Quantization Phenomenon in Dynamical Stochastic Systems

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Abstract

The stochastic dynamical system with the states described by elements u of a Hilbert space is considered. There is a deterministic system considered as its nonperturbed variant. An outcome y is observed under random perturbations. The probability distribution $P(y, u)$ of the measurements results in the fixed states u is analysed. A class of stochastic systems marked by the full determination of the law $P(y, u)$ via equations of the nonperturbed system is found. We also find the distributions $P(y, u)$. These distributions prove to be similar to the quantum laws of probability distribution of observable quantities including the principles of superposition and uncertainty and the phenomenon of quantization.

Keywords: Dynamic System, Stochastic System, Observation, Quantum Mechanics, Superposition Principle

1 Introduction

We find in this paper a class of the stochastic dynamical systems and the observation procedures for which the probability distribution of observable outputs prove to be similar to quantum laws, including the quantum superposition principle (the summation law for probability amplitudes) and the principle of uncertainty. Such laws are new for traditional theory of stochastic systems. The latter supposes the input probability characteristics do not depend on system dynamics. Our results relate to the systems with strong interaction of these components. These systems are analized in the theories of random processes based on the non-commutative theory of probability (Ludwig, 1967), (Pool, 1968) and other, which above all deal with the problems of quantum mechanics foundations. Definition of the stochastic systems and the observation procedures considered here do not include explicitly the postulates of non-commutative probability. On the other hand, this result give a new matherial for discussion of the quantum mechanics foundations. The special and even mysterious, in some sense, quantum postulates mentioned above

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can be explained in terms of the general system theory on the basis of sufficiently evident requirements imposed on the system and on the obsevarion procedure. These requirements, taken separately, stay within the limits of the conventional scientific experience related to systems of a quite different nature.

2 Problem Statement

We consider a stochastic dynamical system S with the states described by elements u of the set M. Let Y be an observable outcome of this system which takes real values $y \in R$.

We introduce an observation procedure for the outcome Y in the given state u, which consists in a sequence of measurements. Let the values of y in the measurements be random, and let the law of probability distribution be defined for every u in the form of the probability measure:

$$P(Q,u) \geq 0, \quad \forall Q \subset R, u \in M; \qquad P(R,u) = 1, \quad \forall u. \tag{1}$$

There is a deterministic system D considered as a nonperturbed variant of system S. The equations of the system D, which determine the trajectory $u(t)$ on a given segment of time t, are assigned, and the outcome Y is expressed in view of this equations as the functional

$$y = J(u) \tag{2}$$

The challenge is *to find a class of the systems and observation procedures characterized by the full determination of the law $P(Q,u)$ via the system D equations. We also find the characteristic properties of these distributions $P(Q,u)$. These distributions prove to be similar to quantum laws of probability distribution of observable quantities.*

The traditional problem of stochastic dynamics is to find the probability distribution $P(Q,t)$ when the characteristics of the random perturbations are preassigned. The relation between the forms $P(Q,t)$ and $P(Q,u)$ is similar, to a certain degree, to relation between the program and feedback forms of control in the theory of controlled dynamical systems.

Let the observation be organized so that the average result of the measurements coincides with the theoretical value (2):

$$J(u) = \int y dP, \quad \forall u \in M. \tag{3}$$

Such an observation procedure is termed *correct in the* average. In this type of observation, a significant perturbation of system in each measurement is allowed. But these perturbations are filtered by averaging operation.

3 Infinitesimal Systems

Let us assume that the states u are elements of a Hilbert space H, which is, generally, complex, with the product (u_1, u_2). We postulate *the infinitesimality* of the considered dynamical system. Or more precisely, we suppose the following:

- The representation

  $$J(u) = J(0) + Ku + (u, Lu) + O(u) \tag{4}$$

  for all system characteristics Y is true. Here K is the linear functional, $H \to R$; L is the linear operator which is Hermitian due to the reality of the values of $y = J(u)$; $O(u)u^{-2} \to 0, u \to 0$.

- The states $u \in M$ are small enough for the assumption $O(u) = 0$ and the linear representation of the system equations to be valid. We suppose also that these equations are homogenous.

In what follows, we shall consider the observable physical quantities Y which are described by quadratic forms only, i.e., $J(0) = Ku = 0$,

$$J(u) = (u, Lu) \tag{5}$$

For example, the energy and angular momentum in infinitesimal mechanics; the energy, momentum, and angular momentum of the electromagnetic field; all physical observable quantities in quantum theory are represented by (5). True, the postulate of infinitesimality is not used in the electrodynamics and quantum mechanics but it does not contradict them, and there exist models that include such a postulate (see, for example, (Krotov, 1997)). The linear operator L will be called *the operator of the observable outcome* Y.

Because of the infinitesimality postulate, the laws $P(Q, u)$ at every fixed $Q \subset M$ also have to be expressed similar to (4):

$$P(Q, u) = P(Q, 0) + B(Q)u + (u, A(Q)u) + O(Q, u) \tag{6}$$

The representation (6) has to satisfy Eqs. (1) and (3). The minimal order which ensures the reality and nonnegativeness of $P(Q, u)$ in H is two, therefore $B(Q)u = 0$. Equation (3) can also be satisfied in the class of quadratic forms. However, there is no solution for Eq. (1) in this class. Let the conditions on the set M of permissible states include the equality

$$|u|^2 = b, \tag{7}$$

where $b > 0$ is a given sufficiently small real number. In other words, the permissible states are normalized and the set M is a sphere of a sufficiently small radius in H.

The conditions (1), (3) can then be solved in the class of real (nonnegative) quadratic forms:

$$P(Q,u) = (u, A(Q)u), \tag{8}$$

where $A(Q)$ for fixed Q is a linear Hermitian nonnegative operator. We assume here $P(Q,0) = 0$. The equations of the system are homogeneous, therefore we can set $b = 1$ not restricting the generality. Here we limit the analysis to this case. Similar to the quantum systems, a dynamical invariant of the system D can be considered here as the Hilbert's norm $\|u\|$. Then Eq. (7) does not restrict the set of possible states.

For example, we can consider the energy as $\|u\|^2 = (u,u)$, when observing a conservative mechanical system.

4 Probability Distribution of Measurements Results

4.1 Theorem

We consider the dynamical system that satisfies the conditions mentioned above. Let the space H be of finite dimensionality n. Let there be a series of the pairs $(y_k, u_k), k = 1, \ldots, n$, such that:

$$P_y(y_k, u_k) = 1, \quad k = 1, \ldots, n,$$

where $P_y(\xi, u) = P(Q : y = \xi, u)$. Then:

- The series $\{u_k\}$ forms an orthogonal basis of the space H.
- The probability distribution for the observable values y in a state u has the following form:

$$P_y(y_k, u) = |(u_k, u)|^2, \quad k = 1, \ldots, n; \tag{9}$$

$$P(Q,u) = \sum P_y(y_k, u), \quad y_k \in Q, \tag{10}$$

  for any set $Q \subset R$ which does not include the points $y = y_k$.

- The operator L of the observable outcome Y has eigenvalues and eigenvectors which coincide with $y_k, u_k, k = 1, \ldots, n$.

4.2 Proof

- Let us write a diagonal representation of the nonnegative quadratic form (8):

$$P(Q,u) = \sum_{i=1}^{m} \alpha_i(Q)|\nu_i, u|^2. \tag{11}$$

where $\alpha_i(Q)$ are the positive eigenvalues of the operator $A(Q)$, $m \leq n$, and $\nu_i(Q)$, $i = 1, \ldots, m$, are its corresponding (orthogonal) eigenvectors. Let us denote by $\Pi(Q)$ the subspace spanned by $\{\nu_i(Q)\}$. We have

$$P_y(y_k, u_k) = 1 = \max_{|u|^2=1} P_y(y_k, u), \quad k = 1, \ldots, n. \tag{12}$$

From this it follows that, firstly,

$$u_k \in \Pi(y_k), \quad k = 1, \ldots, n, \tag{13}$$

and, secondly, $P_y(y_l, u_k) = 0, \quad l \neq k$.

But the latter means, by virtue of Eq. (11), that the vector u_k is orthogonal to the spaces $\Pi(y_l)$. And by virtue of Eq.(13), it is orthogonal to the vectors u_l, $l \neq k$. Therefore, the vectors u_k ,$k = 1, \ldots, n$, are mutually orthogonal and form a basis of the space H.

- The space $\Pi(y_l)$ is orthogonal to the system of $n - 1$ orthogonal vectors $\{u_k\}$, $k \neq l$. This means that it is one-dimensional, $m = 1$, and by virtue of Eq.(13) its basis coincides with the vector u_l. This eigenvector of the operator $A(y_l)$ corresponds by virtue of Eq (12), to the unit eigenvalue. Therefore, $P_y(y_k, u) = |(u_k, u)|^2$. Then

$$\sum_{k=1}^{n} P_y(y_k, u) = \sum_{k=1}^{n} |u_k, u|^2 = |u|^2 = 1.$$

Therefore, $P(Q, u) = 0$ for all $Q \subset R$ which do not include the points $y = y_k$, and

$$P(Q, u) = \sum P_y(y_k, u), \quad y_k \in Q.$$

- By virtue of Eqs (3), (4), (9), and (10) we have

$$J(u) = (u, Lu) = \sum_{k=1}^{n} y_k |u_k, u|^2, \forall u.$$

But the latter equality means that the condition (3) of the theorem is satisfied. We have proved the theorem.

4.3 Remarks

- It is easy to see that the requirement that the space H be finite-dimensional in the theorem can be replaced by the requirement that the set $\{u_k\}$ contain sufficient set of elements. Specifically, the states u_k must constitute a basis of the space H. The law (9), (10) remains valid.

5 Relation to Principles of Quantum Mechanics

A similarity between these results and the postulates of quantum mechanics is evident. The names assigned to the properties 4 and 5 emphasize this similarity. The state u of a quantum system is a complex wave function. The state is normalized and described by linear homogeneous equations. The functional representations $J(u)$ of the observable physical quantities are, in general, a consequence of these equations. For example, the representations of the momentum, energy, angular momentum, spin, and charge are their dynamical invariants related to the corresponding symmetry properties. The representations of the coordinates of a particle and the functions of these coordinates follow directly from the statistical interpretation of the wave function. By virtue of the same interpretation, the functionals J(u) are interpreted as the averages of physical quantities.

5.1 Superposition and Uncertainty Principles

The main constructive postulate of quantum mechanics is the superposition principle (or the summation law for probability amplitudes), which literally coincides with the property 5. In addition to the concrete representation $J(u)$, it determines the probability distribution law $P(Q, u)$ of the observable values. This special and even mysterious, in some sense, quantum postulate can therefore be explained in terms of the general system theory on the basis of sufficiently evident requirements imposed on the system and on the observation procedure. These requirements, taken separately, stay within the limits of the conventional scientific experience related to systems of a quite different nature. These requirements are: the coincidence of the observable average and the theoretical value of outcome, the applicability of infinitesimal constructions and the presence of the states where the observable quantity is deterministic.

Systems possessing these properties are also subject to the uncertainty principle in its limited form cited above (the corollary 5) of the Theorem). More informative statements require that the eigenstates of observable quantities (the coordinates and the momentum of a particle in quantum mechanics) be defined in more specific terms.

5.2 Possible Interpretation of Infinitesimality of Quantum Mechanics

The thesis of infinitesimality of quantum mechanics have especially clear interpretation in the de Broglie conception frameworks. It is stipulated by smallness of the pilot wave, (Gueret, Vieger, 1980), (Krotov, 1997).

5.3 A Corollary

We point out a corollary from interpretation of quantum systems as infinitesimal ones: the quantum models with nonlinear equations of wave functions are incorrect, because the superposition and uncertainty principles are invalid in this case.

- Let us consider the follow question: why the Theorem is not applicable to a deterministic system. We have in this case:

$$P_y(y,u) = \begin{cases} 1, & y = J(u) \\ 0, & y \neq J(u) \end{cases}$$

Evidently, there is a series $\{y_k, u_k\}$ $P_y(y_k, u_k) = 1$ required by Theorem. But the function $P_y(y,u)$ does not satisfy the infinitesimality condition (6) by its discontinuity.

4.4 Corollaries

Thus, there exists such a class of dynamical systems that the law of distribution of probability of the results of measurements of its outcome is determined by the equations of the nonperturbed system dynamics. This class is defined by the conditions of the Theorem. Let us point out some properties of these distributions $P(Q,u)$ that appear as corollaries from the Theorem. These properties are determined by the properties of the operator L of the observable outcome Y.

- Property 1. The results of the measurement are concentrated in the eigenvalues of the linear Hermitian operator L. In particular, if its spectrum is discrete, so is the set of possible values of the random quantity Y.

- Property 2. The eigenvalues of the operator L form an orthonormal basis of the space H. The probability of the value y coinciding with the appropriate eigenvalue of L in a given state u equals the square of the absolute value of the projection of u onto the corresponding eigenvector.

- Property 3. The result of measurements is certain in the eigenstates of the operator L, and only in these states, i.e., a single measurement yields the true value of the observable quantity.

- Property 4. The principle of uncertainty. Let two physical quantities Y_1 and Y_2 be measured. Accompanied observation of the two quantities in a single measurement gives their true values only in the eigenstate of both operator L_1 and L_2.

- Property 5. The principle of superposition. Let us form a normalized linear combination $u = c_1u_1 + c_2u_2$, where the deterministic values y_1 and y_2 correspond to the states u_1, u_2. The result of measuring of y in this state u is a random quantity with possible values y_1 and y_2. The probabilities $P_1 = |c_1|^2$ and $P_2 = |c_2|^2$ correspond to these values.

6 Relation to Non–Commutative Probability

The traditional theory of dynamical systems under random perturbations assumes that probability characteristics do not depend on system dynamics. Our results relate to the processes with strong interaction of these components. These processes are analysed in the theories of random processes based on the non–commutative theory of probability (Ludwig, 1967); (Pool, 1968), which above all deal with the problems of foundations of quantum mechanics. According to these theories the principles of superposition and uncertainty follow from the non-traditional probability postulates. According to our results these principles follow from the infinitesimality of the dynamical system and the presence of the states in which the observable quantity is deterministic. It is possible, that these results are not in conflict with each other. But this question is the topic of a special analysis.

This paper develops an approach of (Krotov, 1991).

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