THE CLASS OF DOUBLE MS-ALGEBRAS SATISFYING THE COMPLEMENT PROPERTY

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Abstract

We construct a functor from the category of de Morgan algebras into the category of double MS-algebras which satisfy a certain condition, called the complement property and show that this functor has a left adjoint.

1. Introduction

An MS-algebra \( < A, \lor, \land, \cdot, 0, 1 > \) is an algebra of type \( < 2, 2, 1, 0, 0 > \) such that \( < A, \lor, \land, 0, 1 > \) is a bounded distributive lattice and \( \cdot \) is a unary operation satisfying \( x \leq x^0, \ (x \land y)^0 = x^0 \lor y^0, \ 1^0 = 0 \). These algebras belong to the class of Ockham algebras introduced by Berman [1]. A double MS-algebra is an algebra \( < A, 0, + > \) such that \( < A, 0 > \) and its dual \( < A, + > \) are MS-algebras and for every \( x \in L, \ x^0 = x^\omega, \ x^\omega = x^+ \). T. S. Blyth and J. C. Varlet in [3] pointed out that every de Morgan algebra \( L \) can be represented non-trivially as the skeleton of the double MS-algebra \( L^{\omega} = \{ (a, b) \in L \times L : a \leq b \} \). In this paper we consider the class of double MS-algebras satisfying the complement property which is related to the construction due to T. S. Blyth and J. C. Varlet and investigate the role of the complete-

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ment property in the theory of double MS-algebras. Our main result is that the complement property associates a double MS-algebra $K(L)$ with each de Morgan algebra $L$. More precisely, we show that $K$ is a functor from the category $M$ of de Morgan algebras and de Morgan algebra homomorphisms into the category $V$ of double MS-algebras which satisfy a certain condition called the complement property and double MS-algebra homomorphisms and that the functor $K$ has a left adjoint $L$. As a consequence of the construction we obtain an important property of the left adjoint $L$ of $K$: for each double MS-algebra $A$ satisfying the complement property, the congruence lattice of $A$ is isomorphic to the congruence lattice of the lattice $L(A)$.

The main tools we use in the proof of the results mentioned above are the duality between double MS-algebras and certain ordered topological spaces developed by T. S. Blyth and J. C. Varlet in [5]. We recall here the main results that we shall need.

Since double MS-algebras are bounded distributive lattices, they are dually equivalent to some suitable category of Priestley spaces (i.e., compact totally order disconnected spaces) and order-preserving continuous functions. In fact, a double MS-space is a Priestley space $X$ endowed with two continuous order-reversing maps $g_1, g_2 : X \to X$ satisfying the following conditions:

1. $g_1(x) \leq x$
2. $g_1(x) \geq x$
3. $g_1(g_2(x)) = g_2(x)$
4. $g_2(g_1(x)) = g_1(x)$.

If $<X, g_1, g_2>$ is a double MS-space, then we can define two unary operations $^o, ^+$ on $O(X)$, the lattice of clopen decreasing sets of $X$, by setting

$P = X \setminus g_2^{-1}(1)$, $P^* = X \setminus g_1^{-1}(1)$

for each $I \in O(X)$, and thereby obtain a double MS-algebra. Conversely, if $<A, ^o, ^+>$ is a double MS-algebra, then we can define two maps $g_1, g_2$ on the ordered set $X(A)$ of prime ideals of $A$ by setting

$g_1(p) = \{a \in A : a^o \in p\}$, $g_2(p) = \{a \in A : a^+ \in p\}$

for each $p \in X(A)$, and thereby obtain a double MS-space, which will be denoted by $S(A)$. These constructions give a dual equivalence.

A subset $Y$ of a double MS-space $<X, g_1, g_2>$ is said to be a $g_i$-invariant subset if it satisfies the condition

$x \in Y \Rightarrow g_i(x) \in Y$, $i \in I = \{1, 2\}$.

According to Uryuša in [8], if $A$ is a double MS-algebra, then the congruence lattice of the MS-algebra $<A, ^+>$ is dually isomorphic to the lattice of all closed $g_1$-invariant subsets of the MS-space $<X, g_1>$. Dually, the congruence lattice of the MS-algebra $<A, ^+>$ is dually isomorphic to the lattice of all closed $g_2$-invariant subsets of the dual MS-space $<X, g_2>$. Therefore, the congruence lattice of the double MS-algebra $A$ is dually isomorphic to the lattice of all closed $g_i$-invariant subsets of the double MS-space $<X, g_1, g_2>$. If $\theta(Y)$ is the congruence associated with the closed $g_i$-invariant subset $Y$, then

$b = c(\theta(Y))$ iff $B \cap Y = C \cap Y$.
where B, C are the clopen decreasing subsets that represent b, c.

Clearly, \( g_1(X) \) is a closed \( g_1 \)-invariant subset of \( X \) and \( g_1(X) = g_1^2(X) \). \( g_2(X) \) is a closed \( g_2 \)-invariant subset of \( X \) and \( g_2(X) = g_2^2(X) \).

2. The complement property

**Definition 2.1.** A double MS-space \( <X, g_1, g_2> \) is said to satisfy the **complement property** if the equations \( g_1(X) \cup g_2(X) = X \), \( g_1(X) \cap g_2(X) = \emptyset \) hold. In other words, \( X \) is the disjoint union of \( g_1(X) \) and \( g_2(X) \). A double MS-algebra \( A \) is said to satisfy the complement property if the double MS-space \( S(A) \) satisfies it.

In what follows the complement property will be denoted by (CP).

**Example 2.1.** For every Boolean algebra \( <B, \lor, \land, ', 0, 1> \), let \( B^{2(2)} = \{ (a, b) \in L \times L : a \leq b \} \), \( B^{2(2)} \) is a double Stone algebra which satisfies (CP), where the pseudocomplement of \((a, b)\) is \((b', b')\), the dual pseudocomplement of \((a, b)\) is \((a', a')\).

**Theorem 2.1.** Let \( A \) be a double MS-algebra and \( <X, g_1, g_2> = S(A) \). Then the following conditions are equivalent:

(i) \( A \) satisfies (CP).

(ii) Given \( b, c \in A \) such that \( b = b^0 \), \( c = c^0 \) and \( b \leq c \), then there exists a unique element \( d \in A \) such that \( d^+ = b \), \( d^0 = c \).

**Proof.** We are going to show that the following conditions (1) and (3) are equivalent, and that so are conditions (2) and (4).

(1) \( g_1(X) \cup g_2(X) = X \)
(2) \( g_1(X) \cap g_2(X) = \emptyset \)

(3) Given \( b, c \in A \) such that \( b = b^0 \), \( b^+ = c^+ \), then \( b = c \).

(4) Given \( b, c \in A \) such that \( b = b^0 \), \( c = c^0 \), \( b \leq c \), then there exists an element \( d \in A \) such that \( d^+ = b \), \( d^0 = c \).

The following B, C, D represent \( b, c, d \) respectively.

(1) \iff (3). Since \( b^+ = c^+ \), \( b^0 = c^0 \) if \( b^+ = c^+ \), \( b^0 = c^0 \).

If \( X \setminus g_1^2(B) = X \setminus g_1^2(C) \), \( X \setminus g_2^1(B) = X \setminus g_2^1(C) \)

If \( g_1^2(B) = g_1^2(C) \), \( g_1^1(B) = g_1^1(C) \)

If \( g_2^2(B \setminus C) \cup (C \setminus B) = \emptyset \), \( g_2^1(B \setminus C) \cup (C \setminus B) = \emptyset \)

If \( (B \setminus C) \cup (C \setminus B) \subseteq X \setminus g_1^2(X) \), \( (B \setminus C) \cup (C \setminus B) \subseteq X \setminus g_2^1(X) \)

If \( (B \setminus C) \cup (C \setminus B) \subseteq X \setminus (g_1(X) \cup g_2(X)) \)

If \( B \cap (g_1(X)) = C \cap (g_1(X) \cup g_2(X)) \)

If \( b = c \) (\( \emptyset \).

If (3) holds, that is, \( c = b \), then \( \emptyset \cup g_1(X) \cup g_2(X) = \emptyset \) (i.e., the zero congruence), \( g_1(X) \cup g_2(X) = X \). If (1) holds, that is, \( g_1(X) \cup g_2(X) = X \), then \( \emptyset \cup g_1(X) \cup g_2(X) = \emptyset \) (i.e., the zero congruence), \( g_1(X) \cup g_2(X) = X \).

(2) \iff (4). If (2) holds and \( b = b^0 \), \( c = c^0 \), \( b \leq c \), then \( B = X \setminus g_1^2(X \setminus g_1^2(B)) \), \( C = X \setminus g_1^2(X \setminus g_1^2(C)) \), \( B \subseteq C \), that is, \( B = g_1^2(B) \), \( C = g_1^2(C) \). Setting
$D = (C \cap g_1(X)) \cup B$, therefore we have that $D$ is a clopen decreasing set of $S(A)$, $g_1(X) \cap g_1(X) = \emptyset$ implies that $D \cap B \subseteq C \cap g_1(X) \subseteq X \setminus g_1(X)$, $C \setminus D = C \setminus (B \cup g_1(X)) \subseteq C \setminus g_1(X) \subseteq X \setminus g_1(X)$, hence we obtain that $X \setminus g_1(D) = X \setminus g_1(B)$, $X \setminus g_1(C) = X \setminus g_1(D)$, that is, $d^* = b^*$, $d^* = e^*$, $d^{++} = b^{++} = b$, $d^{oo} = b = c$. Therefore (4) holds.

If (4) holds and there exists an element $d \in A$ such that $d^{++} = b = b^0$, $d^{oo} = c = c^0$, that is, $d^* = b^*$, $d^* = e^*$, then $X \setminus g_1(B) = X \setminus g_1(C)$, hence, we have that $B \subseteq D \subseteq C$ from the fact that $b = d^{++} \leq d \leq d^{oo} = c$, and so $B \subseteq X \setminus g_1(X)$, $C \setminus D \subseteq X \setminus g_1(X)$, $C \setminus B \subseteq X \setminus g_1(X) \cap g_1(X)$, $\emptyset = b = 0$, $c = 1$, then $B = \emptyset$, $C = X$, it follows that $g_1(X) \cap g_1(X) = \emptyset$.

**Theorem 2.2.** Let $<L, \lor, \land, \neg, 0, 1>$ be a de Morgan algebra and let $K(L) = \{(a, b) \in L \times L \mid a \leq b\}$. For every $(a, b) \in K(L)$, define $(a, b)^0 = (b', b')$ and $(a, b)^* = (a', a')$. Then $<K(L), ^0, ^*>$ is a double MS-algebra satisfying (CP).

**Proof.** By Theorem 2.3 in [3], $<K(L), ^0, ^*>$ is a double MS-algebra.

Let $(a, b), (c, d) \in K(L)$ such that $(a, b)^0 = (a, b)^0$, $(c, d)^0 = (c, d)^0$ and $(a, b) \leq (c, d)$. Then $a = b$, $c = d$ and $a \leq c$, hence $(a, c) \in K(L)$, $(a, c)^* = (a, c)$, $(a, c)^0 = (a, c)^0$. Suppose that there exists some $(a_1, c_1) \in K(L)$ such that $(a_1, c_1)^* = (a_1, c_1)^* = (a_1, c_1)^0 = (a_1, c_1)^0 = (a, c)^0$. This means that $a = a_1$, $c = c_1$, i.e., $(a, c) = (a_1, c_1)$. According to Theorem 2.1, $K(L)$ satisfies (CP).

**Lemma 2.1.** Let $L_1, L_2$ be de Morgan algebras and let $h : L_1 \rightarrow L_2$ be a $0$-$1$-preserving de Morgan algebra homomorphism. For each $(x, y)$ in $K(L_2)$ define $K(h)((x, y)) = (h(x), h(y))$. Then $K(h)$ is a double MS-algebra homomorphism from $K(L_1)$ into $K(L_2)$.

According to Theorem 2.2 and Lemma 2.1, we obtain a functor $K$ from the category of de Morgan algebras and de Morgan algebra homomorphisms into the category of double MS-algebras and MS-algebra homomorphisms.

**Lemma 2.2.** Let $A$ be a double MS-algebra and let $L(A) = \{x \in A \mid x = x^{oo}\}$. Then $L(A)$ is a de Morgan algebra.

**Lemma 2.3.** Let $A_1$ be double MS-algebras and let $h$ be a double MS-algebra homomorphism. define $L(h)(x) = h(x)$ for each $x \in L(A)$. Then $L(h)$ is a de Morgan algebra homomorphism from $L(A)$ into $L(A_1)$.

**Proof.** It suffices to show that the definition of $L(h)$ is reasonable. For $x \in L(A)$, $x \in A$ and $x = x^{oo}$, We have $h(x) = h(x^{oo}) = h(x)^{oo}$, thus $h(x) \in L(A_1)$.
MS-algebras and double MS-algebra homomorphisms into the category of de Morgan algebras and de Morgan algebra homomorphisms.

The following will be devoted to prove that $L$ is a left adjoint of $K$.

**Theorem 2.3.** Let $L$ be a de Morgan algebra and let $L(K(L)) = \{ x \in K(L) : x = x^0 \}$. Then $L(K(L)) \leq L$.

**Proof.** It is plain that $L(K(L))$ is a de Morgan algebra, whose elements have the form $(a, a)$. $a \in L$. Define $P_L : (a, a) \rightarrow a$. It is easy to show that $P_L$ is an isomorphism from $L(K(L))$ into $L$.

**Theorem 2.4.** Let $A$ be a double MS-algebra satisfying (CP). Define $J_A(a) = (a^{++}, a^0)$ for each $a \in A$. Then $J_A$ is a double MS-algebra isomorphism from $A$ into $K(L(A))$.

**Proof.** Since $(a^{++})^0 = a^{++}$ and $(a^0)^0 = a^0$, we deduce that $a^{++}, a^0 \in L(A)$. By observing that $a^{++} \leq a^0$, we have $(a^{++}, a^0) \in K(L(A))$, which implies that the definition of $J_A$ is reasonable. Since

$J_A(a \lor b) = ((a \lor b)^{++}, (a \lor b)^0) = (a^{++} \lor b^{++}, a^0 \lor b^0) = (a^{++}, a^0) \lor (b^{++}, b^0) = J_A(a) \lor J_A(b),$

$J_A(a \land b) = ((a \land b)^{++}, (a \land b)^0) = (a^{++} \land b^{++}, a^0 \land b^0) = (a^{++}, a^0) \land (b^{++}, b^0) = J_A(a) \land J_A(b),$

$J_A$ is a lattice homomorphism.

Moreover, $J_A(a^0) = (a^{++}, a^{00}) = (a^0, a^0)$, $(J_A(a^0))^0 = (a^{++}, a^0)^0 = (a^{00}, a^{00}) = (a^0, a^0)$, hence $J_A(a^0) = (J_A(a))^0$. Similarly, $J_A(a^{++}) = (J_A(a))^{++}$. So we obtain that $J_A$ is a double MS-algebra homomorphism. To see that $J_A$ is also an isomorphism, suppose $(a^{++}, a^0) = (b^{++}, b^0)$. By Theorem 2.1, we have $a = b$. This means that $J_A$ is one-one. Since, for any $(x, y) \in K(L(A))$, that is, $x, y \in L(A)$ and $x \leq y$, by Theorem 2.1, there exists some $c \in A$ such that $c^{++} = x$, $c^0 = y$, and hence, $J_A(c) = (x, y)$. This means that $J_A$ is an onto mapping.

According to Theorems 2.3 and 2.4, it is easy to check that the mappings $J_A$ and $P_L$ define natural transformations $J : 1_V \rightarrow KL$ and $P : KL \rightarrow 1_M$, where $1_V$ and $1_M$ are the identity functors in the categories $V$ and $M$ respectively. More precisely, we have:

**Theorem 2.5.** $<L, K, J, P>$ is an adjunction, with unit $J$ and counit $P$.

**Proof.** Since we have already noted that $J : 1_V \rightarrow KL$ and $P : KL \rightarrow 1_M$ are natural transformations, according to a result in [7, ch. iv, Theorem 2(1)], to complete the proof, we have to show that the following two conditions hold, where $1_X$ denotes the identity for the object $X$:

1. For each de Morgan algebra $L$, $K(P_{KL}) J_{KL} = 1_{KL}$, and

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(1) For each de Morgan algebra $\mathcal{L}, \mathcal{K}, \mathcal{L}(\mathcal{K}) \mathcal{K}(\mathcal{L}) = 1_{\mathcal{K}(\mathcal{L})}$, and
(2) For each double MS-algebra $A$ satisfying (CP), $\mathcal{L}(\mathcal{K})(\mathcal{L}(\mathcal{K})) = 1_{\mathcal{L}(\mathcal{K})}$.

The prove (1), let $a \leq b \in \mathcal{L}$, $\alpha \leq b$. Since
$$J_{\mathcal{K}(\mathcal{L})}(a, b) = ((a, b)^{\alpha}, (a, b)^{\alpha^\omega}) = ((a, a), (b, b)),$$
we have
$$\mathcal{K}(\mathcal{L}(\mathcal{K}))(\mathcal{K}(\mathcal{L}))(a, b) = (\mathcal{L}(\mathcal{K})(a, a), \mathcal{L}(\mathcal{K})(b, b)) = (a, b).$$

To prove (2), let $a \in \mathcal{L}(\mathcal{K})$, that is, $a \in \mathcal{L}$, $a = a^\omega$, since
$$J_a(a) = (a^{\alpha^+}, a^{\alpha^\omega}) = (a, a) \in \mathcal{L}(\mathcal{K}(\mathcal{L})).$$
$$\mathcal{L}(\mathcal{K})((\mathcal{L}(\mathcal{K}))(a)) = \mathcal{L}(\mathcal{K})(a) = a.$$

**Theorem 2.6.** Let $(X, g_1, g_2)$ be a double MS-space satisfying (CP). Then the correspondence $\mathcal{T} \rightarrow \mathcal{T} \cup g_1^-(\mathcal{T})$ establishes an isomorphism from the lattice of all the closed $g_1^-$-invariant subsets of $g_1(X)$ onto the lattice of all the $g_1^-$-invariant subsets in $X$.

**Proof.** Since $X$ is a compact totally-ordered connected topological space and $g_1(X)$ is a closed set in $X$, hence, is compact and the continuous mapping $g_1|(X) \subseteq X$ is closed.

Let $\mathcal{T}$ be a closed $g_1^-$-invariant subset of $g_1(X)$. It is plain that $\mathcal{T} \cup g_2^-(\mathcal{T})$ is a closed set in $X$. To prove $\mathcal{T} \cup g_2^-(\mathcal{T})$ is a $g_1^-$-invariant subset in $X$, let $x \in \mathcal{T} \cup g_2^-(\mathcal{T})$. If $x \in \mathcal{T}$, then $g_1(x) \in g_1(X) = T_1$ if $x \in g_2^-(\mathcal{T})$, let $x = g_2^-(t)$, $t \in \mathcal{T}$, then $g_1(x) = g_1(g_2^-(t)) = g_1^+(t) \in T$. This implies that $\mathcal{T} \cup g_2^-(\mathcal{T})$ is $g_1^-$-invariant. Similarly, $\mathcal{T} \cup g_2^-(\mathcal{T})$ is $g_2^-$-invariant, hence, we obtain $\mathcal{T} \cup g_2^-(\mathcal{T})$ is a $g_1^-$-invariant subset in $X$.

For each $x \in g_2^-(\mathcal{T}) \cap g_1(X)$, i.e., there exist $t \in \mathcal{T}$, $x_0 \in X$ such that $x = g_2^-(t) = g_1(x_0)$, we have $g_1(x) = g_1(x_0) = g_1^+(t) \in T$, and so, $g_1(x) \in T$. Since $g_1^+(x) = g_1(g_1(x_0)) = g_1(x_0) = x$, $x \in \mathcal{T}$, therefore, $g_2^-(\mathcal{T}) \cap g_1(X) \subseteq \mathcal{T}$ and then, $(\mathcal{T} \cup g_2^-(\mathcal{T})) \cap g_1(X) = \mathcal{T}$. From this equality we obtain that $\mathcal{T} \cup g_2^-(\mathcal{T}) \subseteq \mathcal{T} \cup g_2^-(\mathcal{T})$, if $\mathcal{T} \subseteq \mathcal{S}$. Finally, to see that the mapping is onto, note that if $Y$ is a $g_1^-$-invariant subset in $X$, then $Y = (Y \cap g_1(X)) \cup (Y \cap X \setminus g_1(X))$. It is easy to show that the following two conditions hold,

(1) $Y \cap g_1(X)$ is a $g_1^-$-invariant subset of $g_1(X)$.
(2) $Y \cap X \setminus g_1(X) = g_2(Y \cap X \setminus g_1(X))$.

The prove (1), note that if $y \in Y \cap g_1(X)$, then $y \in Y$, $y \in g_1(X)$ and hence $g_1(y) \in Y$, $g_1(y) \in g_1(g_1(X)) \subseteq g_1(X)$, therefore, $g_1(y) \in Y \cap g_1(X)$.

To see (2), note that by (CP), $g_1(X) \cup g_1(X) \cap g_1(X) = \emptyset$, $\emptyset \subseteq g_1(X)$. If $t \in Y \cap X \setminus g_1(X)$, then there exists $x_0 \in X$ such that $t = g_1(x_0)$. Since $Y$ is $g_1^-$-invariant, $g_1(x) \in Y$, and so, $g_1(g_1(x_0)) = g_1(x_0) \subseteq Y$, $t = g_1(g_1(x_0)) \subseteq g_1(Y \cap g_1(X))$. Suppose alternatively that $t \in g_1(Y \cap g_1(X))$, then there exists $y \in Y$ such that $t = g_1(y)$, $y \in Y \cap g_1(X)$, since $Y$ is $g_1^-$-invariant, $t \in Y$, $t = g_1(y) \in g_1(g_1(X))$. Hence, $t \subseteq Y \cap g_1(g_1(X))$. $\square$

From Theorem 2.6 and topological duality for double MS-algebras, we have,

**Corollary 2.1.** The lattices $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{L}(\mathcal{A}))$ are isomorphic for each double MS-algebra $A$ satisfying (CP). $\square$

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Remark. Since the only subdirectly irreducible de Morgan algebras are the chains with two and three elements and the four-element de Morgan algebra with two fixed points, from the above Corollary we obtain at once that the only subdirectly irreducible double MS-algebras satisfying (CP) are the following:

\[ a = a^+ = b^+ = c^+ \]

\[ g = g^+ = b^+ = d^+ = e^+ \]

\[ a = a^+ = b^+ = c^+ \]

\[ e = e^+ = f^+ = g^+ \]

\[ 0 = 0^+ = c^+ \]

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