Some Transforms for Modelling of Computing Anticipatory Systems

Miomir S. Stanković, Radomir S. Stanković*
Dept. of Mathematics, Faculty of Occupational Safety
Čarnojevićeva 10, 18 000 Niš, Yugoslavia
Fax: 381 18 49-694, e-mail: mstan@znrfak.znrfak.ni.ac.yu
*Dept. of Computer Science, Faculty of Electronics
Beogradska 14, 18 000 Niš, Yugoslavia
Fax: 381 18 46-180, e-mail: stanko@503c1.elfak.ni.ac.yu

Abstract

In this paper, we introduce and investigate transforms mapping the space of continuous signals into the space of discrete signals. These transforms map the derivative of an arbitrary order into the forward, the backward, and the average difference. The introduced transforms permit to derive discrete modells for continuous time systems described by differential equations. Since a transform mapping the derivatives into the average differences is provided, modeling of continuous time systems by computing anticipatory systems is included. Application of introduced transforms is illustrated by deriving the discrete modells for harmonic oscillators.

Keywords: Anticipatory systems, Forward differences, Backward differences, Average differences, Harmonic oscillators.

1 Introduction

Mathematical modells of many dynamic systems are given in the form of differential equations. Numerical simulation on computer of such systems is only possible through a discrete model of these equations.

Such discretization assumes that functions are replaced by sequences. In this case, the derivative is replaced by a difference, that may be

1. Forward difference

$$\Delta f_n = f_{n+1} - f_n, \tag{1}$$

2. Backward difference

$$\nabla f_n = f_n - f_{n-1}, \tag{2}$$

3. Average difference

$$\delta f_n = \frac{1}{2}(f_{n+1} - f_{n-1}). \tag{3}$$

A slight generalization of these differences is achieved through the following operators

$$\begin{aligned}
\Delta_h f_n &= \frac{1}{h}(f_{n+h} - f_n), \\
\nabla_h f_n &= \frac{1}{h}(f_n - f_{n-h}), \\
\delta_h f_n &= \frac{1}{2h}(f_{n+h} - f_{n-h}),
\end{aligned}$$

with $h \in Q$, Q-the field of rational numbers.

If h is replaced by $-h$, the forward difference becomes the backward difference, and conversely, the backward difference becomes the forward difference, while the average difference is time reverse. Successive application of the forward difference to the backward difference, or the opposite, produces the second order difference

$$\Delta_h \nabla_h f_n = \nabla_h \Delta_h f_n = \delta^2_{h/2} f_n = \frac{1}{2h^2}(f_{n+h} - 2f_n + f_{n-h}),$$

which is also reverse to the replacement of h by $-h$, and vice versa.

Numerical simulations on computer of systems described by differential equations is only possible from a discrete model of these equations. Classically, discrete models of dynamic systems are defined in a recursive way as

$$x_{n+1} = f(n, x_{n-1}, x_n, p),$$

where x_n is the state variable of the system, n is the time, and p is a parameter. Therefore, the future state of the variable x at the time $n + 1$ is a function f of this variable at the past and/or present time.

Discretization of differential equation systems causes sometimes numerical instabilities. It is shown by Dubois (1998a), that this instability can be controlled by defining incursive discretization, which is an extension of the recursion as defined by Dubois (1995), Dubois (1998b). An incursive system computes the future state of the variable x as a function f of this variable at the past and/or present time, but also at the future time. This defines a self-referential system which is an anticipatory system of itself.

By using that property, the average difference is expressed in terms of the forward and backward differences as

$$\delta_h f_n = \frac{1}{2}(\Delta_h - \nabla_h) f_n.$$

Dubois (1998) introduced a generalized difference $d_h f_n$ defined as the weighted sum of forward and backward derivatives,

$$d_h f_n \;=\; (w\Delta_h + (1-w)\nabla_h) f_n = \frac{1}{h}(w f_{n+h} + (1-2w) f_n + (w-1) f_{n-h}), \quad (4)$$

where the weight w is defined in the interval $[0,1]$.

This definition includes the forward, the backward, and the average difference for $w=1$, $w=0$, and $w=1/2$, respectively.

In this paper, we define some transforms which map the space of continuous-time signals into the space of discrete-time signals. By using these transforms, the derivative of an arbitrary order is mapped into the forward, the backward, and the average difference, respectively. These transforms map the integral, which is the inverse operator for the derivative, into a sum that is the inverse operator of the corresponding difference. The continuous-time convolution is mapped into the discrete convolution, which permits the use of the introduced transforms in systems theory. We briefly investigate some of the basic properties of the introduced transforms and define their inverse transforms.

2 Transforms Mapping Derivatives into Forward Differences

Let $C^\infty(R)$ be the set of real functions having continuous derivatives of all orders. Denote by $S_f \subset C^\infty(R)$ the set of functions such that $f \in S_f$ iff there exist constants $\alpha, M > 0$, such that $\left|f^{(k)}(0)\right| < \alpha M^k$ for each $k \in N_0$. Denote by $c^\infty(R)$ the space of real sequences (f_n), and by S_n the set of real sequences (f_n), such that $(f_n) \in S_n$ iff there exist constants $\beta, N > 0$ such that $\left|\Delta^k f_0\right| < \beta N^k$ for each $k \in N_0$, where Δ is the forward difference (1).

Definition 1 *The transform $V_F : S_f \to S_n$ of a function $f(t)$ is the sequence (f_n) determined by*

$$V_F f = (f_n) = \left(\frac{d^n}{dx^t} e^x f(t) \,|_{t=0}\right), \quad n \in N_0. \quad (5)$$

Thus, $f \in S_f$ is mapped into the sequence whose elements are determined as

$$f_n = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(0).$$

Table 1 shows some functions and their V_F-transforms. Basic properties of V_F-transform are given in the following theorem.

Theorem 1 *If $V_F f = (f_n)$, then*

1. $V_F(f^{(k)}(t)) = (\Delta^k f_n)$,
2. $V_F\left(\int_0^t f(x)dx\right) = (\sum_{k=1}^{n} f_i)$,
3. $V_F(x^k f^{(k)}(t)) = (n^{(k)}\Delta^k f_{n-k})$,

where $n^{(k)} = n(n-1)\cdots(n-k+1)$.

The V_F^{-1}-transform inverse for the V_F-transform is defined as follows.

Definition 2 *The transform $V_F^{-1} : S_n \to S_f$ a sequence (f_n) is the function $f(t)$*

$$V_F^{-1}(f_n) = f(t) = \sum_{k=0}^{n} \frac{\Delta^k v_0}{k!} t^k.$$

Thus,

$$f(t) = e^{-t}\sum_{k=0}^{+\infty} \frac{v_k}{k!} t^k.$$

For two sequences $(f_n), (g_n) \in S_n$, the convolution is defined by

$$f_n * g_n = \sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^{m-1}\begin{pmatrix} m \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix} f_j g_{i-j}. \tag{6}$$

For thus defined convolution, the following theorem can be proved in a way similar to that used in (Stanković, 1982).

Theorem 2 *Let $(f_n), (g_n) \in S_v$. Then, for an operator $\star$,*

$$V_F^{-1}(f_n \star g_n) = V_F^{-1}(f_n) V_F^{-1}(g_n) \tag{7}$$

iff $\star$ is defined by (6).

Let $V_F f = (f_n)$, and $V_F g = (g_n)$. Then, by (5) and (7),

$$V_F(f(t)g(t)) = (f_n * g_n).$$

In a similar way, we define a transform mapping the derivative into the difference Δ_h. This transform maps $f \in S_f$ into the sequence whose elements are determined as

$$f_n = \sum_{k=0}^{\infty} h^k \begin{pmatrix} n/h \\ k \end{pmatrix} f^{(k)}(0).$$

Table 1: V_F-transform

$f(t)$	$f_n = V_F f(t)$
t^m	$n^{(m)} = n(n-1)(n-2)\cdots(n-m+1)$
e^{at}	$(1+a)^n$
$\sin at$	$(1+a^2)^{n/2}\sin(n\mathrm{arctg}(a))$
$\cos at$	$(1+a^2)^{n/2}\cos(n\mathrm{arctg}(a))$
$\frac{e^{-t}}{1-t}$	$n!$

3 Transforms Mapping Derivatives into Backward Differences

Let B_f be the set of real-valued functions $f(t)$ defined on the interval $(0,\infty)$ for which

$$\int_0^{+\infty} e^{-t}|f(t)|^2 dt < +\infty,$$

and B_n the set of sequences with the property that

$$\sum_{k=0}^{+\infty} |\Delta^k 2^n f_n \,|_{n=0}\,| < +\infty.$$

Definition 3 *The transform $V_B : B_f \to B_n$ of a function $f(t)$ is the sequence (f_n) defined by*

$$V_B f = (f_n) = \left(\frac{1}{n!}\int_0^{+\infty} e^{-t} t^n f(t) dt\right), \qquad n \in N_0.$$

If f is restricted to the set S_f, then

$$V_B f = (f_n) = \sum_{k=0}^{+\infty} \binom{n+k}{k} f^{(k)}(0).$$

The sequence (f_n) assigned to $f \in B_f$ by the V_B-transform is related to the coefficients a_n of the series representation of $f(t)$ in terms of the orthogonal set of Laguerre functions $l_n(t)$ where

$$l_n(t) = \sqrt{2}(-1)^n e^{-t} L_n(2t)\},$$

and $L_n(t)$ is the Laguerre polynomial

$$L_n(t) = \sum_{k=0}^{+\infty} (-1)^k \binom{n}{k} \frac{t^k}{k!}.$$

The sequence

$$a_n = \int_0^{+\infty} l_n(t) f(t) dt,$$

is called the Laguerre transform of $f(t)$, (Keilson and Nunn, 1997).

It was shown in (Stanković, 1982), that elements of these sequences are related as follows

$$\begin{aligned} a_n &= \sqrt{2}(-1)^n \sum_{k=0}^{n} (-2)^k \binom{n}{k} v_k, \\ v_n &= \frac{\sqrt{2}}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} a_k. \end{aligned}$$

Table 2 shows some functions and their V_B-transforms. Basic properties of the V_B-transform are given in the following theorem.

Theorem 3 *If $V_B f = (f_n), V_B g = (g_n)$, then*

1. $V_B\left(\frac{d^k f(t)}{dt^k}\right) = \left(\nabla^k f_n - (-1)^k \sum_{i=0}^{k} \binom{k-i}{i} f^{(i)}(0)\right),$

2. $V_B\left(\int_0^t f(x)dx\right) = \left(\sum_{i=0}^{n} f_i\right),$

3. $V_B\left(\int_0^t f(\tau)g(t-\tau)d\tau\right) = \left(\sum_{i=0}^{n} f_i g_{n-i}\right).$

A transform inverse for the V_B-transform is defined as by using its relation to the Laguerre functions.

Definition 4 *The transform $V_B^{-1} : B_n \to B_f$ of a sequence (f_n) is the function $f(t)$ determined by*

$$V_B^{-1}(f_n) = f(t) = \sqrt{2} \sum_{k=0}^{+\infty} \sum_{i=0}^{k} \binom{k}{i} 2^i \Delta^i f_0 l_k(t).$$

In a similar way, we define a transform which maps the derivative into the difference ∇_h. This transform maps f into the sequence whose elements are determined by

$$f_n = \frac{1}{\Gamma(\frac{n}{h}+1)} \int_0^{\infty} e^{-t} t^{\frac{n}{h}} f(t) dt.$$

Table 2: V_B-transform

$f(t)$	$f_n = V_B f(t)$
t^m	$n(m) = n(n+1)\cdots(n+m-1)$
e^{at}	$\frac{1}{(1-a)^{n+1}}$
$\sin at$	$(1+a^2)^{-(n+1)/2}\sin((n+1))\mathrm{arctg}(a)$
$\cos at$	$(1+a^2)^{-(n+1)/2}\cos((n+1))\mathrm{arctg}(a)$
$\ln t$	$\Psi(n+1)$, $\Psi(n+1) = \frac{\Gamma'(n+1)}{\Gamma(n+1)}$

4 Transforms Mapping Derivatives into Average Differences

Let $f \in S_f$. Denote by D_n the set of real sequences (f_n) such that $(f_n) \in D_n$ iff there exist constants $\beta, N > 0$ such that $\left|\delta^k f_0\right| < \beta N^k$ for each $k \in N_0$, where δ is the average difference (3).

Definition 5 *The transform $V_A : S_f \to D_n$ of a function $f(t)$ is the sequence (f_n) determined by*

$$V_A f = (f_n) = \left(\sum_{k=0}^{[n/2]} \binom{n-k}{k} \frac{d^{n-2k}}{dx^{n-2k}} f(x) \mid_{x=0} \right),$$

where $[q]$ denotes the greatest integer of q. Thus, f is mapped into the sequence whose elements are determined by

$$f_n = \sum_{k=0}^{[n/2]} \binom{n-k}{k} f^{(n-2k)}(0).$$

Table 3 shows some functions and their V_A-transforms. Basic properties of V_A transform are given in the following theorem.

Theorem 4 *If $V_A f = (f_n)$, then*

1. $V_A(f^k(t)) = (\delta^k f_n)$,
2. $V_A(\int_0^t f(x)dx) = (\sum_{i=0}^{\infty} f_{n-2k-1})$.

A transform inverse for the V_A-transform is defined as follows.

Definition 6 *The transform $V_A^{-1} : D_n \to S_f$ of a sequence (f_n) is the function $f(t)$ determined by*

$$V_A^{-1}(f_n) = f(t) = \sum_{k=0}^{n} \frac{\delta^k f_0}{k!} t^k.$$

Table 3: V_A-transform

$f(t)$	$f_n = V_B f(t)$
t^m	$n \prod_{k=0}^{m-1}(n + \frac{m}{2} - k)$
e^{at}	$F_n(a)$
$\sin at$	$\frac{2}{\sqrt{4-a^2}} \sin(n \mathrm{arctg} \frac{a}{\sqrt{4-a^2}})$
$\cos at$	$\frac{2}{\sqrt{4-a^2}} \cos(n \mathrm{arctg} \frac{a}{\sqrt{4-a^2}})$

In Table 3, $F_n(a)$ is the Fibonacci polynomial defined by the recursive relation, (Ricci, 1995)

$$F_{n+1}(a) = aF_n(a) + F_{n-1}(a),$$

with $F_{-1}(a) = F_1(a) = 1$.

Derivation of a transform mapping the derivative into the average difference $\delta_h f_n$ requires the use of fractional derivatives. In that order, it is necessary to develop first many related methods used in formulation of the V_A for δf_n, which is quite beyond the scope of this paper.

5 Discretization of Harmonic Oscillator

Differential equation of the linear harmonic oscillator

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0, \tag{8}$$

can be written as

$$\frac{dx(t)}{dt} = v(t), \tag{9}$$

$$\frac{dv(t)}{dt} = -\omega^2 x(t), \tag{10}$$

in defining by $x(t)$ the position and $v(t)$ the velocity of a particle in a linear harmonic potential, where t is the time and ω the position. The analytical solution of this system is

$$x(t) = x(0)\sin(\omega t + \phi), \tag{11}$$

$$v(t) = \omega x(0)\cos(\omega t + \phi), \tag{12}$$

which is given by oscillations of the period $T = 2\pi/\omega$, the amplitude of which being determined by the initial position and velocity $x(0)$ and $v(0)$ at the time $t = 0$. This system shows an orbital stability.

The V_F-transform maps the differential equations (8) into the following system of difference equations in terms of the forward difference

$$\Delta^2 x_n + \omega^2 x - n = 0, \tag{13}$$

which can be written as

$$\begin{aligned} x_{n+1} &= x_n + v_n, & (14)\\ v_{n+1} &= v_n - \omega^2 x_n. & (15) \end{aligned}$$

From Table 1, application of V_F-transform to the solutions (11) and (12) of the linear harmonic oscillator (8), produces solutions of (13)

$$\begin{aligned} x_n &= x_0(1+\omega^2)\sin(nasctg(\omega+\phi)),\\ v_n &= \omega x_0(1+\omega^2)^{n/2}\cos(n\mathrm{arctg}(\omega+\phi)), \end{aligned}$$

where x_0 is the initial condition.

This is a recursive discretization of the linear harmonic oscillator. With such discretization, the harmonic oscillator shows no more the orbital stability. The derived recursive system is unstable in presence of oscillations with growing amplitudes.

Similar, V_B-transforms maps the differential equation (8) into the difference equation in terms of the backward difference

$$\nabla^2 x_n + \omega^2 x_n = 0, \tag{16}$$

which can be written as

$$\begin{aligned} x_n &= x_{n-1} + v_n, & (17)\\ v_n &= v_{n-1} - \omega^2 x_n, & (18) \end{aligned}$$

i.e.,

$$\begin{aligned} x_{n+1} &= x_n + v_{n+1}, & (19)\\ v_{n+1} &= v_n - \omega x_{n+1}. & (20) \end{aligned}$$

From Table 2, application of V_B-transform to (11) and (12) produces solutions of the system (19) and (20)

$$\begin{aligned} x_n &= x_0(1+\omega^2)^{-(n+1)/2}\sin((n+1)\mathrm{arctg}(\omega+\phi)),\\ v_n &= \omega x_0(1+\omega^2)^{-(n+1)/2}\cos((n+1)\mathrm{arctg}(\omega+\phi)). \end{aligned}$$

This is a recursive discretization of the harmonic oscillator with the knowledge of the previous elements of the sequence in calculation of the n-th element of the sequence. In this discretization, the value of the proposition x_{n+1} is propagated to the equation of the velocity (see (20)). As is noted in Dubois (1998a), this is called an incursive system, since the value of the velocity at the future time step v_{n+1} depends on the future value of the position x_{n+1}. Thus, this is an inclusive recursion denoted as the incursion (Dubois, 1998b). Such model of the harmonic oscillator expresses the orbital stability.

The V_A-transform maps the differential equation (8) into the difference equation in terms of the average difference

$$\delta^2 x_n + \omega^2 x_n = 0, \tag{21}$$

which can be written as

$$\begin{aligned} x_{n+1} &= x_n + v_{n+1}, \quad &(22)\\ v_{n+1} &= v_n - \omega^2 x_n. \quad &(23) \end{aligned}$$

From Table 3, application of V_A-transform to (11) and (12) produces solutions of (21)

$$\begin{aligned} x_n &= \frac{2x_0}{\sqrt{4-\omega^2}} \sin\left(n \operatorname{arctg} \frac{\omega}{\sqrt{4-\omega^2}} + \phi\right), \\ v_n &= \frac{2\omega x_0}{\sqrt{4-\omega^2}} \cos\left(n \operatorname{arctg} \frac{\omega}{\sqrt{4-\omega^2}} + \phi\right). \end{aligned}$$

This is an incursive discretization of the linear harmonic oscillator. This model expresses an orbital stability for $\omega \in (-2, 2)$.

6 Conclusion

Transform discussed in this paper, map the derivative of an arbitrary order into the forward, the backward, and the average differences of the same order. That permits to derive discrete modells for continuous time systems in terms of the corresponding differences which enables simulation on computers.

Further investigations should be devoted to the application of fractional derivatives in definition of transforms for derivation of discrete models for continuous time systems. That should permit definition of differences with fractional shift as well as a study of the influence of the fractional shift to the stability of discrete modells. These results may be useful in analysis of systems with delay.

References

Stanković, M.S., (1982) "On a convolution product for the transform which maps derivatives into differences", *Fibonacci Quarterly*, 20, 334-343.

Stanković, M.S., Stanković, R.S., (1978) "Generalized Poisson transform", *Publ. Elect. Eng. Fac. Univ. Belgrade*, No. 118 - No. 128, 63-72.

Dubois, D., (1995) "Total incursive control of linear, non-linear and chaotic systems", in: Lasker, G.E., (ed.), *Advances in Computer Cybernetics*, Vol. II, International Institute for Advanced Studies in Systems Research and Cybernetics, Univ. of Winsdor, Canada, 1995, 167-171.

Dubois, D., (1998a) "Incursive harmonic oscillator related to quantum mechanics", *Cybernetics and Systems'98*, Vol. 1, Vienna, 129-133.

Dubois, D., (1998b) "Introduction to computing anticipatory systems", http://www.ulg.ac.be, *Centre for Hyperincursion and Anticipation in Ordered Systems*, CHAOS asbl., Institute of Mathematics, University of Liege, Belgium.

Keilson, J., Nunn, W.R., (1997) "Laguerre transformation as a tool for the numerical solution of integral equations of convolution type", *Appl. Math. Comput.*, Vol. 5, 313-359.

Ricci, P.E., (1995) "Generalized Lucas polynomials and Fibonacci polynomials", *Riv. Mat. Univ. Parma*, Vol. 4, 137-146.