Strong Influence of Small Fluctuations in Nonlinear Systems

P.S. Landa
Department of Physics,
Lomonosov Moscow State University,
119899 Moscow, Russia
planda@td.lpi.ac.ru

Abstract. Recent investigations showed that even weak noise acting upon a nonlinear dynamical system can have a pronounced effect on its behaviour resulting in the transitions to a new state and in qualitative change in the system's properties, e.g., the transformation of an unstable equilibrium state into a stable one, and vice versa, the occurrence of multistability, noise-induced transport (stochastic ratchets), so-called stochastic resonance, and so on. The phenomenon of noise-induced transport is closely allied to the well-known problem of fluctuational transitions from one stable state to another. The theory of such transitions and examples of the phenomena indicated are considered.

Keywords: fluctuations, ratchets, stochastic resonance, noise-induced oscillations

1 Introduction

The problems of fluctuational transitions of a nonlinear system from one stable steady state to another in response to weak noise, that can be either of internal or of external origin, have long attracted the attention of physicists and chemists (Pontryagin et al., 1933; Kramers, 1940; Landa and Stratonovich, 1962). These transitions play a dominant role in the route to dynamical chaos via intermittency (Landa and Stratonovich, 1987; Landa and McClintock, 2000), in the excitation of noise-induced oscillations (Landa et al., 1997; Landa and Zaikin, 1998, 1999), and in the fluctuational transport (Millonas and Dykman, 1994; Landa, 1998; Astumian and Moss, 1998). Many researchers believe that the fluctuational transitions are of crucial importance in stochastic resonance as well (Gammaitoni et al., 1998). However, it is the author's opinion that such is not the case.

Noise can not only cause the transition of a nonlinear system from one stable steady state to another but also induce these states in themselves. Such is indeed the case when noise-induced oscillations are excited.

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2 Fluctuational Transitions of Nonlinear Systems from one Stable Steady State to another

The problem of fluctuational transitions from one stable steady state of a system to another under influence of weak noise can be reduced to the statistical problem of the probability of the first attainment of a boundary by a Brownian particle moving in a given force field (Pontryagin, 1933; Landa and Stratonovich, 1962). As an example, let us consider a Brownian particle in a force field corresponding to double-well potential $U(x)$. The motion of such a particle is conveniently described by the equation

$$m\ddot{x} + \gamma\dot{x} + \gamma f(x) = \gamma\xi(t), \tag{1}$$

where m is the mass of the particle, $f(x) = dU/dx = -ax+bx^3$, $\xi(t)$ is a sufficiently wide-band random process of intensity K.

For $\xi(t) \equiv 0$ eq. 1 has two stable singular points $x_{1,2} = \pm\sqrt{a/b}\,, \dot{x}_{1,2} = 0$ and one unstable $x_0 = 0\,, \dot{x}_0 = 0$. These points corresponds to steady states of the particle. In the absence of fluctuations, the particle, being in one of the stable steady states, cannot pass from one to another without external action of some kind. In the presence of weak noise, however, the particle executes small random oscillations in the vicinity of one of the stable steady states and, from time to time, undergoes a transition to another stable steady state. If the noise is sufficiently weak (the condition for the smallness of the noise intensity will be specified later), such transitions occur only very rarely. Thus the particle remains in the vicinity of the corresponding stable steady state over a long period, and the probability distribution consequently has a chance to reach its stationary value. We will consider namely this case.

The statistical problem of the probability of the first attainment of a boundary is significantly simplified for a single stochastic equation of the first order. Let us show that in two limiting cases eq. 1 can be reduced to such an equation.

(i) The damping factor γ is sufficiently large, so that we can rewrite eq. 1 as

$$\mu\ddot{x} + \dot{x} + f(x) = \xi(t), \tag{2}$$

where $\mu = m/\gamma$ is a small parameter. As is shown in (Stratonovich, 1963), in the first approximation with respect to μ the two-dimensional Fokker–Planck equation corresponding to eq. 2 can be reduced to one-dimensional equation

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\Big(1+\mu f'(x)\Big)\left(f(x)w(x,t) + \frac{K}{2}\frac{\partial w(x,t)}{\partial x}\right)\right], \tag{3}$$

where $f'(x) = df(x)/dx$. We note that the derivation of the corresponding one-dimensional equation in higher approximations with respect to μ is given in (Landa and McClintock, 2000). Eq. 3 can be rewritten in the form of the Fokker–Planck equation as

$$\frac{\partial w(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\left(\Big(1+\mu f'(x)\Big)f(x) - \frac{\mu K}{2}f''(x)\right)w(x,t)\right] +$$

$$\frac{K}{2}\frac{\partial}{\partial x^2}\Big(\big(1+\mu f'(x)\big)w(x,t)\Big). \tag{4}$$

This equation is associated with the following Langevin equation:

$$\dot{x} = -\big(1+\mu f'(x)\big)f(x) + \frac{\mu K}{4}f''(x) + g(x)\xi(t), \tag{5}$$

where $g(x) = \big(1+\mu f'(x)/2\big)/\gamma$.

(ii) Let the damping factor γ be sufficiently small and $\gamma f(x)$ is of order of $m\ddot{x}$. In this case the particle energy E, which is described by

$$E = \frac{m\dot{x}^2}{2} + \gamma U(x), \tag{6}$$

is a slowly varying function of t. The stable steady states correspond to minima of the function $U(x)$, and the unstable state corresponds to maximum of this function. A transition from one stable steady state to another can occur when $U(x)$ attains its maximal value. Multiplying both sides of eq. 1 by $\dot{x}$ we find the following exact equation for E:

$$\dot{E} = -\gamma\big(\dot{x}^2 - \dot{x}\xi(t)\big). \tag{7}$$

So, combining eqs. 6 and 7 we obtain the two stochastic equations

$$\begin{aligned} \dot{x} &= \sqrt{\frac{2\big(E-\gamma U(x)\big)}{m}}, \\ \dot{E} &= -\gamma\left(\frac{2\big(E-\gamma U(x)\big)}{m} - \sqrt{\frac{2\big(E-U(x)\big)}{m}}\,\xi(t)\right). \end{aligned} \tag{8}$$

If the correlation time of the noise is sufficiently small, we can write the two-dimensional Fokker–Planck equation corresponding to eqs. 8:

$$\begin{aligned} \frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}\left(\sqrt{\frac{2\big(E-\gamma U(x)\big)}{m}}\,w\right) + \frac{2\gamma}{m}\frac{\partial}{\partial E}\Bigg[\bigg(\big(E-\gamma U(x)\big)- \\ \frac{\gamma K}{4}\bigg)w\Bigg] + \frac{\gamma K}{m}\frac{\partial^2}{\partial E^2}\Big(\big(E-\gamma U(x)\big)w\Big) = 0. \end{aligned} \tag{9}$$

A solution of eq. 9 can be represented as

$$w(x,E,t) = w(E,t)w(x|E), \tag{10}$$

where $w(x|E)$ is the conditional probability distribution. Because E is a slowly varying function of t, the conditional probability distribution $w(x|E)$ has time to follow the variation of E. Therefore it can be approximately found from the following equation (Stratonovich, 1963):

$$\frac{\partial}{\partial x}\left(\sqrt{\frac{2\big(E-\gamma U(x)\big)}{m}}\, w(x|E)\right)=0. \tag{11}$$

Taking account of the normalization condition, a solution of eq. 11 can be written as

$$w(x|E)=\begin{cases}\dfrac{\sqrt{m}\omega(E)}{\pi\sqrt{2\big(E-\gamma U(x)\big)}}, & \text{for } \gamma U(x)\le E,\\[2ex] 0 & \text{for } \gamma U(x)>E,\end{cases} \tag{12}$$

where

$$\omega(E)=\frac{\pi}{\sqrt{m}}\left(\int\limits_{\gamma U(x)\le E}\frac{dx}{\sqrt{2\big(E-\gamma U(x)\big)}}\right)^{-1}$$

is the oscillation frequency of a particle with energy E. Thus, it follows from 10 and 12 that

$$w(x,E,t)=\frac{\sqrt{m}\omega(E)}{\pi\sqrt{2\big(E-\gamma U(x)\big)}}\, w(E,t). \tag{13}$$

Substituting 13 into eq. 9 and integrating over x we obtain the following one-dimensional Fokker–Planck equation for $w(E,t)$ (Stratonovich, 1963):

$$\begin{aligned}\frac{\partial w(E,t)}{\partial t}=\frac{\gamma}{m}\bigg\{&\frac{\partial}{\partial E}\left(\left(\omega(E)J(E)-\frac{\gamma K}{2}\right)w(E,t)\right)\\ &+\frac{\gamma K}{2}\frac{\partial^2}{\partial E^2}\Big(\omega(E)J(E)w(E,t)\Big)\bigg\},\end{aligned} \tag{14}$$

where

$$J(E)=\frac{\sqrt{m}}{\pi}\int\limits_{\gamma U(x)\le E}\sqrt{2\big(E-\gamma U(x)\big)}\,dx$$

is the action. We note that

$$\frac{dJ(E)}{dE}=\frac{1}{\omega(E)}. \tag{15}$$

It is easily shown that the following Langevin equation can be related to the Fokker–Planck equation 14:

$$\dot{E}=-\gamma\left[\frac{\omega(E)J(E)}{m}-\frac{\gamma K}{2m}\left(1-\frac{1}{2}\frac{d\omega(E)J(E)}{dE}\right)-g(E)\xi(t)\right], \tag{16}$$

where $g(E) = \sqrt{\omega(E)J(E)/m}$.

So, we see that in both of the considered particular cases eq. 1 can be approximately reduced to a single Langevin equation of the form

$$\dot{z} + f(z) = g(z)\xi(t), \tag{17}$$

where $\xi(t)$ is white noise of intensity K. Let us assume that $f(z)$ vanishes at the points $z = z_0$ and $z = z_1$ and is positive for $z_0 < z < z_1$. This implies that the point z_0 is a stable steady state and that z_1 is an unstable steady state.

The Fokker–Planck equation for the probability density $w(z,t)$ is

$$\frac{\partial w(z,t)}{\partial t} = \frac{\partial}{\partial z}\left[\left(f(z) - \frac{Kg(z)g'(z)}{2}\right)w(z,t)\right] + \frac{K}{2}\frac{\partial^2}{\partial z^2}\big(g^2(z)w(z,t)\big). \tag{18}$$

The stationary solution of Eq. (18) satisfying the condition for zero probability flux is

$$w_{\mathrm{st}}(z) = \frac{C}{g(z)}\exp\left(-2\int_{z_0}^{z}\frac{f(u)}{\kappa g^2(u)}\,du\right), \tag{19}$$

where the constant C is determined from the normalization condition. It is easy to verify that, for small noise intensity $\kappa g^2(z)$, the function $w_{\mathrm{st}}(z)$ peaks at the points corresponding to stable steady states, in particular, at the point z_0.

Let us calculate the probability for the passage of the system from a certain point z' lying in the range from z_2 to z_1, where $z_2 \le z_0$, through the boundary $z = z_1$. Clearly, for sufficiently small noise intensity, the probability of such passage must be independent of the initial point z', provided only that this point is not located too close to the boundary. Let us denote a solution of eq. 18, satisfying the conditions

$$w(z,0) = \delta(z - z'), \quad w(z_1,t) = 0,$$

by $w(z,z',t)$. Then the probability that z does not attain the boundary $z = z_1$ in a time t is

$$P(t,z') = \int_{z_2}^{z_1} w(z,z',t)\,dz. \tag{20}$$

One of the methods of calculating $P(t,z')$ was suggested in (Landa and Stratonovich, 1962). It is known that the probability density $w(z,z',t)$ as a function of z' is described by the equation conjugate to eq. 18, namely

$$\frac{\partial w(z,z',t)}{\partial t} = F(z')\frac{\partial w(z,z',t)}{\partial z'} + \frac{Kg^2(z')}{2}\frac{\partial^2 w(z,z',t)}{\partial z'^2}, \tag{21}$$

where

$$F(z) = -f(z) + \frac{Kg(z)g'(z)}{2}. \tag{22}$$

Integrating eq. 21 over z from z_2 to z_1, and taking account of 20, we find the equation for the probability $P(t, z')$ [1]:

$$\frac{\partial P(t,z)}{\partial t} = F(z)\frac{\partial P(t,z)}{\partial z} + \frac{\kappa g^2(z)}{2}\frac{\partial^2 P(t,z)}{\partial z^2}. \tag{23}$$

Let us represent $\partial P(t,z)/\partial t$ in terms of the characteristic function

$$\Theta(iv, z) = -\int_0^\infty \frac{\partial P(t,z)}{\partial t}\, \mathrm{e}^{ivt}\, dt. \tag{24}$$

Expanding both sides of the expression 24 as a power series in iv we obtain

$$\Theta(iv, z) = \sum_{k=0}^{\infty} (iv)^k m_k(z), \tag{25}$$

where

$$m_k(z) = -\frac{1}{k!}\int_0^\infty t^k \frac{\partial P(t,z)}{\partial t}\, dt \tag{26}$$

is the kth moment of the attainment time. Because $P(\infty, z) = 0$ and $P(0, z) = 1$, then $m_0(z) = P(\infty, z) - P(0, z) = -1$. Differentiating both sides of eq. 23 with respect to t, multiplying by e^{ivt} and integrating over t from 0 to ∞, we obtain the following equation for the characteristic function $\Theta(iv, z)$:

$$-iv\Theta = F(z)\frac{\partial \Theta}{\partial z} + \frac{\kappa g^2(z)}{2}\frac{\partial^2 \Theta}{\partial z^2}. \tag{27}$$

Substituting 25 in eq. 27 we can obtain the equations for all of the moments of the attainment time. In particular, for the mean first attainment time $M(z) \equiv m_1(z)$ we find

$$\frac{\kappa g^2(z)}{2}\frac{d^2 M}{dz^2} + F(z)\frac{dM}{dz} + 1 = 0. \tag{28}$$

This equation, as well as eq. 23, was first derived in (Pontryagin et.al., 1933).

To solve eq. 28 we must set two boundary conditions. One of these is immediately evident, it is

$$M(z_1) = 0. \tag{29}$$

The second boundary condition depends on the character of the boundary $z = z_2$ (Landa and Stratonovich, 1962). If it is perfectly reflecting, and the requirements that $g(z_2) \neq 0$, $|f(z_2)| < \infty$ and $z_2 \neq -\infty$ are fulfilled, then $\left.\frac{dM}{dz}\right|_{z=z_2} = 0$. If one of these requirements is not fulfilled then we must use as the second boundary condition the requirement of boundedness of the function $M(z)$ at the point $z = z_2$.

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[1] Below we substitute unprimed z in place of primed z'.

A solution of eq. 28, in view of 22, satisfying the condition 29 is (Pontryagin et.al., 1933)

$$M(z) = \frac{2}{K} \int_{z}^{z_1} \frac{1}{g(z')} \exp\left(2 \int_{z_0}^{z'} \frac{f(u)}{K g^2(u)} du\right)$$

$$\times \int_{z_2}^{z'} \frac{1}{g(z)} \exp\left(-2 \int_{z_0}^{z} \frac{f(u)}{K g^2(u)} du\right) dz\, dz'$$

$$+C \int_{z}^{z_1} \frac{1}{g(z')} \exp\left(2 \int_{z_0}^{z'} \frac{f(u)}{K g^2(u)} du\right) dz', \tag{30}$$

where the constant C is determined from the second boundary condition. In all examples considered by us the second boundary condition causes C to be equal to zero.

By using the condition of the noise intensity smallness, the expression 30, for $C = 0$, can be reduced approximately to

$$M(z) \approx \frac{2}{K} \int_{z_2}^{z_1} \frac{1}{g(z)} \exp\left(-2 \int_{z_0}^{z} \frac{f(u)}{\kappa g^2(u)} du\right) dz$$

$$\times \int_{z_2}^{z_1} \frac{1}{g(z)} \exp\left(2 \int_{z_0}^{z} \frac{f(u)}{\kappa g^2(u)} du\right) dz. \tag{31}$$

If the conditions

$$|z_1 - z_0| \gg \sqrt{Q(z_0)}, \quad |z_2 - z_0| \gg \sqrt{Q(z_0)}, \quad |z - z_1| \gg \sqrt{-Q(z_1)}, \tag{32}$$

where

$$Q(z) = \frac{K}{2} \left\{ \frac{d}{dz} \left(\frac{f(z)}{g^2(z)} \right) \right\}^{-1},$$

are fulfilled (the first from these conditions should be considered as the condition for the smallness of the noise intensity), the integrals in the expression 31 can be calculated approximately by using a method similar to the saddle-point technique. We thus obtain

$$M(z) \approx \pi \frac{\sqrt{-Q(z_0) Q(z_1)}}{\kappa g(z_0) g(z_1)} \exp\left(2 \int_{z_0}^{z_1} \frac{f(u)}{\kappa g^2(u)} du\right). \tag{33}$$

We see from 33 that, in the approximation considered, the mean first attainment time $M(z)$ is independent of z and exponentially dependent of the ratio of the potential barrier height to the noise intensity. The expression 33, as applied to eq. 5, takes the form:

$$M \approx \frac{\pi}{2\sqrt{2}\, a} \left[1 + \mu a \left(1 - \frac{3K}{32 \Delta U}\right)\right] \exp\left(\frac{2\Delta U}{K}\right), \tag{34}$$

where $\Delta U = U(0) - U(\pm\sqrt{a/b}) = a^2/(4b)$ is the height of the potential barrier. It follows from the condition of the smallness of the noise intensity that $K \ll \Delta U$. Therefore the expression 34 shows that the mean first attainment time increases with increasing particle mass. It should be noted that this result is valid only for small noise and in the first approximation with respect to μ.

If the second condition of 32 is not fulfilled, e.g. $z_2 = z_0$, then an approximate calculation of the integrals in the expression 31 can be performed in another way. As an example, let us consider eq. 16. For this equation, in view of 15, we can rewrite the expression (31) as

$$M \approx \frac{2m}{\gamma^2 K} \int_{E_0}^{E_1} \frac{1}{\omega(E)} \exp\left(-\frac{2}{\gamma K} E\right) dE \int_{E_0}^{E_1} \frac{1}{J(E)} \exp\left(\frac{2}{\gamma K} E\right) dE, \tag{35}$$

where $E_0 = \gamma U(\pm\sqrt{a/b}) = -\gamma^2 a^2/(4b)$, $E_1 = U(0) = 0$. Taking into account that $\exp\big(\mp 2/\gamma K)\, E\big)$ have the most values for $E = E_0$ and $E = E_1$, respectively, and, for $K \ll \Delta U$, decline rapidly , in the first integral of 35 we can substitute $\omega(E_0)$ in place of $\omega(E)$ and in the second integral we can substitute $J(E_1)$ in place of $J(E)$. In so doing we obtain

$$M \approx \frac{Km}{2\omega(E_0)J(E_1)} \exp\left(\frac{2\Delta U}{K}\right), \tag{36}$$

where $\omega(E_0) = \sqrt{2a\gamma/m}$ is the frequency of small oscillations in the vicinity of the stable steady state. Because $J(E_1) \sim \sqrt{m}$, within the weak-noise approximation, the mean first attainment time is proportional to the particle mass. Thus, we obtain that in both of the particular cases the mean first attainment time increases as the particle mass increases.

The value of $M(z)$ is equal to the mean time at which the coordinate z first attains the boundary $z = z_1$. If the potential at this boundary has a smooth maximum, then the probability of passing through the boundary (p) is equal to the probability $(1-p)$ of returning back again, i.e. $p = 1/2$. Hence the mean time of the passage through the boundary T has to be equal to $2M$. It can be shown that for $p \neq 1-p$

$$T = \frac{M}{p}. \tag{37}$$

3 Fluctuational Transport (Stochastic Ratchets)

In recent years noise-induced transport phenomena for Brownian particles have attracted considerable interest, usually in the context of biological and chemical problems (see, for example, (Astumian and Bier, 1996; Jülicher et al. (1997); Bier (1997); Astumian and Moss, 1998)).

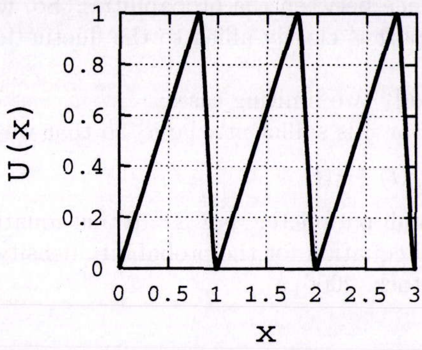

FIGURE 1. An example of the ratchet-like potential $U(x)$.

Systems in which noise-induced transport occurs are often called stochastic ratchet-like devices by analogy with mechanical device "ratchet and pawl" described and considered by Feynman (Feunman, 1963). Feynman showed that in the case of thermodynamic equilibrium the ratchet on average is at rest – it advances and retreats by an equal number of teeth on the wheel – as it must be because of the Second Law of Thermodynamics. It is interesting that similar considerations were discussed by Smoluchowski (Smoluchovski, 1912) well before Feynman. However, if the system is not in thermodynamic equilibrium state then directional motion of the wheel is possible.

Most commonly, consideration of noise-induced transport is restricted to the so-called overdamped case, when the mass of the Brownian particle can be neglected and its motion is described by a first order differential equation of the form

$$\dot{x} + f(x) = \zeta(x,t) + \xi(t), \tag{38}$$

where $\zeta(x,t)$ is either a regular or a random process with zero mean value, which just disturbs the thermodynamic equilibrium, $\xi(t)$ is white noise of intensity K imitating thermal fluctuations, $f(x) = (dU/dx)$ is an asymmetric periodic function of x, and $U(x)$ is a ratchet-like potential. An example of such a potential is shown in Fig. 1.

We consider here a more general equation of the form

$$m\ddot{x} + \gamma\dot{x} + \gamma f(x) = \gamma\zeta(x,t) + \gamma\xi(t), \tag{39}$$

where m is the particle mass, but restrict ourselves by the case that $\zeta(x,t) = \varphi(t) = B\cos\omega t$, where the frequency ω is sufficiently small. If the amplitude B is small then, in the absence of the noise $\xi(t)$, the particle is to be found in one of the potential wells; such a state is stable. As the noise is present, transitions from one well to another can occur. Directional motion of the particle is possible if the probabilities of transitions in opposite directions are different. The force $B\cos\omega t$

just causes this difference between the probabilities. So, we see that the problem of noise-induced transport is closely allied to the fluctuational transition problem considered above.

As previously, we study two limiting cases:

(i) The damping factor γ is sufficiently large, so that we can rewrite eq. 39 as

$$\mu\ddot{x} + \dot{x} + f(x) = \varphi(t) + \xi(t), \tag{40}$$

where $\mu = m/\gamma$ is a small parameter. For $\mu = 0$ this equation reduces to eq. (38).

The one-dimensional equation for the probability density $w(x,t)$ can be written as (Landa and McClintock, 2000)

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\sum_{n=0}^{\infty}\mu^n w_{1n}, \tag{41}$$

where

$$w_{10} = \big(f(x) - \varphi(t)\big)w + \frac{K}{2}\frac{\partial w}{\partial x}, \quad w_{11} = f'(x)w_{10},$$

$$w_{12} = \left(2\big(f'(x)\big)^2 + 2\big(f(x) - \varphi(t)\big)f''(x) + \frac{3K}{4}f'''(x)\right) +$$

$$\left(\big(f(x) - \varphi(t)\big)f'(x) + \frac{7K}{4}f''(x)\right)\frac{\partial w_{10}}{\partial x} + \frac{K}{2}f'(x)\frac{\partial^2 w_{10}}{\partial x^2},$$

. . . .

The series in the right-hand side of eq. 41 is convergent asymptotically.

It should be emphasized that generally eq. 41 cannot be transformed to a Fokker–Planck equation from the second approximation on. This can be done only in the case of small ω_0, when we can use the quasistationary approximation.

Starting from eq. 41 in the third approximation, for two different shapes of the function $f(x)$, we calculated the mean velocity of a particle in relation to its mass and the noise intensity K (Landa and McClintock, 2000). We found that for moderately large K the flux reversal occurs.

The aforesaid is well illustrated by the examples of the dependencies of $\overline{\langle\dot{x}\rangle}/B^2$ on μ for different values of K/U_0 calculated in the first, second and third approximations (Fig. 2). We see that the difference between the results is small only for $\mu < 0.002$ but for such values of μ the flux reversal is possible only for large values of K/U_0.

(ii) Let the damping factor γ be sufficiently small. In this case we can take as a slow variable the particle energy E, which we define as

$$E = \frac{m\dot{x}^2}{2} + \gamma V(x,t), \tag{42}$$

where $V(x) = \int f(x)\,dx - x\varphi(t) = U(x) - x\varphi(t)$. The variables E and x are described by the equations

$$\dot{x} = \sqrt{\frac{2\big(E - \gamma V(x,t)\big)}{m}},$$

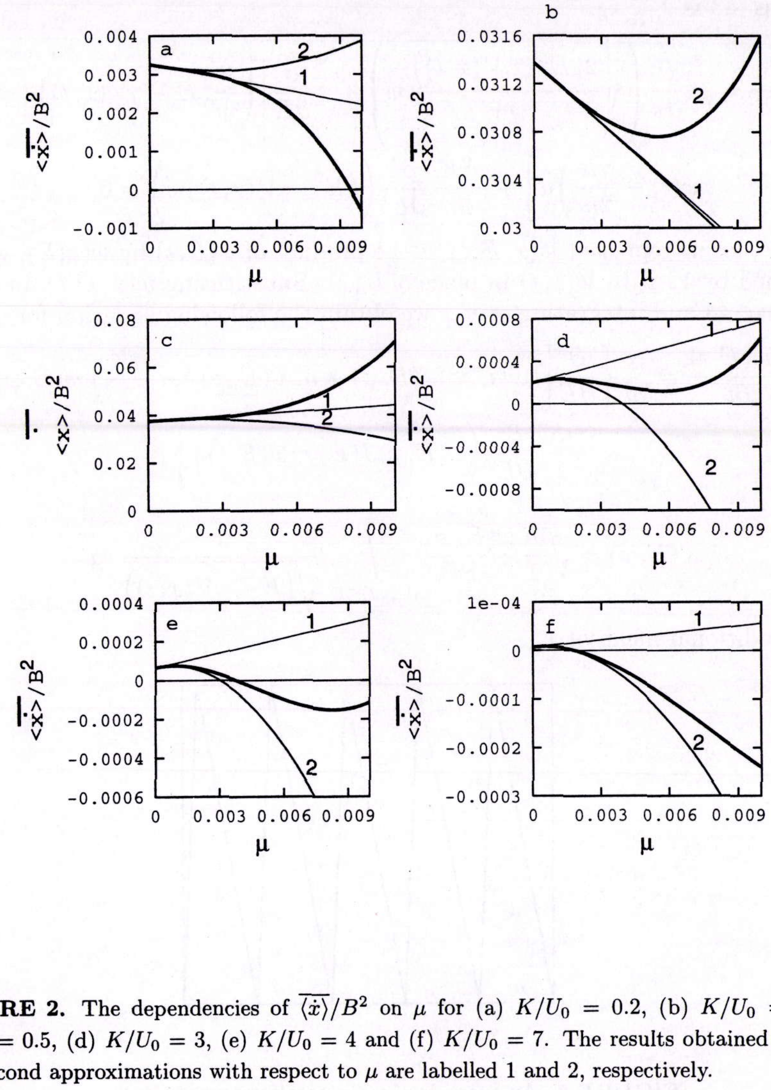

FIGURE 2. The dependencies of $\overline{\langle\dot{x}\rangle}/B^2$ on μ for (a) $K/U_0 = 0.2$, (b) $K/U_0 = 0.35$, (c) $K/U_0 = 0.5$, (d) $K/U_0 = 3$, (e) $K/U_0 = 4$ and (f) $K/U_0 = 7$. The results obtained in the first and second approximations with respect to μ are labelled 1 and 2, respectively.

$$\dot{E} = -\gamma\left(\frac{2\big(E-\gamma V(x,t)\big)}{m} + x\dot{\varphi}(t) - \sqrt{\frac{2\big(E-\gamma V(x,t)\big)}{m}}\,\xi(t)\right). \tag{43}$$

The Fokker–Planck equation for the probability distribution $w(x,E,t)$ corresponding to eqs. 43 is

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial x}\left(\sqrt{\frac{2\big(E-\gamma V(x,t)\big)}{m}}\,w\right) + \gamma\frac{\partial}{\partial E}\left[\left(\frac{2}{m}\big(E-\gamma V(x,t)\big)+ x\dot{\varphi}(t) - \frac{\gamma K}{2m}\right)w\right] + \frac{\gamma^2 K}{m}\frac{\partial^2}{\partial E^2}\Big(\big(E-\gamma V(x,t)\big)w\Big) = 0. \tag{44}$$

As before, let us represent $w(x, E, t)$ as the product of $w(E, t)$ by $w(x|E)$, where is determined by 12 with $V(x, t)$ in place of $U(x)$. Substituting $w(x, E, t)$, in view of 12, into eq. 44 and integrating over x we obtain the following equation for $w(E, t)$:

$$\frac{\partial w(E,t)}{\partial t} = \frac{\gamma}{m}\left\{\frac{\partial}{\partial E}\left[\left(\omega(E,\varphi)J(E,\varphi) + mX(E,\varphi)\dot{\varphi} - \frac{\gamma K}{2}\right)w(E,\varphi)\right] + \frac{\gamma K}{2}\frac{\partial^2}{\partial E^2}\Big(\omega(E,\varphi)J(E,\varphi)w(E,t)\Big)\right\}, \tag{45}$$

where

$$X(E,\varphi) = \sqrt{\frac{m}{2}}\frac{\omega(E,\varphi)}{\pi}\int\limits_{\gamma V(x,t)\le E}\frac{x}{\sqrt{\big(E-\gamma V(x,t)\big)}}\,dx$$

is the conditional mean of x.

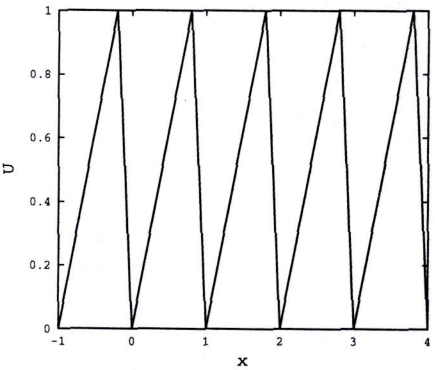

FIGURE 3. An example of a saw-tooth potential $U(x)$.

More often than not researchers of noise-induced transport consider a saw-tooth potential $U(x)$ shown in Fig. 3. In this case we can set

$$f(x) = \begin{cases} a_1 & \text{for } nL < x < nL + x_1, \\ -a_2 & \text{for } nL - x_2 < x < nL, \end{cases} \tag{46}$$

where $n = 0, \pm1, \pm2, \ldots$, $L = x_1 + x_2$ is the period of the function $f(x)$. It is easily shown that, in the absence of the disturbances $\varphi(t)$ and $\xi(t)$, the points $x = nL$

and $x = nL + x_1 = (n+1)L - x_2$ correspond to stable and unstable equilibrium states, respectively. For this saw-tooth potential

$$\omega(E,\varphi) = \frac{\pi\gamma\big(a_1-\varphi\big)\big(a_2+\varphi\big)}{(a_1+a_2)\sqrt{2mE}}, \quad J(E,\varphi) = \frac{2(a_1+a_2)E\sqrt{2mE}}{3\pi\gamma\big(a_1-\varphi\big)\big(a_2+\varphi\big)},$$

$$X(E,\varphi) = \frac{2(a_2-a_1+2\varphi)E}{3\gamma(a_1-\varphi)(a_2+\varphi)}. \tag{47}$$

Substituting 47 into eq. 45 we obtain

$$\frac{\partial w(E,t)}{\partial t} = \frac{\gamma}{m}\left\{\frac{\partial}{\partial E}\left[\left(\frac{2E}{3} + \frac{2mE(a_2-a_1+2\varphi)}{3\gamma(a_1-\varphi)(a_2+\varphi)}\dot{\varphi} - \frac{\gamma K}{2}\right) w(E,t)\right] + \frac{\gamma K}{3}\frac{\partial^2 (Ew)}{\partial E^2}\right\}. \tag{48}$$

It follows from 48 and 23 that the probabilities $P_r(t,E)$ and $P_l(t,E)$ that E does not attain the boundaries $E_{1,2} = \gamma U_0\big(1 \mp \varphi(t)/a_{1,2}\big)$, respectively, are described by the equations

$$\frac{\partial P_{r,l}}{\partial t} + \frac{\gamma}{m}\left(\frac{2E}{3} + \frac{2mE(a_2-a_1+2\varphi)}{3\gamma(a_1-\varphi)(a_2+\varphi)}\dot{\varphi} - \frac{\gamma K}{2}\right)\frac{\partial P_{r,l}}{\partial E} - \frac{\gamma^2 KE}{3m}\frac{\partial^2 P_{r,l}}{\partial E^2} = 0. \tag{49}$$

For the calculation of the probability $P_r(t,E)$ we substitute into eq. 49 $E = E' - \gamma U_0 \varphi(t)/a_1$. As a result we obtain

$$\frac{\partial P_r}{\partial t} + \frac{\gamma}{m}\left(\frac{2E'}{3} - \frac{\gamma K}{2}\right)\frac{\partial P_r}{\partial E'} - \frac{\gamma^2 KE'}{3m}\frac{\partial^2 P_r}{\partial E'^2} = \frac{2\gamma^2 U_0}{3ma_1}\left(\frac{\partial P_r}{\partial E'} - \frac{\gamma K}{2}\frac{\partial^2 P_r}{\partial E'^2}\right)\varphi(t) - \left(E' - \frac{\gamma U_0 \varphi}{a_1}\right)\frac{2(a_2-a_1+2\varphi)}{3(a_1-\varphi)(a_2+\varphi)}\frac{\partial P_r}{\partial E'}\dot{\varphi}. \tag{50}$$

A similar equation we can write for $P_l(t,E')$.

In the case of small B we can seek a solution of eq. 50 as a power series in B:

$$P_r(t,E) = P_0(t,E) + P_{r1}(t,E)B + P_{r2}(t,E)B^2 + \dots . \tag{51}$$

Expanding the right side of eq. 50 in powers of B and substituting 51 into the equation found we obtain the following equations:

$$\frac{\partial P_0}{\partial t} + \frac{\gamma}{m}\left(\frac{2E'}{3} - \frac{\gamma K}{2}\right)\frac{\partial P_0}{\partial E'} - \frac{\gamma^2 KE'}{3m}\frac{\partial^2 P_0}{\partial E'^2} = 0, \tag{52}$$

$$\frac{\partial P_{r1}}{\partial t} + \frac{\gamma}{m}\left(\frac{2E'}{3} - \frac{\gamma K}{2}\right)\frac{\partial P_{r1}}{\partial E'} - \frac{\gamma^2 KE'}{3m}\frac{\partial^2 P_{r1}}{\partial E'^2} =$$

$$\frac{2\gamma^2 U_0}{3ma_1}\left(\frac{\partial P_0}{\partial E'} - \frac{\gamma K}{2}\frac{\partial^2 P_0}{\partial E'^2}\right)\cos\omega t + \frac{2E'(a_2-a_1)\omega}{3a_1a_2}\frac{\partial P_0}{\partial E'}\sin\omega t. \tag{53}$$

$$\frac{\partial P_{r2}}{\partial t} + \frac{\gamma}{m}\left(\frac{2E'}{3} - \frac{\gamma K}{2}\right)\frac{\partial P_{r2}}{\partial E'} - \frac{\gamma^2 K E'}{3m}\frac{\partial^2 P_{r2}}{\partial E'^2} =$$
$$\frac{2\gamma^2 U_0}{3ma_1}\left(\frac{\partial P_{r1}}{\partial E'} - \frac{\gamma K}{2}\frac{\partial^2 P_{r1}}{\partial E'^2}\right)\cos\omega t + \frac{2E'(a_2 - a_1)\omega}{3a_1 a_2}\frac{\partial P_{r1}}{\partial E'}\sin\omega t +$$
$$\frac{E'(a_1^2 + a_2^2) - \gamma U_0 a_2(a_2 - a_1)}{3a_1^2 a_2^2}\frac{\partial P_0}{\partial E'}\omega\sin 2\omega t. \tag{54}$$

Correspondingly, we can set

$$M_r(E) = M_0(E) + M_{r1}(E)B + M_{r2}(E)B^2 + \dots . \tag{55}$$

The value of $M_0(E)$ can be calculated from eq. 52. It is

$$M_0(E) = \frac{3m}{2\gamma}\int\limits_{\gamma U_0}^{E}\frac{1}{E}\left\{\sqrt{\frac{\pi\gamma K}{8E}}\left[\Phi\left(\sqrt{\frac{2U_0}{K}}\right) - \Phi\left(\sqrt{\frac{2E}{\gamma K}}\right)\right] - \right.$$
$$\left.\sqrt{\frac{\gamma U_0}{E}}\exp\left(\frac{2(E-\gamma U_0)}{\gamma K}\right) + 1\right\} dE, \tag{56}$$

where

$$\Phi(z) = \frac{2}{\sqrt{\pi}}\int\limits_0^z e^{-t^2}\,dt$$

is the error integral.

Let us express $P_0(t,E)$ and $P_{r1}(t,E)$ in terms of the characteristic functions by

$$P_0(t,E) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Theta_0(iv,E)}{iv}e^{-ivt}\,dv,$$
$$P_{r1}(t,E) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\frac{\Theta_{11}(iv_1,E)}{iv_1}e^{-iv_1 t}\,dv_1 + \int_{-\infty}^{\infty}\frac{\Theta_{12}(iv_2,E)}{iv_2}e^{-iv_2 t}\,dv_2\right), \tag{57}$$

where $v_{1,2} = v \pm \omega$. Substituting 57 into eq. 53 and equating the coefficients of $\exp\left(-iv_{1,2}t\right)$ we obtain

$$-iv_1\Theta_{11}(iv_1,E) + \frac{\gamma}{m}\left(\frac{2E}{3} - \frac{\gamma K}{2}\right)\frac{\partial\Theta_{11}}{\partial E} - \frac{\gamma^2 KE}{3m}\frac{\partial^2\Theta_{11}}{\partial E^2} =$$
$$\frac{\gamma^2 U_0 v_1}{3ma_1 v}\left(\frac{\partial\Theta_0}{\partial E} - \frac{\gamma K}{2}\frac{\partial^2\Theta_0}{\partial E^2}\right) + \frac{iv_1\omega E(a_2-a_1)}{3a_1 a_2 v}\frac{\partial\Theta_0}{\partial E}, \tag{58}$$

$$-iv_2\Theta_{12}(iv_2,E) + \frac{\gamma}{m}\left(\frac{2E}{3} - \frac{\gamma K}{2}\right)\frac{\partial\Theta_{12}}{\partial E} - \frac{\gamma^2 KE}{3m}\frac{\partial^2\Theta_{12}}{\partial E^2} =$$
$$\frac{\gamma^2 U_0 v_2}{3ma_1 v}\left(\frac{\partial\Theta_0}{\partial E} - \frac{\gamma K}{2}\frac{\partial^2\Theta_0}{\partial E^2}\right) - \frac{iv_2\omega E(a_2-a_1)}{3a_1 a_2 v}\frac{\partial\Theta_0}{\partial E}. \tag{59}$$

Using the expansion 25 and taking into account eq. 52 and the evident condition that $m_{10}(E) = 0$ we obtain the following equation for $M_{r1}(E)$:

$$\frac{\gamma}{m}\left(\frac{2E}{3} - \frac{\gamma K}{2}\right)\frac{dM_{r1}}{dE} - \frac{\gamma^2 KE}{3m}\frac{d^2M_{r1}}{dE^2} = \frac{\gamma U_0}{a_1 E}\left(\frac{\gamma^2 K}{2m}\frac{dM_0}{dE} + 1\right). \tag{60}$$

Its solution is

$$M_{r1}(E) = \frac{3mU_0}{\gamma K a_1}\int\limits_E^{\gamma U_0}\frac{1}{E'^{3/2}}\exp\left(\frac{2E'}{\gamma K}\right)$$
$$\times\int\limits_{\gamma U_0}^{E'}\frac{1}{\sqrt{E}}\left(1 + \frac{\gamma^2 K}{2m}\frac{dM_0}{dE}\right)\exp\left(-\frac{2E}{\gamma K}\right)dE\,dE'. \tag{61}$$

The value of $M_{r1}(E)$ should be avareged having regard to the stationary probability distribution for E. The latter can be found from eq. 48 for $\varphi(t) = 0$. It is

$$w(E) = C\sqrt{2E\gamma K}\exp\left(-\frac{2E}{\gamma K}\right), \tag{62}$$

where C is found from the normalization condition

$$\int\limits_0^{\gamma U_0} w(E)\,dE.$$

It follows from here that

$$C = \left(\frac{2}{\gamma K}\right)\left[\frac{\sqrt{\pi}}{2}\,\Phi\left(\sqrt{\frac{2U_0}{K}}\right) - \sqrt{\frac{2U_0}{K}}\exp\left(\frac{2U_0}{K}\right)\right]^{-1}. \tag{63}$$

Thus, we can write

$$\langle M_{r1}\rangle = \frac{3mU_0}{\gamma K a_1}\left[\frac{\sqrt{\pi}}{2}\,\Phi\left(\sqrt{\frac{2U_0}{K}}\right) - \sqrt{\frac{2U_0}{K}}\exp\left(\frac{2U_0}{K}\right)\right]^{-1}\times$$
$$\int\limits_0^{2U_0/K}\sqrt{z}e^{-z}\int\limits_z^{2U_0/K}\frac{e^{z'}}{z'^{3/2}}\int\limits_{2U_0/K}^{z'}\frac{1}{\sqrt{z''}}\left(1 + \frac{\gamma}{m}\frac{dM_0}{dz''}\right)e^{-z''}\,dz''\,dz'\,dz, \tag{64}$$

where

$$\frac{dM_0}{dz} = \frac{3m}{2\gamma z\sqrt{z}}\left\{\frac{\sqrt{\pi}}{2}\left[\Phi\left(\sqrt{\frac{2U_0}{K}}\right) - \Phi(\sqrt{z})\right]\right.$$
$$\left.-\sqrt{\frac{2U_0}{K}}\exp\left(z - \frac{2U_0}{K}\right) + \sqrt{z}\right\}, \tag{65}$$

It can be shown that $\langle M_{r1}\rangle$ is negative. Therefore we can rewrite the expression 64 as

$$\langle M_{r1}\rangle = -\frac{3m}{\gamma a_1}F(U_0/K), \tag{66}$$

where

$$F(U_0/K) = \frac{U_0}{K}\left[\frac{\sqrt{\pi}}{2}\Phi\left(\sqrt{\frac{2U_0}{K}}\right) - \sqrt{\frac{2U_0}{K}}\exp\left(\frac{2U_0}{K}\right)\right]^{-1} \times$$
$$\int_0^{2U_0/K}\sqrt{z}e^{-z}\int_z^{2U_0/K}\frac{1}{z'^{3/2}}e^{z'}\int_{z'}^{2U_0/K}\frac{1}{\sqrt{z''}}\left(1+\frac{\gamma}{m}\frac{dM_0}{dz''}\right)e^{-z''}\,dz''\,dz'\,dz, \tag{67}$$

The expression for $\langle M_{l1}\rangle$ can be obtained from 66 by substitution of a_2 in place of a_1.

It is evident that in the first approximation with respect to B the mean particle velocity is

$$\overline{\langle\dot{x}\rangle} = \frac{L}{2\big(\langle M_0\rangle + \langle M_{r1}\rangle B\big)} - \frac{L}{2\big(\langle M_0\rangle + \langle M_{l1}\rangle B\big)}$$
$$= \frac{3mL(a_2-a_1)}{2\gamma a_1 a_2\langle M_0\rangle^2}BF(U_0/K). \tag{68}$$

We see that in the case under consideration the mean particle velocity is different from zero even in the first approximation with respect to B, as differentiated from the case (i). Because $\langle M_0\rangle$ is proportional to m, the mean particle velocity decreases as the particle mass increases. The direction of the mean velocity depends on the sign of the difference $a_2 - a_1$.

4 Stochastic Resonance

In the last few years, a substantial body of journal papers and reviews has evolved which traces the role of stochastic resonance in different physical, chemical, biological, and other phenomena, see for example (Moss et al., 1994; Wiesenfeld and Moss, 1995; Gammaitoni et al., 1998; Anishchenko et al., 1999; Klimontovich, 1999). In the simplest case stochastic resonance is defined as follows: Let one have a device described by the equation

$$\dot{x} + f(x) = F(t) + \xi(t), \tag{69}$$

where $F(t) = A\cos\omega t$ is a weak input signal, $f(x) = dU(x)/dx$, $U(x) = -x^2/2 + x^4/4$ is a symmetric double-well potential, $\xi(t)$ is white noise of intensity K, i.e. $\langle\xi(t)\xi(t+\tau) = K\delta(\tau)$. It is known from numerical simulations that the response of the system $69Q(K,\omega)$ to the signal of the frequency ω is a nonmonotone function of the noise intensity K and has a maximum at a certain value of $K = K_m$. The most of the researchers of stochastic resonance (see, for example, (Gammaitoni et al., 1998; Anishchenko et al., 1999) belive that the dependence of Q on K peaks when there is a certain relation between the signal frequency ω and the mean frequency of the fluctuational transitions from one stable steady state to another (the so-called Kramers rate), which increases as the noise intensity increases. However, it is the author's opinion, that this assertion is false (Landa and Zaikin, 2000). If this were so, we would obtain the similar dependence of Q on ω at a fixed value of K/U_0; whereas this dependence is monotonically dropping. The nonmonotone character of the dependence $Q(K)$ at a fixed value of ω can be explained only by the nonmonotone change of the mean slope of the function $f(x)$ (of the system effective rigidity) as K changes. This idea is in agreement with the Klimontovich hypothesis that stochastic resonance in an overdamped bistable oscillator is caused by the nonmonotone change of the bandwidth of the low-frequency filter described by eq. 69 (Klimontovich, 1999). As follows from our studies, similar nonmonotone change of the effective rigidity may be obtained not only in response to noise, but in response to a high-frequency additive force as well (Landa et al., 2000).

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