QUANTUM HOLOGRAPHY AND MAGNETIC RESONANCE TOMOGRAPHY: AN ENSEMBLE QUANTUM COMPUTING APPROACH

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Abstract. Coherent wavelets form a unified basis of the multichannel perfect reconstruction analysis-synthesis filter bank of high resolution radar imaging and clinical magnetic resonance imaging (MRI). The filter bank construction is performed by the Keplerian temporospatial phase detection strategy which allows for the stroboscopic and synchronous cross sectional quadrature filtering of phase histories in local frequency encoding multichannels with respect to the rotating coordinate frame of reference. The Kepplerian strategy and the associated filter bank construction take place in symplectic affine planes which are immersed as coadjoint orbits of the Heisenberg two-step nilpotent Lie group G into the foliated three-dimensional real projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$. Due to the factorization of transvections into affine dilations of opposite ratio, the Heisenberg group G under its natural sub-Riemannian metric acts on the line bundle realizing the projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$. Its elliptic non-Euclidean geometry without absolute quadric, associated to the unitary dual $\hat{G}$, governs the design of the coils inside the bore of the MRI scanner system. It determines the distributional reproducing kernel of the tracial read-out process of quantum holograms excited and coexisting in the MRI scanner system. Thus the pathway of this paper leads from Keppler's approach to projective geometry to the Heisenberg approach to the sub-Riemannian geometry of quantum physics, and finally to the enormously appealing topic of ensemble quantum computing.

Keywords: Quantum computing, quantum holography, magnetic resonance tomography, sub-Riemannian geometry, harmonic analysis on the Heisenberg group

Magnetic resonance imaging (MRI) scanners are cognitive systems which reconstruct cross-sectional images of objects and perform the reconstructive amplification by coherent quantum *stochastic resonance* as a form of multichannel parallelism. The reconstructive process from multichannel phase histories is through probing the magnetic moments of nuclei employing strong magnetic flux densities and radiofrequency radiation. The whole process of MRI is based on perturbing the equilibrium magnetization of the object with a train of pulses and observing the resulting time-evolving response signal produced as a free induction decay (FID) in a coil.

Nuclear spins and the arrays of quantum bits ("qubits") they represent can be manipu-

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lated in a multitude of different ways in order to extract site information about molecular structure and dynamic information about molecular motion. Due to the spin dynamics, a preparation of the sample can be achieved such that the reconstructive amplification process by coherent quantum stochastic resonance is a well posed problem. With NMR tomography it is possible to observe, non-invasively, cross-sections through objects, and thus obtain synthetic aperture radar (SAR) like image information about density, flow, and spectrally localized chemical composition (Gillies [25]). The preparation procedures of NMR and MRI turn the reconstruction into a well posed problem. Specifically, an application of the blood oxygen level dependent (BOLD) contrast method of human brain mapping to morphological cranial anatomy (Sanders [39]) allows for an observation of activation of the brain *in vivo*.

The moment of birth of the temporal magnetic resonance phenomenon was marked by Felix Bloch's *dynamical* approach. The great Felix Bloch (1905–1983), the first graduate student and assistant to Werner Karl Heisenberg in Leipzig, outlined the NMR experiment in his source paper of 1946 as follows (Fukushima [24], Mattson [34]):

"The first successful experiments to detect magnetic resonance by electromagnetic efects have been carried out recently and independently at the physics laboratories of Harvard and Stanford Universities. The considerations upon which our work was based have several features in common with the two experiments, previously mentioned, but differ rather essentially in others. In the first place, the radiofrequency field is deliberately chosen large enough so as to cause at resonance a considerable change of orientation of the nuclear moments. In the second place, this change is not observed by its relatively small reaction upon the driving circuit, but by directly observing the induced electromotive force in a coil, due to the precession of the nuclear moments around the constant field and in a direction perpendicular both to this field and the applied r-f field. This appearance of a magnetic induction at right angles to the r-f field is an effect which is of specifically nuclear origin and it is the main characteristic feature of our experiment. In essence, the observed perpendicular nuclear induction indicates a rotation of the total oscillating field around the constant magnetic field."

Because the computer performance was severely limited at the time of the discovery of NMR spectroscopy, and the fast Fourier transform (FFT) algorithm was not available to Bloch and his coworkers, the enormously appealing perspective to spin isochromat computers is not present in his dynamical approach. Such a machine performs a calculation using quantum parallelism at the molecular level and then amplifies the results to the macroscopic level via coherent quantum stochastic resonance as a form of multichannel parallelism. Recent experiments in neurobiology verified the amplification effect of stochastic resonance in the information transfer performed by weak signals in biological neural networks (Douglass [20]).

From the dynamical approach to NMR spectroscopy, however, Hahn's spin echo method popped up. Due to its favorable signal-to-noise ratio, his spin echo pulse sequence is extensively used both in clinical MRI, NMR spectroscopy and NMR microscopy (Callaghan [6]). It plays a major role in the emulation of quantum computers by NMR spectroscopy (Cory [11], Gershenfeld [26], Lloyd [33]).

The immersion aspect of the *spectroscopic* approach has been summarized by Nicolaas Bloembergen, Edward Mills Purcell (1912–1997), and Robert V. Pound as follows (Fukushima [24], Mattson [34]):

"In nuclear magnetic resonance absorption, energy is transferred from a radiofrequency circuit to a system of nuclear spins immersed in a magnetic field, H_0, as a result of transitions among the energy levels of the spin system."

"The exposure of the system to radiation, with consequent absorption of energy, tends to upset the equilibrium state previously attained, by equalizing the population of the various levels. The new equilibrium state in the presence of the radiofrequency field represents a balance between the processes of absorption of energy by the spins, from the radiation field, and the transfer of energy to the heat reservoir comprising all other internal degrees of freedom of the sustance containing the nuclei in question."

"Finally we review briefly the phenomenological theory of magnetic resonance absorption, before describing the experimental method. The phenomenon lends itself to a variety of equivalent interpretations. One can begin with static nuclear paramagnetism and proceed to paramagnetic dispersion, or one can follow Bloch's analysis, contained in his paper on nuclear induction, of the dynamics of a system of spins in an oscillating field, which includes the absorption experiments as a special case. We are interested in absorption, rather than dispersion or induction, in the presence of *weak* oscillating fields, the transitions induced by which can be regarded as non-adiabatic. We therefore prefer to describe the experiment in optical terms."

Bloch and Purcell shared the 1952 Nobel Prize in Physics in recognition of their pioneering achievements in condensed matter. The methods due to Bloch and Purcell are not only of high intellectual beauty leading finally to quantum computing, they also place an analytic method of high efficacy in the hands of scientists. Therefore, during the next quarter of a century NMR spectroscopy flourished, and more than 1000 NMR units were manufactured. The award of the Nobel Prize in Chemistry to Richard Robert Ernst in 1991 later served to highlight the fact that high resolution NMR spectroscopy is not only an essential physical technique for chemists and biochemist, but also offers a fascinating application of non-commutative Fourier analysis to system theory. Ernst summarized the application of Fourier transform spectroscopy to NMR as follows (Fukushima [24], Mattson [34]):

"It is well-known that the frequency response function and the unit impulse response of a linear system forms a Fourier transform pair. Both functions characterize the system entirely and thus contain exactly the same information. In magnetic resonance, the frequency response function is usually called the spectrum and the unit impulse is represented by the free induction decay. Although a spin system is not a linear system, Lowe and Norberg (1957) have proved that under some very loose restrictions the spectrum and the free induction decay after a 90^o pulse are Fourier transform of each other. The proof can be generalized for arbitrary flip angles."

"For complicated spin systems in solution, the spectrum contains the information in a more explicit form than does the free induction decay. Hence it is generally assumed that recording the impulse response does not give any advantages compared to direct spectral techniques. The present investigations show that the impulse response method can have significant advantages, especially if the method is generalized to a series of equidistant identical pulses instead of a single pulse. In order to interpret the result, it is usually necessary to go to a spectral representation by means of a Fourier transformation. The numerical transformation can conveniently be handled by a digital computer or by an analog Fourier analyzer."

"Here are some features of the pulse technique: (1) It is possible to obtain spectra in a much shorter time than with the conventional spectral sweep technique. (2) The achievable sensitivity of the pulse experiment is higher. All spins with resonance frequencies within a certain region are simultaneously excited, increasing the information content of the experiment appreciably compared with the spectral sweep technique where only one resonance is observed at a time."

Based on the work of the Nobel laureates Pauli, Bloch, Bloembergen, Purcell, Gabor and Ernst, a whole new science culminating in Fourier transform MRI has been created where none existed before (Mattson [34], Stark [46], Schempp [45]). This new science of ensemble quantum computing needs its own mathematical foundation based on elliptic geometric analysis. Surprisingly, spin isochromat computing by NMR spectroscopy has its deep roots in the Kepplerian dynamics of physical astronomy.

The Kepplerian temporospatial phase detection strategy of physical astronomy is derived from the quadrature conchoid trajectory stratification and the second fundamental law of planetary motion analysis (Stephenson [47]), as displayed in Keppler's Astronomia Nova of 1609. The dynamics of the quadrature conchoid trajectory stratification which seems to have almost escaped notice in literature, is best understood from the viewpoint of projective geometry. Although Keppler described the projective approach to astronomical observations in the Paralipomena of 1604, he is not recognized, along with Desargues, as one of the pioneers of projective geometry which then culminated in Poncelet's investigations.

Projective geometry, which is since about the mid 1980's standard in the computer vision and robotics literature, allows for the stroboscopic and synchronous cross sectional quadrature filtering of phase histories in local frequency encoding multichannels with respect to the rotating coordinate frame of reference (Freeman [23]), and provides the implementation of a matched filter bank by orbit stratification in a symplectic affine plane. An application of this procedure leads to the landmark observation of the earliest SAR pioneer, Carl A. Wiley, that motion is the solution of the high resolution radar imagery and phased array antenna problem of holographic recording by quasi-optical systems (Leith [32], Wu [52]). Whereas the Kepplerian temporospatial strategy is realized in SAR imaging by the range Doppler principle (Cutrona [13], Leith [31]), it is the Lauterbur spectral localization principle (Schempp [45]) which takes place in clinical MRI. Having Damadian's approach to tumor detection in mind, Lauterbur wrote the following observation into his 1971 notebook under the title of "Spatially Resolved Nuclear Magnetic Resonance Experiments" (Mattson [34]):

"The distribution of magnetic nuclei, such as protons, and their relaxation times and diffusion coefficients, may be obtained by imposing magnetic field gradients (ideally, a complete set of orthogonal spherical harmonics) on a sample, such as an organism or a manufactured object, and measuring the intensities and relaxation behavior of the resonance as functions of the applied magnetic field. Additional spatial discrimination may be achieved by the application of time-dependent gradient patterns so as to distinguish, for example, protons that lie at the intersection of the zero-field (relative to the main magnetic field) lines of three linear gradients."

"The experiments proposed above can be done most conveniently and accurately by measurements of the Fourier transform of the pulse response of the system. They should be capable of providing a detailed three-dimensional map of the distributions of particular classes of nuclei (classified by nuclear species and relaxation times) within a living organism. For example, the distribution of mobile protons in tissues, and the differences in relaxation times that appear to be characteristic of malignant tumors, should be measurable in an intact organism."

Thus the Lauterbur spectral localization is based on affine dilations. These are implemented on a modular stratification basis by *linear* magnetic field gradient matrices into which transvections admit factorizations (Schempp [45]). The measurements of the one-dimensional Fourier transform have been refined by the two-dimensional Fourier transform spectroscopy contributed by Ernst, and the spin-warp version of Fourier transform MRI developed by W.A. Edelstein, J.M.S. Hutchinson, and J. Mallard of the Aberdeen University group in Scotland.

With regard to possible applications of his spectral localization method, Lauterbur drew the following conclusions without explicit citation of Damadian's paper (Mattson [34]):

"Applications of this technique to the study of various inhomogeneous objects, not necessarily restricted in size to those commonly studied by magnetic resonance spectroscopy, may be anticipated. A possible application of considerable interest at this time would be the *in vivo* study of malignant tumors, which have been shown to give proton nuclear magnetic resonance signals with much longer water spin–lattice relaxation times than those in the corresponding normal tissues."

At the background of both the SAR and MRI high resolution imaging techniques lies the construction of a multichannel coherent wavelet reconstruction analysis–synthesis filter bank of matched filter type (Davies [15], Rihaczek [38], Freeman [23]). Beyond these applications to local frequency encoding subbands, the Kepplerian temporospatial phase detection strategy leads to the concept of Feynman path integral or summation over phase histories.

As approved by quantum electrodynamics, geometric quantization allows for a semi-classical approach to the interference pattern of quantum holography and the spin excitation profiles of MRI (Schempp [42], [43], [44], [45]). Implementation of interference needs, of course, phases coherency and therefore the transition to the frequency domain by a *duality* procedure. Indeed, the unitary dual $\hat{G}$ of the Heisenberg group G consisting of the equivalence classes of irreducible unitary linear representations of G allows for a coadjoint orbit foliation of symplectic affine leaves $\mathcal{O}_\nu$ $(\nu \neq 0)$, spatially located as a stack of tomographic slices, and decomposing the dual vector space $\mathrm{Lie}(G)^\star$ of the real Heisenberg Lie algebra $\mathrm{Lie}(G)$ (Schempp [40]). This fact is a consequence of the Kirillov homeomorphism

$$\hat{G} \longrightarrow \mathrm{Lie}(G)^\star/\mathrm{CoAd}_G(G)$$

which establishes the canonical foliation of the three-dimensional super-encoding projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$.

- The connected, simply connected Heisenberg two-step nilpotent Lie group G admits a realization by a faithful matrix representation $G \longrightarrow \mathbf{SL}(3, \mathbf{R})$.

In terms of standard coordinates, the Heisenberg group G is realized by the set of unipotent matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbf{R} \right\}$$

under the matrix multiplication law of the *dual pairing* presentation

$$\begin{pmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 + x_2 & z_1 + z_2 + x_1.y_2 \\ 0 & 1 & y_1 + y_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The form of rank one defining the non-commutative matrix multiplication of G is neither antisymmetric nor non-degenerate. However, it is *cohomologous* to the non-degenerate alternating determinant form. It suffices to use any form which is cohomologous to a non-degenerate alternating bilinear form to define the multiplication law of G.

The differential operator associated to the natural sub–Riemannian metric of G is the sub–Laplacian $\mathcal{L}_G$ on G. Notice that the sub–Riemannian geometry is to the sub–Laplacian in the sub–elliptic realm what Riemannian geometry is to the Laplacian in the elliptic realm. The solutions of the Hamilton–Jacobi equations associated to the principal symbol of $\mathcal{L}_G$ project onto the constant velocity *locally* length minimizing curves in G.

- The geodesics with respect to the natural sub–Riemannian metric of G are the Heisenberg helices.

In terms of the coordinates of the dual pairing presentation of G, the sub–Laplacian $\mathcal{L}_G$ takes the form of a Hörmander sum of squares

$$\boxed{\mathcal{L}_G = -\frac{1}{2}\Big((\frac{\partial}{\partial x} - y\frac{\partial}{\partial z})^2 + (\frac{\partial}{\partial y} + x\frac{\partial}{\partial z})^2\Big)}$$

The Heisenberg group G has two presentations that are particularly important in applications. It is standard that the radical of a bundled alternating bilinear form is the only invariant of the bundled form. Therefore, the dual pairing presentation of G is isomorphic to the *basic* presentation of the Heisenberg group G which is given by the multiplication law of the unipotent matrices

$$\left\{\begin{pmatrix} 1 & \bar{w} & \frac{1}{2}|w|^2 + zi \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \Big| w \in \mathbf{C}, z \in \mathbf{R}\right\}.$$

Computations are usually easiest in the basic presentation of G because the straight lines through the origin are the one–parameter–subgroups. Due to the planetary orbit stratification, the Kepplerian temporospatial phase detection strategy leads to the basic presentation (Schempp [45]).

- There is a realization of the Heisenberg group G by a faithful matrix representation $G \longrightarrow \mathbf{Sp}(4, \mathbf{R})$ defining the image group as an extension via matrix multiplication.

In terms of the left–invariant vector fields

$$W = \frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial z}, \qquad \bar{W} = \frac{\partial}{\partial \bar{w}} + w\frac{\partial}{\partial z},$$

the sub–Laplacian $\mathcal{L}_G$ takes the form

$$\boxed{\mathcal{L}_G = -\frac{1}{2}(W\bar{W} + \bar{W}W)}$$

The spectrum of the sub–elliptic operator $\mathcal{L}_G$ is absolutely continuous with uniform multiplicity on the longitudinal center frequency axis $\mathbf{R}$, transverse to the symplectic affine plane $\mathbf{R} \oplus \mathbf{R}$. The density of the spectrum on the longitudinal center frequency axis $\mathbf{R}$ is given by the Pfaffian of G.

- The symmetries of the sub–elliptic differential operator $\mathcal{L}_G$ are reflected in the time reversal, which is implicit in the spin echo methods and the conjugation of the gradient echo imaging methods.

The coordinate functions

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of G define *transvections* or shearings of a three-dimensional real vector space (Dieudonné [19]). The multiplicative group of longitudinal *dilations* transforms the transvections into the transvections

$$\begin{pmatrix} 1 & ax & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & ay \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & a^2z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for $a \neq 0$. Conversely, it is easy to verify that transvections admit factorizations into affine dilations of opposite ratio. With respect to the sub-Riemannian metric of G, each dilation multiplies lengths by the fixed value $|a|$. The existence of dilations shows that small neighborhoods are similar to large neighborhoods in G. The reflections implementing the time reversal implicit in the spin echo method and the gradient echo technique are given by the improper matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the rotating coordinate frame of reference, and

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

in the laboratory coordinate frame of reference, respectively.

The contact geometry of the quotient projection

$$G \longrightarrow G/\text{center}$$

gives rise to the contact one-form

$$\eta = \mathrm{d}z + \frac{1}{2}(x.\mathrm{d}y - y.\mathrm{d}x)$$

where

$$\mathrm{d}\eta = \mathrm{d}x \wedge \mathrm{d}y$$

holds with respect to the laboratory coordinate frame of reference.

If the unipotent matrices $\{P, Q, I\}$ denote the canonical basis of the three-dimensional real vector space $\mathrm{Lie}(G)$, where the elementary matrices

$$\exp_G P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \exp_G Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \exp_G I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are given by the matrix exponential diffeomorphism

$$\exp_G : \mathrm{Lie}(G) \longrightarrow G,$$

the coadjoint action of G on $\mathrm{Lie}(G)^\star$ is given by

$$\mathrm{CoAd}_G \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore the action CoAd_G reads in terms of the coordinates $\{\alpha, \beta, \nu\}$ with respect to the dual basis $\{P^\star, Q^\star, I^\star\}$ of the real vector space dual $\mathrm{Lie}(G)^\star$ as follows:

$$\mathrm{CoAd}_G \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} (\alpha P^\star + \beta Q^\star + \nu I^\star) = (\alpha - \nu y)P^\star + (\beta + \nu x)Q^\star + \nu I^\star.$$

- In radar imaging, $\nu \neq 0$ denotes the center frequency of the transmitted coherent pulse train, whereas in clinical MRI the center frequency ν is the frequency of the rotating coordinate system defined by tomographic slice selection.

In clinical MRI, the Pfaffian of G allows to select the tomographic slice by an application of linear magnetic field gradients.

- The pitch of the Heisenberg helices is inversely proportional to the polarity of the linear slice select gradients.

For an illustration of the Heisenberg helix after excitation by a nutation $\frac{\pi}{2}$ pulse. The pitch of the Heisenberg helix indicates the energy gain due to the longitudinal relaxation effect. This is typical of a single–frequency FID.

The linear varieties

$$\boxed{\mathcal{O}_\nu = \mathrm{CoAd}_G(G)(\nu I^\star) = \mathbf{R}P^\star + \mathbf{R}Q^\star + \nu I^\star \qquad (\nu \neq 0)}$$

actually are symplectic affine planes in the sense that they are in the natural way compatibly endowed with both the structure of an affine plane and a symplectic structure. Therefore the trivial line bundle $\mathbf{R} \oplus \mathbf{R}$ on the symplectic affine plane $\mathcal{O}_\nu \hookrightarrow \mathrm{Lie}(G)^\star\ (\nu \neq 0)$ of *connection* differential 1–form

$$\boxed{\nu.(x.\mathrm{d}y - y.\mathrm{d}x)}$$

and *rotational curvature* differential 2–form

$$\boxed{\omega_\nu = \nu.\mathrm{d}x \wedge \mathrm{d}y}$$

in the cohomology group

$$\bigwedge^2(\mathcal{O}_\nu) \cong \mathrm{H}^2(\mathbf{R} \oplus \mathbf{R}, \mathbf{R}) \qquad (\nu \neq 0)$$

forms the predestinate planar mathematical structure to implement the Kepplerian temporospatial phase detection strategy over the bi–infinite stratigraphic time line $\mathbf{R}$ of time cycles or repetition times. The closed exterior differential 2–form

$$\boxed{\omega_\nu = \frac{1}{2}\nu i\, \mathrm{d}w \wedge \mathrm{d}\bar{w} \qquad (\nu \neq 0)}$$

is a representative of the magnetic moment referred to in Bloch's dynamical approach. In the NMR experiment, the intrinsic dynamics is due to the driving flat radiofrequency circuit.

In MRI, the symplectic affine linear varieties $\mathcal{O}_\nu \hookrightarrow \mathrm{Lie}(G)^\star\,(\nu \neq 0)$ are predestinate to carry quantum holograms or spin excitation profiles acting as multichannel perfect reconstruction analysis–synthesis filter banks (Schempp [41], Farre [22]). The quantum holograms are implemented by the frequency modulation action of G.
The stationary singular plane $\mathcal{O}_\infty \hookrightarrow \mathrm{Lie}(G)^\star$ of equation

$$\nu = 0$$

consists of the single point orbits or focal points

$$\mathcal{O}_\infty = \{\varepsilon_{(\alpha,\beta)} \mid (\alpha,\beta) \in \mathbf{R} \oplus \mathbf{R}\}$$

corresponding to the one–dimensional representations of G. The elements of the plane $\mathcal{O}_\infty$ are the analogs of the resonance "sweet spots" of the conventional spectral sweep technique employed in the early NMR spectroscopy, as well as the prototype whole–body MRI scanner. The world's first whole-body scanner, dubbed "Indomitable" by Damadian to capture the spirit of its seven–year construction (Schempp [45]), provided a technique named FONAR to achieve the first MRI scan of the human body *in vivo*, and to convince the medical community that MRI scanning was, in fact, a reality.

The infinite dimensional irreducible unitary linear representations of G associated to the symplectic affine leaves $\mathcal{O}_\nu\,(\nu \neq 0)$ *collapse* down to characters of G. The state–vector reduction, or collapse of coherent wavelet can be described by the transition

$$\boxed{\omega_\nu \rightsquigarrow \varepsilon_{(\alpha,\beta)}\, \mathrm{d}\alpha \otimes \mathrm{d}\beta \qquad (\nu \neq 0)}$$

As an energetic edge, the confocal plane at infinity $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\infty)$ plays a fundamental role in the energetic structure of observation (Farre [22]), and specifically in the coherent optical processing of radar data (Cutrona [13]), morphological MRI studies, and neurofunctional MRI detection for the mapping of the activities of the human brain to morphological cranial anatomy. From there the reconstructive amplification via the multichannel parallelism of coherent quantum stochastic resonance takes place.

The quantum holograms which are generated by neurofunctional MRI experiments represent "matière à pensée" (Changeux [9]), or shadows of the mind implemented by the rotationally curved planes of immanence in the philosophy of constructivism (Deleuze [16], [17]), or symplectic affine planes of incidence (Farre [22]).

"Ein maschinelles Gefüge[1] ist den Schichten zugewandt, reinen Intensitäten, die sie zirkulieren läßt um die Selektion der "Konsistenzebene" zu sichern und der sich die Subjekte zuordnen, welchen sie einen Namen nur als Spur einer Intensität läßt."

The shadows of the mind emulated by MRI scanner systems seem to provide a promising conceptual approach to the missing science of *consciousness*.

[1] "agencement" *in French*

- The canonical foliation of the three-dimensional super-encoding projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ allows to deduce the phase coherent wavelet collapse phenomenon.
- There exists no equivalent of the state-vector reduction, or coherent wavelet collapse phenomenon in the realm of classical physics.
- The amount of information that can be extracted from a spin isochromat computer is limited by the phenomemon of phase coherent wavelet collapse.

In contrast to the coadjoint orbit visualization of the unitary dual $\hat{G}$, the standard bra-ket procedures of quantum mechanics provide no implication that there be any way to deduce the collapse phenomenon as an instance of the deterministic Schrödinger evolution. Whereas the weak containment of the identity representation 1 in the tensor product representation provides a geometric symmetry condition for the decryption of quantum information from the holographic encryption, there is in standard quantum mechanics no clear rule as to when the probabilistic collapse rule should be invoked, in place of the deterministic Schrödinger evolution. This establishes the extraordinary *power* of the coadjoint orbit visualization in terms of the three-dimensional real projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$, and the confocal plane at infinity $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\infty)$ included.

In order to define the transvectional G-action of $\hat{G}$, it is convenient to immerse the $\mathrm{CoAd}_G(G)$-foliation of $\mathrm{Lie}(G)^\star$ with typical fiber $\mathbf{R} \oplus \mathbf{R}$ into its projective completion $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ by the bi-infinite stratigraphic time line $\mathbf{R}$. It will be shown that the concept of projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ which is helpful in the realms of computerized geometric design, computer vision and robotics, is also useful in non-invasive radiodiagnostics.

- The intrinsic construction provides the foliated three-dimensional super-encoding projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ as the projective completion of the dual vector space of the affine dual of any of the tomographic slices $\mathcal{O}_\nu \hookrightarrow \mathrm{Lie}(G)^\star$ $(\nu \neq 0)$ by the bi-infinite stratigraphic time line $\mathbf{R}$.
- The stratigraphic time line $\mathbf{R}$ records simultaneously the time cycles or repetition times of the MRI protocol as well as the superposition of spin up and spin down states, and the arrays of qubits they are representing in coexistence.
- The unitary dual $\hat{G}$ of the Heisenberg group G can be immersed into the foliated three-dimensional projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$. The confocal plane $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\infty)$ is the plane at infinity of $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$. The two-dimensional projective varieties $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu)$ $(\nu \neq 0)$ are contained in its complement.
- Due to the factorization of transvections into affine dilations of opposite ratio, the Lauterbur spectral localization controls the transvectional action of G on the line bundle model $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ of $\hat{G}$.
- The three-dimensional elliptic non-Euclidean space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ is homeomorphic to the compact unit sphere $\mathbf{S}_3 \hookrightarrow \mathbf{R}^4$ under antipodal point identification via the action of the group $\{\mathrm{id}, -\mathrm{id}\}$.
- The three-dimensional elliptic non-Euclidean space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ is homeomorphic to the compact solid ball $\mathbf{B}_3 \hookrightarrow \mathbf{R}^3$ with the antipodal (diametrically opposite) points of its boundary $\mathbf{S}_2 = \partial \mathbf{B}_3$ identified.

It follows from the classification of the coadjoint orbits of G in the foliated projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^*)$ the highly remarkable fact that there exists no finite dimensional irreducible unitary linear representation of G having dimension > 1. Hence the irreducible unitary linear representations of G which are not unitary characters are infinite dimensional and unitarily induced. Their coefficient cross sections for the Hilbert bundle sitting over the bi-infinite stratigraphic time line $\mathbf{R}$ define the holographic transforms which sum the free induction decays.

- In the data acquisition process, the holographic transform collects the decaying response wavelets of the radiofrequency pulse trains in quantum holograms, or FIDs in spin excitation profiles.

Let C denote the one-dimensional center of G, transverse to the plane carrying the quantum holograms or spin excitation profiles. Then

$$C = \mathbf{R}.\exp_G I$$

is spanned by the central transvection $\exp_G I$. In coordinate-free terms, G forms the non-split central group extension

$$C \lhd G \longrightarrow G/C$$

where the plane G/C is transverse to the line C. Thus G is defined to be the unique central extension

$$\{0\} \longrightarrow \mathbf{R} \longrightarrow G \longrightarrow \mathbf{R} \oplus \mathbf{R} \longrightarrow \{0\}$$

which does not contain any line $\mathbf{R}$ as a direct factor. This condition of not containing $\mathbf{R}$ as a factor is equivalent to the 2-cocycle of the extension which always can be taken to be alternating bilinear, being non-degenerate. The uniqueness follows from the fact that every pair of non-degenerate such forms are congruent in $\mathbf{GL}(2, \mathbf{R})$, the outer automorphism group of the plane $\mathbf{R} \oplus \mathbf{R}$.

The irreducible unitary linear representations of G associated to the projective coadjoint orbits $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu) \hookrightarrow \mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^*)$ $(\nu \neq 0)$ are unitarily induced in stages by the unitary characters of closed normal abelian subgroups which provide a fibration of G sitting over the bi-infinite stratigraphic time line $\mathbf{R}$. The elements $w \in \mathcal{O}_1$ of the typical fiber are represented by complex numbers of the form

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

including the differential phase

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

and the local frequency

$$\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}$$

as real coordinates with respect to the frame of reference rotating with center frequency $\nu \neq 0$. The alternating matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of Pfaffian

$$\mathrm{Pf}(J) = 1$$

acts as imaginary unit of the basic presentation of G. It generates the special orthogonal group $\mathbf{SO}(2,\mathbf{R}) \hookrightarrow \mathbf{O}(2,\mathbf{R})$. Together with the reflection defined by the data routing matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the non diagonalizable matrix J generates the orthogonal group $\mathbf{O}(2,\mathbf{R})$. The group $\mathbf{O}(2,\mathbf{R})$ can be lifted to the group of all isometries of G with respect to the natural sub-Riemannian metric of G.

It becomes obvious that $|w|^2 = \det w$ and that the rotational curvature differential 2-form ω_ν of $\mathcal{O}_\nu$ is exactly the standard symplectic form of $\mathbf{R}\oplus\mathbf{R}$, dilated by the center frequency $\nu \neq 0$. Thus G implements the *oscillator driven* dynamical system

$$\mathbf{R} \lhd \mathbf{R} \oplus \mathbf{R}$$

of longitudinal center frequency axis $\mathbf{R}$, transverse to the symplectic affine plane $\mathbf{R}\oplus\mathbf{R}$. Keppler described the idea of an oscillator driven cyclic clockwork as an act of profanation:

"Mein Ziel ist es, zu zeigen, daß die himmlische Maschinerie nicht von der Art eines göttlichen Lebewesens, sondern von der eines Uhrwerks ist, daß die ganze Mannigfaltigkeit ihrer Bewegungen von einer einfachsten magnetischen körperlichen Kraft herrührt, so wie alle Bewegungen des Uhrwerks allein von dem es treibenden Gewicht."

The $\mathbf{R}$-linear isomorphism

$$(x,y) \rightsquigarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

of $\mathcal{O}_1$ onto the realification $\mathbf{C}(\mathbf{R}\oplus\mathbf{R})$ of the field $\mathbf{C}$ of complex numbers suggests an extension from two dimensions to three dimensions via the real quaternion skew-field $\mathbf{H}$. The $\mathbf{R}$-linear mapping

$$(w,w') \rightsquigarrow \begin{pmatrix} w & w' \\ -\bar{w}' & \bar{w} \end{pmatrix}$$

provides an isomorphism from the image of $\mathbf{C}^2$ onto $\mathbf{H}$. In terms of the matrices of this type, the multiplication in $\mathbf{H}$ reads

$$\begin{pmatrix} w_1 & w_1' \\ -\bar{w}_1' & \bar{w}_1 \end{pmatrix} \cdot \begin{pmatrix} w_2 & w_2' \\ -\bar{w}_2' & \bar{w}_2 \end{pmatrix} = \begin{pmatrix} w_1 w_2 - w_1'\bar{w}_2' & w_1 w_2' + w_1'\bar{w}_2 \\ -(\bar{w}_1\bar{w}_2' + \bar{w}_1' w_2) & \bar{w}_1\bar{w}_2 - \bar{w}_1' w_2' \end{pmatrix}$$

The tangent space of $\mathbf{S}_3 \hookrightarrow \mathbf{R}^4$ at the neutral element of $\mathbf{SU}(2,\mathbf{C})$ is isomorphic to the vector space $\mathbf{R}^3$. The isomorphism suggests to introduce the Pauli spin matrices forming the canonical basis of the Lie algebra associated to $\mathbf{SU}(2,\mathbf{C})$, and the real Clifford algebra $\mathcal{C}\ell_{(3,0)}(\mathbf{R})$. These matrices generate analyzing one-parameter subgroups of the group $\mathbf{SU}(2,\mathbf{C})$. The corresponding elements in the skew-field $\mathbf{H}$ are given by the pure or traceless quaternions.

- The group $\mathbf{S}_3$ is the non-trivial covering $\mathrm{Spin}(3,\mathbf{R})$ of the rotation group $\mathbf{SO}(3,\mathbf{R})$. The group $\mathbf{SO}(3,\mathbf{R})$ contains two normal subgroups, both isomorphic to $\mathbf{S}_3$, which give rise to the Clifford translations acting transitively on the foliated projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$.

- Identification of the group $\mathbf{S}_3 \hookrightarrow \mathbf{R}^4$ with the unit sphere of the skew-field $\mathbf{H}$ provides the multi-slice imaging capability of the MRI modality via the abelian groups $\mathbf{SO}(2,\mathbf{R})$ of Clifford translations of tomographic slices in the elliptic non-Euclidean space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$.
- Identification of the unit sphere $\mathbf{S}_3 \hookrightarrow \mathbf{R}^4$ with the compact group $\mathbf{SU}(2,\mathbf{C})$, or the compact homogeneous manifolds $(\mathbf{SU}(2,\mathbf{C}) \times \mathbf{SU}(2,\mathbf{C}))/\mathbf{SU}(2,\mathbf{C})$, or $\mathbf{SO}(4,\mathbf{R})/\mathbf{SO}(3,\mathbf{R})$ provides the design of pairs of surface coils of the MRI scanner bore via zonal spherical harmonics.

The interleaving of data acquisition through multi-slice imaging provides a simple means of acquiring data in all three dimensions, and is widely used in clinical imaging. Due to the multiplanar imaging capability of MRI, direct transverse slices of superior to inferior orientation of the plane normal, sagittal slices of anterior to posterior orientation of the normal, and coronal slices of left to right orientation of the normal, as well as oblique plane selections can be performed without changing the patient's position. In X-ray computed tomography (XCT) imaging, sagittal and coronal images are reconstructed from a set of contiguous images. The orthogonal and oblique scan plane selection offer clinical advantages of MRI over XCT. Actually, MRI is closer to high resolution radar imaging than to XCT. The high soft-tissue contrast resolution is another advantage over XCT. Neuroradiologists think that if history of science was rewritten, and XCT invented after MRI, nobody would bother to pursue XCT imaging. For whole-body imaging radiologists, however, the predictions of XCT's imminent demise and MRI's ascendency no longer seem so prescient.

The bundle-theoretic interpretation of the inducing mechanism gives rise to the pair of isomorphic irreducible unitary linear representations

$$(U^\nu, V^\nu) \qquad (\nu \neq 0)$$

of G unitarily induced in quadrature by the unitary characters of the associated closed normal abelian subgroups of G. The induced Hilbert bundles sitting in quadrature over the bi-infinite stratigraphic time line $\mathbf{R}$, admit for any Fourier transformed pair of exciting phase coherent wavelets

$$(\psi, \varphi)$$

in the frequency modulation space $L^2_{\mathbf{C}}(\mathbf{R})$, and element $z \in C$ the contiguous cross-sections of a phase-splitting network of uncorrelated multichannels in quadrature format (Freeman [23], Farre [22])

$$\boxed{\left(x_0\,,\, e^{2\pi i\nu\left(z-(x-x_0)y\right)}.\psi(-x)\right),\quad \left(y_0\,,\, e^{2\pi i\nu\left(z+x(y-y_0)\right)}.\varphi(y)\right) \qquad ((x_0,y_0)\in T\oplus S)}$$

where $x_0 \in T$ denotes the phase reference of the stroboscopic phase cycling at which system state change. Moreover, $y_0 \in S$ denotes the intermediate frequency reference of the synchronous period cycling clockwork of transitions determined by the computer's programming, and

$$\varphi = \mathcal{F}_{\mathbf{R}}\psi$$

where the phase coherent wavelet φ is the Fourier transform of ψ. The linear representation U^ν of G and its swapped copy V^ν are globally square integrable mod C. Indeed, it is well known that a coadjoint orbit is a linear variety if and only if one (and hence all) of the corresponding irreducible unitary linear representations is globally square integrable modulo its kernel. An equivalent characterization of square integrability mod C is that the Pfaffian of G does not vanish at the center frequency ν.

It is reasonable to regard global square integrability as an essential part of the Stone–von Neumann theorem of quantum physics, because a representation of a nilpotent Lie group is determined by its central unitary character χ_ν if and only if it is globally square integrable mod C. Thus χ_ν allows for selection in the tomographic slice $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu) \hookrightarrow \mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)\,(\nu \neq 0)$ a coordinate frame rotating with center frequency $\nu \neq 0$ via an affine dilation in the longitudinal direction of the line C. The corresponding equivalence classes of irreducible, unitarily induced, linear representations U^ν of G acting on the complex Hilbert space of globally square integrable cross sections for the Hilbert bundle sitting over the bi-infinite stratigraphic time line $\mathbf{R}$ are infinite dimensional and can be realized as Hilbert-Schmidt integral operators with kernels $K^\nu \in L^2(\mathbf{R} \oplus \mathbf{R})$ (Schempp [40], [42], [43], [44], [45]). The derived representation

$$U^\nu(\mathcal{L}_G),$$

evaluated on the sub-Laplacian $\mathcal{L}_G$ of G in the universal enveloping algebra of Lie(G), is the harmonic oscillator Hamiltonian of center frequency $\nu \neq 0$. Due to the global square integrability mod C of U^ν for $\nu \neq 0$, the center of Lie(G) coincides with the center of the universal enveloping algebra of Lie(G).

The center of the product group $\mathbf{S}_3 \times \mathbf{S}_3$ is given by the set

$$\{1,-1\} \times \{1,-1\},$$

and therefore has order 4. It contains the kernel

$$\{1,1\} \times \{-1,-1\}$$

of order 2 of the natural group epimorphism $\mathbf{S}_3 \times \mathbf{S}_3 \longrightarrow \mathbf{SO}(4,\mathbf{R})$.

Due to the antipodal point identification of $\mathbf{S}_3$, the realization of the foliated three-dimensional super-encoding projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ is diffeomorphic to the quotient of $\mathbf{S}_3$ by the action of the group $\{\mathrm{id}, -\mathrm{id}\}$. As a result, the center of $\mathbf{S}_3 \times \mathbf{S}_3$ in the direction plane $\mathbf{R} \oplus \mathbf{R}$ of $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu)\,(\nu \neq 0)$ gives rise to the distributional reproducing kernel

$$1_\nu \otimes 1_\nu$$

on $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu) \hookrightarrow \mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ corresponding to the rotational curvature differential 2-form

$$\omega_\nu \in \mathrm{H}^2(\mathbf{R} \oplus \mathbf{R}, \mathbf{R}) \qquad (\nu \neq 0).$$

It defines the symplectically reformatted two-dimensional Fourier transform

$$\boxed{\star(1_\nu \otimes 1_\nu)}$$

acting as a spectral sweep by *symplectic convolution* (Schempp [44], [45]) on the symplectic spinors of $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu) \hookrightarrow \mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)\,(\nu \neq 0)$. In contrast to the conventional two-dimensional Fourier transform of order 4, the symplectic Fourier transform admits order 2 (Schempp [43], [44]). This corresponds to the involutory *entangling* map $W \rightsquigarrow \bar{W}$ of quantum computation (Schempp [45]).

- The kernel function $K^\nu \in L^2(\mathbf{R} \oplus \mathbf{R})$ associated to the irreducible unitary linear representation U^ν of central unitary character $\chi_\nu = U^\nu \mid C$ implements a multichannel coherent wavelet perfect reconstruction analysis–synthesis filter bank of matched filter type.

In order to paratactically synchronize the rotating coordinate frame to the laboratory frame of reference, the kernel function K^ν has to be composed with the symbol map σ which is defined by the Hopf fibration

$$\mathbf{S}_3 \longrightarrow \mathbf{S}_2$$

with fiber $\mathbf{S}_1$ into Clifford parallel circles $\mathbf{S}_1 \hookrightarrow \mathbf{S}_3$. The Clifford parallelism is understood in the sense of the elliptic non–Euclidean geometry.

- The decomposition of the complement of $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\infty)$ in the real projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^*)$ by the canonical foliation $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu)\,(\nu \neq 0)$ corresponds to the decomposition of the unit sphere $\mathbf{S}_3 \hookrightarrow \mathbf{R}^4$ by the Hopf fibration.

In terms of a partial Fourier cotransform, the symbol of K^ν takes the explicit form

$$\sigma(K^\nu)(x, y) = e^{-2\pi i \nu x y} \bar{\mathcal{F}}^2_{\mathbf{R}\oplus\mathbf{R}} K^\nu(x, y) \quad ((x, y) \in \mathbf{R} \oplus \mathbf{R})$$

The excitation profile, generated by the density f of proton–weighted spin isochromats, takes the form of the symplectic spinor extension

$$U^\nu(f)$$

corresponding to $U^\nu(\mathcal{L}_G)$. If the tempered distribution

$$K^\nu = K^\nu_f$$

represents the kernel associated to $U^\nu(f)$, the symbol $\sigma(K^\nu_f)$ of K^ν_f results from the standard spin echo pulse sequence.

- The continuous affine wavelet transform performing the spectral localization of the proton–weighted spin isochromat density f in the leaf $\mathbf{P}(\mathbf{R} \times \mathcal{O}_\nu)\,(\nu \neq 0)$ by linear gradient stratification lifts to the central spectral transform for the sub–Laplacian $\mathcal{L}_G$.
- The central spectral transform for $\mathcal{L}_G$ diagonalizes the weak action of $\mathcal{L}_G$ on the symplectically reformatted two–dimensional Fourier transform. It gives rise to the distributional reproducing kernel $1 \otimes 1$ for the tracial read–out sweep of quantum holograms in the laboratory frame of reference.
- The Karhunen–Loëve expansion associated to the central spectral transform provides the information distribution within the quantum hologram.
- The reconstructive amplification process is performed by coherent quantum stochastic resonance as a form of multichannel parallelism.
- The multichannel reconstruction of the phase histories in local frequency encoding subbands from the symbol $\sigma(K^\nu_f)$ is performed by the symplectic Fourier transform $\star(1_\nu \otimes 1_\nu)$.

The spin echo method and the Lauterbur spectral localization method are closely related refocusing techniques. Why Lie group theory in the field of spin isochromat computing? Because the the Heisenberg group G allows to describe the *synergy* between radiofrequency pulse trains and linear gradient stratification. This synergy actually is the core of the tracial encoding procedure performed by MRI protocols. The Heisenberg group approach leads to the *explicit* tracial reconstruction formula

$$\boxed{f(x,y) = \frac{1}{2}e^{\pi i\nu xy}\sigma(K_f^\nu) \star (1_\nu \otimes 1_\nu)(\frac{1}{2}x, \frac{1}{2}y)}$$

The two-dimensional Fourier transform method, contributed by the physical chemist Ernst, forms the completion of the Lauterbur spectral localization method. It is remarkable, that the elliptic non-Euclidean geometry of the projective space $\mathbf{P}(\mathbf{R} \times \mathrm{Lie}(G)^\star)$ provides the unifying fundament for both of the achievements.

The Heisenberg group approach leads to the non-local entangling phenomenon of quantum physics (Schempp [41], [45]), and to major application areas of pulse train recovery methods, the corner turn algorithm in the digital processing of high resolution SAR data (Wehner [51]), the spin-warp procedure in clinical MRI via an application of the FFT algorithm, the gradient echo imaging methods, and finally to the variants of the ultra-high-speed echo-planar imaging technique of functional MRI (Schempp [45]). Combined with multi-slice imaging via interleaving of data acquisition, the spin-warp version of Fourier transform MRI is used almost exclusively in current routine clinical examinations (Crooks [12], Reiser [37], Stark [46]). For updated surveys of practical magnetic resonance tomography, see the monographs by Beltran [4], Brown [5], Cardoza [7], and Heuser [27].

The speed with which clinical MRI spread throughout the world as a diagnostic imaging tool was phenomenal. In the early 1980s, it burst onto the scene with even more intensity than XCT imaging in the 1970s. The superiority in spectroscopic sensivity of MRI over XCT imaging was first approved by the non-invasive detection of demyelinating plaques in multiple sclerosis (MS) patients. For the MRI based diagnosis of demyelinating disorders such as MS, several chelates of gadolinium are available for use as intravenous paramagnetic contrast agents (Knaap [29], Paty [36]).

Similarly, MRI is more sensitive than XCT for detecting the epileptogenic zones and is superior to XCT for predicting seizure outcome after surgery.

Whereas at the end of 1981 there were only three working MRI scanner systems available in the United States, presently there are more than 4.000 imagers performing in a non-invasive manner more than 8.5 million examinations per year. Due to its spectroscopic sensitivity and specificity, MRI provides the techniques of choice to assess MS plaques of demyelination in the periventricular white matter, cerebral cortex, cerebellum, brainstem, and spinal cord, and to monitor the short-term as well as the long-term evolution of MS (Beltran [4], Knaap [29]). The contrast developed by lesions depends on the orientation of myelinated white matter tracts relative to the linear magnetic field gradients. With gradients perpendicular to the predominant fiber direction, the lesions are poorly seen. With gradients parallel to the fibers, they are readily seen. XCT imaging is not reliable for the diagnosis of MS.

The speed of growth is a testimony of the clinical significance of this sophisticated technique. Today the modality is firmly established as a core diagnostic tool in the fields of neuroradiology (Ball [1], Barkovich [2], [3], Damasio [14], Jackson [28], Kretschmann [30], Patel [35], Truwit [49]) and musculoskeletal imaging (Chan [8], Deutsch [18], Edelman[21], Stoller [48], Vahlensieck [50]), routinely used in all medical centers in Western Europe and the United States. The ability to display the morphological anatomy of living individuals in remarkable detail has been a tremendous boon to clinical practice. It etablishes that non-trivial mathematics can be applied to the benefit of humankind. Due to the inclusions

$$\mathbf{R} \hookrightarrow \mathbf{C} \hookrightarrow \mathbf{H},$$

the claim that four-dimensional spaces are quite *exceptional*, is no idle talk, at least from the point of view of clinical MRI which offers a fascinating intellectual study in its own right.

Summarizing the significant breakthrough which MRI represents in conjunction with the recent hardware and software developments, the future of clinical MRI as a non-invasive diagnostic imaging modality seems to be bright. With its many advantages, including unrestricted multiplanar imaging capability, high spatial resolution imaging, exquisite contrast imaging of soft tissues, in addition to great versatility offering the ability to image blood flow, motion during the cardiac cycle, temperature effects, and chemical shifts, morphological MRI studies are a well-recognized tool in the evaluation of anatomic, pathologic, and functional processes. Specifically, clinical MRI allows for greater depiction of tumor extension and staging (Edelman [21], Reiser [37], Stark [46]).

Radiologists are skilled at interpreting original cross-sectional scans. Nevertheless, more advanced techniques such as magnetic resonance angiography need computer-based medical three-dimensional imaging. Despite formidable challenges, technical advances have already made it possible to develop multiple surface and volume algorithms to generate clinically useful three-dimensional renderings from MRI data sets.

Although MRI has not reached the end of its development, this diagnostic imaging modality has already undoubtedly saved many lives, and patients the world over enjoy a higher quality of life, thanks to MRI. The previously impenetrable black holes of lung air spaces are finally yielding their secrets to MRI. Utilizing inhaled ^{3}He or ^{129}Xe gases that are hyperpolarized by laser light, MRI scans can be acquired in a breath-hold that promise to reveal new insights into pulmonary anatomy and function. Because the exhaled gases can be recycled, MRI will play a role also in the earlier detection of chest diseases and bronchiectasis, and surgical planning of lung transplantation.

The dramatic advances made in clinical MRI within the last few years, the resulting enhancement of the ability to evaluate morphological and pathologic changes (Crooks [12], Stark [46]), and the non-invasive window on human brain activation offered by neurofunctional MRI studies to the preoperative assessment (Cohen [10], Kretschmann [30], Sanders [39]), demonstrate the *unity* of mathematics, science, and engineering in an impressive manner. This unity of sciences proves that the frontiers between different disciplines are only conventional. The frontiers change according to the state of human knowledge, the understanding of nature, and the computer performance *in silicio* available. They can be penetrated by mathematical methodology which allows to support the semantic filter needed as an essential component of all observations and interpretations in biology and medicine.

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