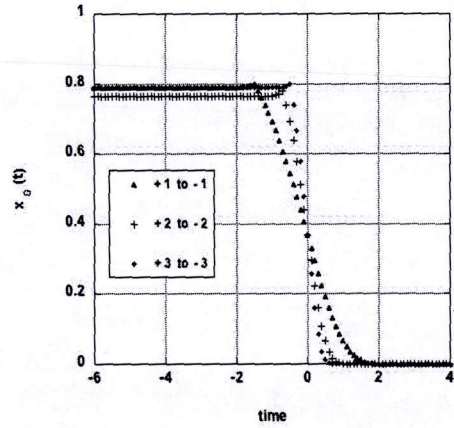
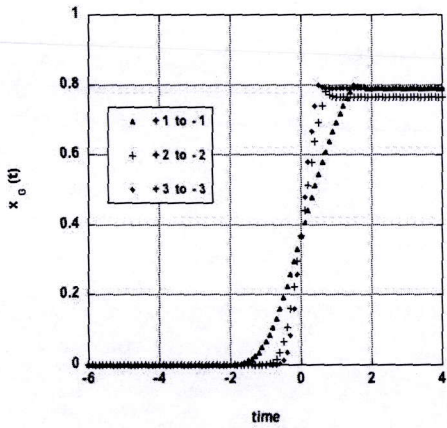


Fig. 4: The four possible cases of sharp turn overs with continuous evolution in a Gompertz approach due to a drastic shift in r values, see insert, at various times t

5.3 Drop or Jump Transitions

On the other hand, the four possible cases of population drops or jumps in a Verhulst approach due to a drastic shift in r values, at some fixed time t are shown in Fig. 5, as if a strong external event had influenced a shift from some r to some $-r$, or the contrary. Similar curves are shown for the Gompertz evolution law in Fig. 6.

5.4 Time and Size Roles

However such drastic changes of signs of the growth rate are rather unlikely or very rough approximation of reality. A smoother evolution of r must be searched for. In this respect, let us recall that the dimension-prone Verhulst differential equation, Eq.(2), is easily integrated,

- if the growth rate r is constant, and
- if the carrying capacity M is a constant.

Let us first observe that there are two equivalent ways to write both Verhulst and Gompertz equations, either emphasizing the *time* role or the *size* role.

5.4.1 Time Role

It is easy to observe that, after an appropriate simple change of variables, the differential equation corresponding to Eq.(3), e.g. for the skewed logistic, Eq.(6), reads

$$\frac{dy}{dt} = \frac{b}{k} y [k - y] v(t), \quad (15)$$

- thereby defining $v(t)$. One can easily integrate both sides of the equation, by the technique of separation of variables, i.e.,

$$\frac{dy}{y [k - y]} = \frac{b}{k} v(t) dt, \quad (16)$$

and easily obtain the skewed logistic in so doing.

5.4.2 Size Role

Another interesting way, for further discussion, is to rewrite the evolution equation, letting the time t in the right hand side be an implicit variable, thus referring to the evolution through the size itself, i. e. let

$$\frac{dy}{dt} = V(y), \quad (17)$$

where $V(y) = ry(1 - y)$ in Verhulst model. Imposing the stability condition $\frac{dV(y)}{dt} = V(y) = 0$, one easily finds out that the r.h.s. is like

$$V(y) = a y + b y^2 + c y^3 + \dots, \quad (18)$$

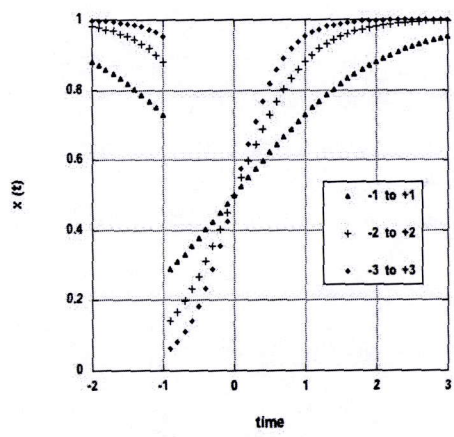
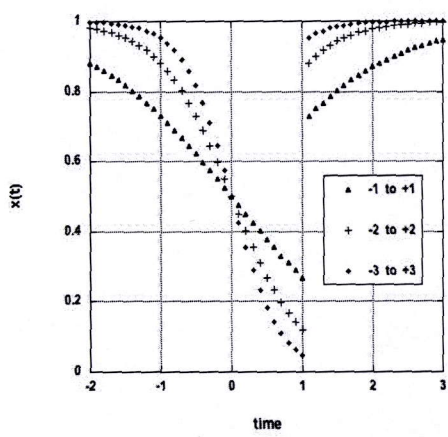
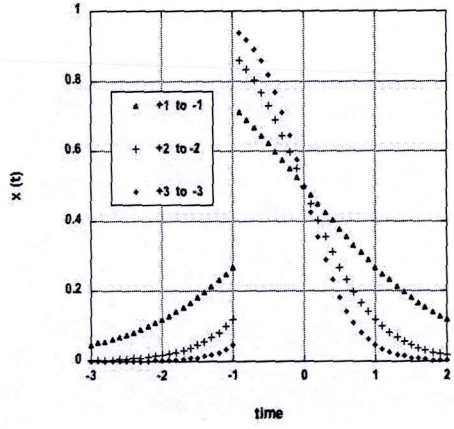
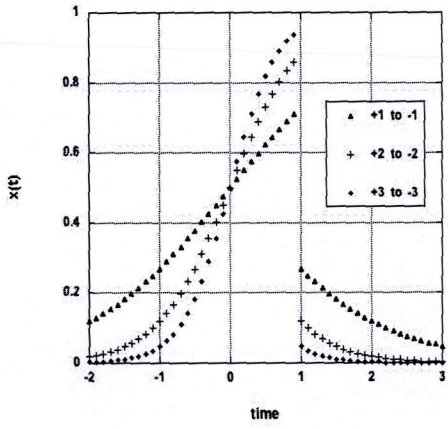


Fig. 5: The four possible cases of population drops or jumps in a Verhulst approach due to a drastic shift in r values, see insert, at some fixed time t ($=+1$, or -1)

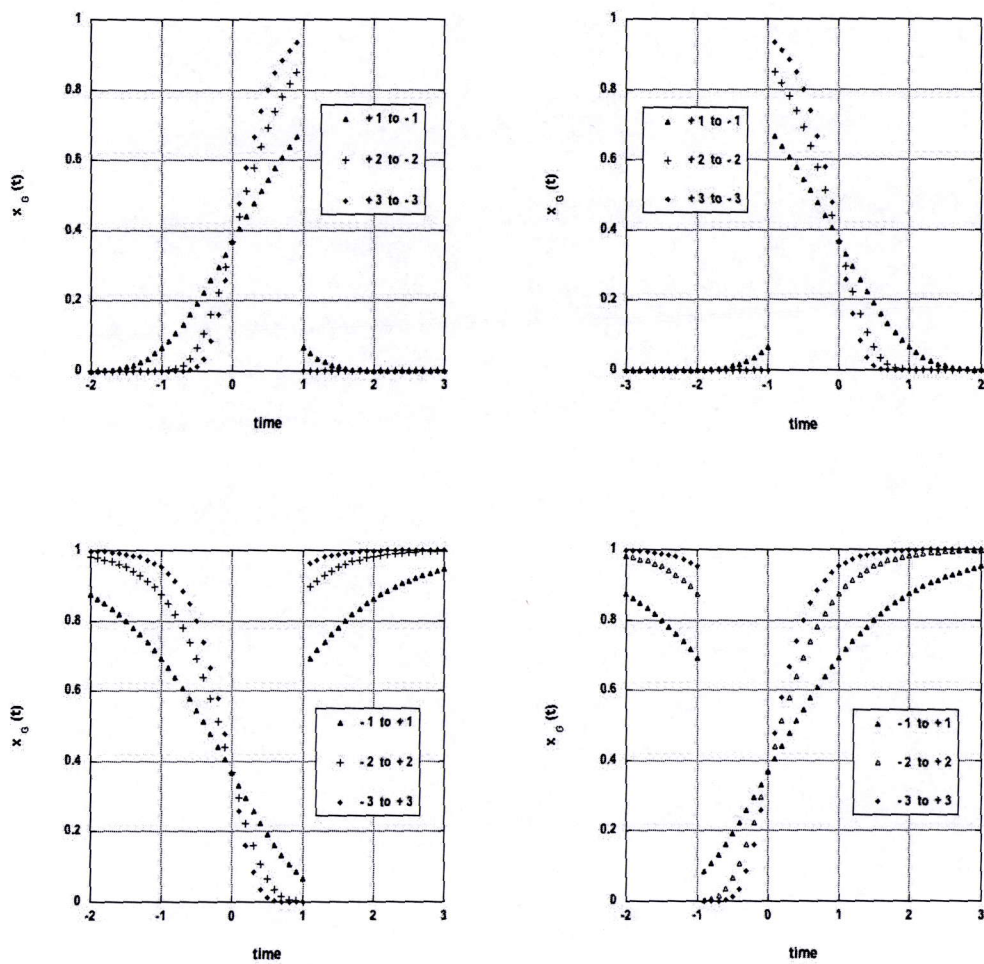


Fig. 6: The four possible cases of population drops or jumps in a Gompertz approach due to a drastic shift in r values, see insert, at some fixed time t ($=+1$, or -1)

i.e. there is no zero order term. Only keeping the cubic term leads to

$$\frac{dy}{dt} = ay(y - y_b)(y - y_c), \quad (19)$$

therefore simply generalizing the Verhulst equation, through the introduction of two inflection points at y_b and y_c .

5.5 Second Order Derivative

Without going into elementary considerations for mathematicians and physicists about the conditions on a differential equation to produce, as its solution, a curve with an extremum, it is immediately obvious, in the present framework that to differentiate once more Eq.(4) and Eq.(9) will produce a maximum, because of the introduction of a second inflection point; - see Fig. 1 (r.h.s). However, the goal is to find a *first* order differential equation (ODE) not a *second* ODE. As hinted here above, a time or size dependence of r and/or k ($\sim M$) seems the simplest way.

6 Growth Rate Role

6.1 Growth Rate Role in Verhulst Model

One may rewrite Verhulst equation, Eq.(3), explicitly expressing the *time* dependence on the right hand side, i.e.

$$\frac{dx}{dt} = \frac{re^{-rt}}{(1 + e^{-rt})^2} = \frac{r}{2(1 + \cosh(rt))}. \quad (20)$$

In order to have a curve with an extremum the most simple way seems to consider its derivative; the extremum being located at the inflection point of the logistic map. One finds

$$\frac{d^2x}{dt^2} = r [1 - 2x] \frac{dx}{dt} = r^2 x (1 - x) (1 - 2x), \quad (21)$$

or emphasizing the time dependence on the right hand side, i.e. explicitly writing

$$\frac{d^2x}{dt^2} = \frac{r^2 e^{rt} (1 - e^{rt})}{(1 + e^{rt})^3}. \quad (22)$$

One can also rewrite the above as

$$\frac{d^2x}{dt^2} = \left[r^2 \frac{1 - e^{rt}}{1 + e^{rt}} \right] x (1 - x) \equiv R_V x (1 - x) \quad (23)$$

in order to reproduce the r.h.s. of Verhulst equation, but with a new r , called R_V having a specific time dependence, i.e. as

$$R_V = \left[r^2 \frac{1 - e^{rt}}{1 + e^{rt}} \right] = -r^2 \tanh(rt), \quad (24)$$

- which is not a birth-death growth rate of any known population, up to now. In so doing one obtains the "generalized" time-Verhulst equation, admitting that the growth rate should be time dependent.

Considering a writing in terms of a size dependent r , one can write

$$\frac{d^2x}{dt^2} \equiv \frac{d}{dt} \left(\frac{dx}{dt} \right) = \rho_V \left(\frac{dx}{dt} \right) \left[1 - \left(\frac{dx}{dt} \right) \right] \quad (25)$$

which has the Verhulst form $\frac{du}{dt} = \rho_V u [1 - u]$, with, comparing Eq.(21) and Eq.(23),

$$\rho_V = \frac{R_V/r}{1 - r x (1 - x)} = \frac{r (1 - 2x)}{1 - rx(1 - x)} \quad (26)$$

where ρ_V is expressed in terms of the population *size*. If one wishes ρ_V to be expressed in terms of the time t , one should replace u ($\equiv x$) and R_V by their expression found in Eq.(4) and Eq.(24) respectively; e. g., $R_V = r^2 (1 - 2x(t))$.

6.2 Growth Rate Role in Gompertz Model

Similar manipulations and algebra can be performed for the basic Gompertz model, i.e.,

$$\frac{d^2x}{dt^2} = r^2 \left[\log \left[\frac{k}{x} \right] - 1 \right] x \log \left[\frac{k}{x} \right] \equiv R_G x \log \left[\frac{k}{x} \right], \quad (27)$$

thereby defining R_G , in terms of its size dependence,

$$R_G = r^2 \left[\log \left[\frac{k}{x} \right] - 1 \right]. \quad (28)$$

Therefore, in order to write

$$\frac{d^2x}{dt^2} \equiv \frac{d}{dt} \left(\frac{dx}{dt} \right) = \rho_G \left(\frac{dx}{dt} \right) \log \left[\frac{k}{(dx/dt)} \right] \quad (29)$$

which has the Gompertz form, one has to introduce ρ_G with

$$\rho_G = \frac{R_G/r}{\log [k/x] - \log [r \log (k/x)]} = r \frac{\log [k/x] - 1}{\log [k/x] - \log [r \log (k/x)]} \quad (30)$$

where ρ_G is expressed in terms of the population size. If one wishes ρ_G to be expressed in terms of the time t , one should replace R_G and (dx/dt) by their expression given in Eq.(28) and Eq.(9). It seems that such a writing is out of necessity here.

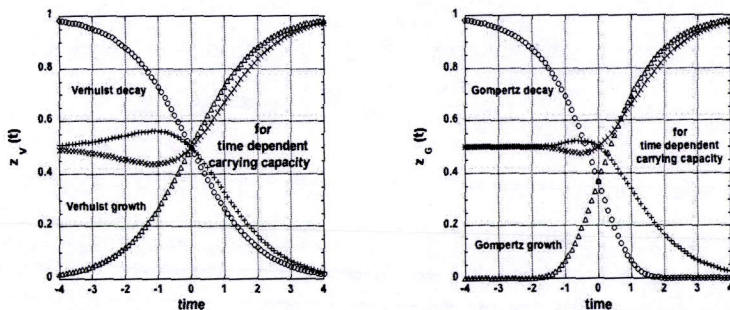


Fig. 7: Possible cases of population evolution in a (l.h.s) Verhulst or a (r.h.s) Gompertz approach when the carrying capacity has a *time* dependent component, as compared to an ordinary growth or decay evolution for constant r or k ($\sim M$); $r = 1$ for all curves; used analytic forms are given in the main text. A bump or a dip can be seen to occur here at finite $t \leq 0$

7 Carrying Capacity Role

Beside the possibility of expressing the growth rate in terms of some time or size dependence, let us recall that k and M in Gompertz and Verhulst model respectively are considered to be constant. To remove such a restriction has already been considered in works referenced to here above. However such a consideration is of highly relevant modern interest in our growing world, and thereby requests some attention.

Note that some population growing in a limited environment can approach the ultimate carrying capacity of that environment in several possible ways: (i) it can adjust smoothly to an equilibrium, (ii) it can overshoot the limit and then dies back, either in a smooth very damped way or (iii) in an oscillatory, more or less damped, way. Let it be emphasized that the population can overshoot the carrying capacity and in the process decreases the ultimate carrying capacity by consuming some necessary nonrenewable resource: e.g., deer or goats often overgraze their range and cause erosion or destruction of the vegetation [31].

7.1 World Model

Related considerations on the carrying capacity are used in Meadows et al. [30] on the "standard" world model. At first it assumes no major change in the physical, economic, or social relationships that have historically governed the development of the world system. Food, industrial output, and population grow exponentially, as according to Malthus [2], until the rapidly diminishing resource base forces a

slowdown in industrial growth. Population growth is finally halted by a rise in the death rate due to decreased food and medical services. However, because of natural delays in the system, the population (and pollution) continues to increase for some time after the peak of industrialization. The evolution of the world features, e.g., its population evolution, has been studied considering plenty of interfering causes, like the doubling of resources, in developments of the "standard" world model [30].

7.2 Carrying Capacity Role in Verhulst and in Gompertz Models

The time and/or size dependence of the so called carrying capacity could be considered in the same spirit as the growth rate. However this is less trivial and apparently can only be tackled through numerical work. Indeed, one can only easily separate the variables when one has either a form $k(x)$, or $k(t)$, in Gompertz model, or $M(N)$, or $M(t)$, in Verhulst model, in Eq.(10) and Eq.(15) respectively, i.e.

$$\frac{dx}{dt} = r x(t) \log \left[\frac{k(x(t))}{x(t)} \right] \quad (31)$$

$$\frac{dN}{dt} = \frac{r}{M(N(t))} N(t) [M(N(t)) - N(t)] , \quad (32)$$

However to go on, some analytic form for k and M is needed. This is leading to more elaborate, less universal, algebraic work, and is outside our present report.

Nevertheless in order to substantiate the role of the carrying capacity time dependence effect in the Verhulst and in Gompertz model, *a posteriori* analytic cases have been considered. A few results are shown in Fig. 7 when a *time* dependent behavior is *a posteriori* included as a mere factor into the original Verhulst or Gompertz laws, r being kept constant. For completeness, the used equations in Fig. 7 read

$$z_V(t) = \frac{1}{2} \left[\frac{e^t}{1+e^t} \right] \left[\pm \frac{e^t - 1}{e^t + 1} + 1 \right] \quad (33)$$

$$z_G(t) = \frac{1}{2} [e^{-e^{-t}}] \left[\pm \frac{e^t - 1}{e^t + 1} + 1 \right] . \quad (34)$$

Bumps or dips are markedly seen, within overall growth or decay features, even in this simple manipulation. A size dependent effect of the carrying capacity either in the Verhulst or Gompertz approach has not been studied.

8 Conclusion

A goal has been to describe some simple way toward one single simple first order differential dynamics equation, recalling that 3-parameter families of curves often describe growth data adequately, like in Verhulst or Gompertz approach. It has

been stressed that extensions of the corresponding equations in order to introduce extrema, or more than one inflection point, are easily found in the second order derivative. However a change of variables is necessary, implying that the growth rate or/and the carrying capacity becomes time (or size) dependent. This corresponds in practice at considering exogenous or/and endogenous causes, in other words, external or internal field interactions with the population, i.e. intra-community interactions in the latter case.

Bass model [38] and Zwanzig model [39] already contain such extrema in an *ad hoc* way, within Verhulst population growth framework. Here, it has been aimed at providing a single diffusion Riccati equation with time/size dependent coefficients of realistic nature, enforcing that the corresponding first ODE is of the Gompertz or Verhulst type. An analytic form for the growth rate coefficients has been derived. The carrying capacity form seems to demand numerical work.

Other ways of interest to describe a population evolution are those taking into account some time delay, as in the world model, and in work on anticipation by Dubois [41] or Rosen [42]. Indeed, in biological and sociological terms, the logistic and Gompertz equations present a logical absurdity, since it requires that cause and effect occur *quasi* simultaneously.

In fine, let it be pointed out that considerations of *discrete* time effects on Gompertz models seem to have not been much studied, - except in [43]: that might be of interest in the framework of CASYS conference reports on chaotic, anticipatory or not, systems. The more so indeed when it is recalled that it was demonstrated by Dubois [44], in the corresponding Verhulst case, that an inclusive recursion is a discrete anticipation removing chaotic behavior, - as if through an intrinsic control parameter.

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Appendix A. Multiparameter Extensions

For completeness, it seems appropriate to recall generalizations of Verhulst and Gompertz laws, beside the so called skewed laws, mentioned in the main text. E.g., one can write indeed

$$G_1(t) = e^{-\rho e^{-t/\sigma}} \quad (35)$$

$$V_1(t) = (1 + \rho e^{-t/\sigma})^{-\theta}, \quad (36)$$

where ρ , σ , and θ are to be positive numbers, - the logistic map corresponds to $\theta=1$.

Ahuja and Nash [37] have further generalized these distributions by introducing an additional parameter $\phi > 0$. The density functions of these distributions are respectively

$$G_2(t; \rho, \sigma, \phi) = \left[\frac{1}{\sigma \Gamma(\phi)} (\rho e^{-t/\sigma})^\phi \right] e^{-\rho e^{-t/\sigma}} \quad (37)$$

$$V_2(t; \rho, \sigma, \theta, \phi) = \left[\frac{1}{\sigma B(\phi, \theta)} (\rho e^{-t/\sigma})^\phi \right] (1 + \rho e^{-t/\sigma})^{-(\theta+\phi)}. \quad (38)$$

However, it seems that there is no physical justification of these parameters to be usefully discussed within the present context.

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