ON THE JAYNES-CUMMINGS HAMILTONIAN
SUPERSYMMETRIC CHARACTERISTICS

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Abstract
We study some degeneracies of the Jaynes-Cummings Hamiltonian eigenstates. More precisely, we underscore operators connecting the degenerated eigenstates or explaining their non-degeneracy. These operators actually are supercharges and the supersymmetry underlying the Jaynes-Cummings model is thus exhibited. We also consider two extensions of the Hamiltonian to show the unicity of their supercharges.

Key-words: Jaynes-Cummings Hamiltonian, degeneracy, supersymmetry.


1 Introduction

The Jaynes-Cummings Hamiltonian ($H_{JC}$) [1] is associated with a model describing the interaction between a spin-$\frac{1}{2}$ particle and a one-mode magnetic field having an oscillating component along one axis and a constant component along another axis [2]. This model, extensively used in quantum optics [3], is one of the simplest examples of quantum systems combining bosons and fermions, a typical feature of supersymmetry [4].

Numerous studies of this model have already been realized. For example, we know that, under some hypotheses, it may be seen as an extension of the supersymmetric harmonic oscillator system [2]. Moreover, the diagonalisation of $H_{JC}$ [2] allows to construct the creation and annihilation operators of $H_{JC}$, and then, to underscore

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the coherent states in the stationary or evolution contexts [2]. Another approach consists in the understanding of \( H_{JC} \) as an element of the \( u(1/1) \) superalgebra [5]. The coherent states of this superalgebra can then be obtained [5]. Two extensions of \( H_{JC} \) can also be considered [6]. We will come back on that point later.

The supersymmetric characteristics of \( H_{JC} \) have only been viewed through the two above-mentioned extensions [6]. We will show here that \( H_{JC} \) has also supersymmetric characteristics by its own.

The main purpose of this paper is to prove that the \( H_{JC} \)-eigenstates are degenerated only for one value of the energy and then to explain this degeneracy.

The contents of this paper are the following. Section 2 is devoted to the energy spectrum and the eigenstates of \( H_{JC} \). Section 3 is divided into two parts: in the first one, we prove the existence of only one supercharge when there is no degeneracy for the \( H_{JC} \)-eigenstates (3.1); then, we prove the existence of two operators connecting the degenerated eigenstates associated with a particular value of the energy (3.2). Finally, we present, in section 4, a few remarks about two extensions of \( H_{JC} \).

Our units are taken with the constant \( \hbar \) equal to unity.

2 Energy spectrum and eigenstates of \( H_{JC} \)

The Jaynes-Cummings model can be described by the Hamiltonian [1]

\[
H_{JC} = \omega(a^+a + \frac{1}{2}) + \frac{\omega_0}{2} \sigma_z + g(a^+ \sigma_- + a \sigma_+),
\]

(2.1)

where \( a^\dagger \) and \( a \) are respectively the creation and annihilation operators of the bosonic harmonic oscillator and where \( \sigma_\pm = \sigma_1 \pm \sigma_2, \sigma_3 \) refer to the Pauli matrices.

In order to find the energy spectrum and the eigenstates of \( H_{JC} \), we have to solve the equation

\[
H_{JC} | \psi \rangle = E | \psi \rangle
\]

(2.2)

in the basis of the vectors

\[
\begin{pmatrix}
0 \\
| n >
\end{pmatrix}
= | n, - > \quad \text{and} \quad \begin{pmatrix}
| n > \\
0
\end{pmatrix} = | n, + >.
\]

(2.3)

If we note \( \Delta \) the difference between the two angular frequencies \( \omega \) and \( \omega_0 \), we obtain results which can be summarized as follows:

(a) For all the values of \( g \), we have to distinguish two cases

(i) either \( E = \frac{\Delta}{2} \) and the corresponding eigenstate is

\[
| E_0 > = | 0, - >.
\]

(2.4)

(ii) or \( E = \omega k \pm \frac{\Delta}{2} r(k), k \in \mathbb{N}_0 \), and the corresponding eigenstates are

\[
| E_k > = \frac{1}{R(k)} (g\sqrt{E} \ | k - 1, + > + \frac{\Delta}{2} r(k) + 1 | k, - >).
\]

(2.5)
where
\[ r(k) = \left(1 + \frac{4g^2k}{\Delta^2}\right)^{1/2} \]
and
\[ R(k) = \left(\frac{\Delta^2}{2} r(k)(1 + r(k))\right)^{1/2}. \]

b) If there exists \( k \in \mathbb{N} \) such that \( \frac{2\Delta}{\omega} = \omega k + \frac{2}{\omega} r(k) \). \( \Delta \) has to be negative and \( g \) has to take the values
\[ g = \pm \sqrt{\omega(\omega k - \Delta)} \]

Then the corresponding eigenstates are
\[ |E_k^+\rangle = \sqrt{\frac{\omega k}{2\omega k - \Delta}} (|k - 1 + > + |\frac{\omega k - \Delta}{\omega k} > |k, - >). \]

3 Explanation of degeneracy

There is degeneracy of the \( H_{JC} \)-eigenstates when \( E = \frac{\Delta}{2} \) only. This fact can be explained through supersymmetry or more precisely through the existence of supercharges.

Let us assume \( \Delta = 0 \). A similar way of thinking in the case \( \Delta \neq 0 \) would lead us to the same conclusions.

First of all, let us search for the supercharges of \( H_{JC} \).

3.1 Supercharges of \( H_{JC} \)

Let us recall that supersymmetric quantum mechanics needs the positive nature of the energies. So we translate \([6]\) \( H_{JC} \) by adding a positive constant \( c \) to it. Thus, in order to find the supercharges of \( H_{JC} \), we have to solve the equation
\[ Q^2 = H_{JC} + c \]
whose solution is

\[ Q = \sqrt{\omega} \sigma_+ + \sqrt{\omega} \sigma_- + \frac{1}{2} \frac{g}{\sqrt{\omega}} I, \]  

(3.2)

fixing the constant \( c \), without loss of generality, as the value

\[ c = \frac{g^2}{4\omega}. \]  

(3.3)

Moreover, the operators \( \sigma_+, \sigma_- \) and \( I \) generating the Clifford algebra \( C_l \) \( C_l(2) \) (characterized here by its fundamental irreducible representation \), we deduce that \( Q = (3.2) \) is the only supercharge of \( H_{JC} + c \).

Furthermore, the effect of \( Q \) on the \( H_{JC} \)-eigenstates explain the non-degeneracy of these states in the general case. As the \( H_{JC} \)-eigenstates are also eigenstates with respect to \( Q \).

In order to understand the eigenstates degeneracy in the case \( E = c \), we have to find operators connecting these states. That is the purpose of the next section.

### 3.2 Existence of operators connecting the \( H_{JC} \) degenerated eigenstates

In the case \( k = 1 \) and \( g = \omega \), the two operators connecting the degenerated eigenstates for \( E = c \) are

\[ P = \begin{pmatrix} f(N) a^{1} & f(N) \\ -g(N) a^{2} & -g(N) a^{1} \end{pmatrix} \quad \text{and} \quad P^\dagger = \begin{pmatrix} a f(N) & -a^2 g(N) \\ f(N) & -a g(N) \end{pmatrix} \]  

(3.4)

where \( f \) and \( g \) are real functions of \( N \). These operators satisfy the typical relations of supersymmetric quantum mechanics [4] i.e.

\[ P^2 = P^\dagger = 0 \quad \{ P, P^\dagger \} = H_{JC} + c \]  

(3.5)

characterizing the \( \text{Lie superalgebra} \ \text{sqm}(2) \), but only on the space generated by \( | E_0 \rangle \) and \( | E_\bar{1} \rangle \) \( = 0 \) with the condition \( f(0) = g(1) \). On the whole Fockspace, the relations of \( \text{sqm}(2) \) are not ascertained.

A similar way of thinking for other values of \( k \) leads us to the same conclusion. Because the two operators connecting degenerated eigenstates only act on the above-mentioned space of these eigenstates, the unicity of \( Q = (3.2) \) is not in the balance again.

### 4 Two generalizations of \( H_{JC} \) in the case \( \Delta = 0 \)

The first one consists in superposing on \( H_{JC} \) a second Hamiltonian \( H_2 \) defined by this expression [6]

\[ H_2 = \omega (a^2 + \frac{1}{2} \sigma_3) + ig (a^2 \sigma_3 - a \sigma_1). \]  

(4.1)
It is unitarily equivalent to $H_{kC}$. The resulting Hamiltonian is thus [6]

$$H = \begin{pmatrix} H_{kC} & 0 \\ 0 & H_2 \end{pmatrix}. \tag{4.2}$$

One supercharge of $H + c$ is given by this expression

$$Q = \sqrt{\omega a} \xi_+ + \sqrt{\omega a^\dagger} \xi_- + \frac{g}{2\sqrt{\omega}} \eta \tag{4.3}$$

where

$$\xi_+ = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{4.4}$$

The odd parity of these operators and all their anticommutation relations lead us to consider five more operators generating with $\xi_+, \xi_-$ and $\eta$ the Lie superalgebra osp(2/2). As there exists only one representation of this superalgebra [8] with 4 by 4 matrices, we can conclude that $Q = (4.3)$ is the only supercharge of $H + c$ connecting the degenerated eigenstates.

The second extension of $H_{kC}$ consists in adding a positive constant $\Delta'$ to $H + c$, where $H = (4.2)$. Also here, there is only one supercharge for $H + c + \Delta'$ which is

$$Q^{\Delta'} = Q + \sqrt{\Delta'} R \tag{4.5}$$

where $Q = (4.3)$ and

$$R = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Indeed, another supercharge should have the form

$$Q^{\Delta'}_1 = Q + \sqrt{\Delta'} R' \tag{4.7}$$

and should satisfy the relations

$$\{Q^{\Delta'}_1, Q^{\Delta'} \} = 0 \tag{4.8}$$

and

$$Q^{\Delta'}_1^2 = H + c + \Delta' \tag{4.9}$$

In other words, the operator $R'$ should have the form

$$R' = \begin{pmatrix} d & 0 & id & 0 \\ 0 & d & 0 & l \\ -id & 0 & -d & 0 \\ 0 & l & 0 & -d \end{pmatrix} \tag{4.10}$$

with

$$l^2 + d^2 = 1. \tag{4.11}$$
Taking the expressions (4.6) and (4.10) for $R$ and $R'$, we have
\[
\{R, R'\} = 2I
\]
and then
\[
\{Q^\Delta, Q^\Delta'\} = 2(H + c + 2I) \neq 0
\]
That is in contradiction with (4.8) and allows us to conclude that $Q^\Delta = (4.5)$ is the only supercharge of $H + c + \Delta'$ connecting the degenerated eigenstates.

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