ON THE COHOMOLOGY OF THE LIE SUPERALGEBRA
OF CONTACT VECTOR FIELDS ON $S^{11}$

B. AGREBAOUI et N. BEN FRAJ

AMS MSC : 47G99 58A50 17B56

Key words : Pseudodifferentiel operator
            Super cercle
            Cohomology
            Contact vector fields

Abstract

We investigate the first cohomology space attached to the embedding of the
Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the supercircle $S^{11}$ in the Lie
Superalgebra of superpseudodifferential operators. Following the paper [11], we
show that this space is four-dimensional with only even cocycles and we calculate
explicitly four 1-cocycles representing non-trivial generating cohomology classes.
1 Introduction

The classifications of multi-parameter deformations of homomorphisms of Lie algebras and in particular representations have been studied in many recent papers [1, 2, 10, 11]. The first cohomology space classify the infinitesimal deformations, while the obstructions are living in the second cohomology space. The study of multi-parameter deformations of the standard embedding of the Lie algebra $\text{Vect}(S^1)$ of vector fields on the circle $S^1$ inside the Lie algebra $\mathfrak{pdo}$ of pseudodifferential operators on $S^1$ was carried out in [11]. In this paper we address ourselves to the computation of the first cohomology space of the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the supercircle $S^{1|1}$ with coefficients in the Lie superalgebra $S\mathfrak{pdo}$ of superpseudodifferential operators on $S^{1|1}$. It is a first step towards that classification for the natural embedding of $\mathcal{K}(1)$ inside $S\mathfrak{pdo}$. Namely, we first compute the first cohomology space of the $\mathcal{K}(1)$-module of tensor densities $\mathfrak{f}_1 = \{ F\alpha^\lambda, F \in C^\infty(S^{1|1}) \}$, where $\alpha = dx + \theta d\theta$ is the contact 1-form and the action of $\mathcal{K}(1)$ is given by Lie derivatives. The first cohomology space of $\mathcal{K}(1)$ with coefficients in the Poisson superalgebra $S\mathcal{p}$ of superpseudodifferential symbols of $S\mathfrak{pdo}$ will be a corollary of the later one, since, $S\mathcal{p}$ has a decomposition to a direct sum of modules of tensor densities. After that we compute the first cohomology space in the $\mathcal{K}(1)$-module $S\mathfrak{pdo}$, using the same method as in [11]. The main result of this paper can be stated as follows (Theorem (6.1)): The first cohomology space $H^1(\mathcal{K}(1), S\mathfrak{pdo})$ is four-dimensional and it is generated by the 1-cocycles (6.1): $\Theta_0, \Theta_1, \Theta_2$ and $\Theta_3$. In our approach to the proof of Theorem (6.1), we follow the lines by [11]. That is we apply successive differentials of the spectral sequences corresponding to the complex $C^*(\mathcal{K}(1), S\mathcal{p})$.

Acknowledgements It is a pleasure to thank Valentin Ovsienko who introduced us to the question of cohomology computations in Lie superalgebras of vector fields. We also thank Pierre Lecomte and Claude Roger for helpful discussions. The first author thanks also Pierre Lecomte for his fruitful invitation to visit his service Geothalg at University of Liège.

2 Superpseudodifferential operators on $S^{1|1}$

2.1 Lie superalgebra structure

We first recall the definition of the algebra of superpseudodifferential operators on the supercircle $S^{1|1}$(cf. [4, 9]).
The supercircle $S^{11}$ is the superextension of the circle $S^1$ with local coordinates $(x, \theta)$, where $x \in S^1$ and $\theta = 0$. A $C^\infty$ function on $S^{11}$ has the form $F = f(x) + 2g(x)\theta$ with $f, g \in C^\infty(S^1)$. The vector field $\eta = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}$ on $S^{11}$ sends $F$ to $\eta(F) = 2g(x) + f'(x)\theta$ so that $\eta^2 = \frac{1}{2}[\eta, \eta] = \frac{\partial}{\partial x}$. The usual Leibniz rule $\frac{\partial}{\partial x} \circ f = f'(x) + f(x)\frac{\partial}{\partial x}$ on $C^\infty(S^1)$, is replaced on $C^\infty(S^{11})$ by:

$$\eta \circ F = \eta(F) + \sigma(F)\eta$$

(2.1)

where the involution $\sigma$ is the grading automorphism on $C^\infty(S^{11})$, equal to 1 on the even part and to $-1$ on the odd part (in other words, $\eta$ is a superderivation).

The formula (2.1), generalises by induction on $m$ to the graded Leibniz formula

$$\eta^m \circ F = \sum_{k=0}^{\infty} \binom{m}{k}_s \eta^k(\sigma^{m-k}(F))\eta^{m-k}$$

(2.2)

for all integers $m \geq 0$, where the supersymmetric binomial coefficients $\binom{m}{k}_s$ are defined by:

$$\binom{m}{k}_s = \begin{cases} \binom{m}{k} & \text{if } k \text{ is even or } m \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

with $[x]$ is usual denoting the integral part of a real number $x$, and for $l \in \mathbb{Z}_{\geq 0}$, the binomial coefficient $\binom{l}{2} = x(x-1) \cdots (x-l+1)$. Let us introduce the superalgebra of superpseudodifferential operators $\mathcal{SPDO}$ on $S^{11}$ by:

$$\mathcal{SPDO} = \{ \sum_{k \in \mathbb{Z}_{\geq 0}} F_k \eta^{m-k}, \ w \in \mathbb{Z}, F_k \in C^\infty(S^{11}) \},$$

where the composition of superpseudodifferential operators is generated by the graded Leibniz formula (2.2):

$$F \eta^m \circ G \eta^n = \sum_{k=0}^{\infty} \binom{m}{k}_s F \eta^k(\sigma^{m-k}(G))\eta^{m+n-k}, \ m, n \in \mathbb{Z} \text{ and } F, G \in C^\infty(S^{11}).$$

(2.3)

As usual, the composition of operators induces a Lie superalgebra structure on $\mathcal{SPDO}$ with the super-commutator defined on homogeneous elements by:

$$[A, B] = A \circ B - (-1)^{p(A)p(B)} B A,$$

where we let $p$ denote the function of parity.

### 2.2 Symbols of superpseudodifferential operators on $S^{11}$

In this subsection, we will define the Poisson bracket of superpseudodifferential symbols. We first list some definitions and notations from [11]. Let $\mathcal{P}(S^1)$ be the ring of symbols of pseudodifferential operators on $S^1$

$$A(x, \xi) = \sum_{-\infty}^{n} a_i(x)\xi^i,$$
where \( a_i(t) \in C^\infty(S^1) \), and the variable \( \xi \) corresponds to \( \frac{\partial}{\partial \xi} \). The space \( \mathcal{P}(S^1) \) is a Poisson Lie algebra with the bracket given by

\[
\{A(x, \xi), B(x, \xi)\} = \frac{\partial}{\partial \xi} A(x, \xi) \frac{\partial}{\partial x} B(x, \xi) - \frac{\partial}{\partial x} A(x, \xi) \frac{\partial}{\partial \xi} B(x, \xi),
\]

where the multiplication is naturally defined.

Analogously, we introduce in the super-case, the super-commutative ring

\[
\mathcal{SP} = C^\infty(S^{1\|1}) \otimes (C[\xi, \xi^{-1}]) \otimes C[\xi, \xi^{-1}][\zeta]
\]

of symbols of superpseudodifferential operators on \( S^{1\|1} \)

\[
S(x, \xi, \zeta) = \sum_{n=0}^{\infty} F_n(x) \xi^n + \left( \sum_{n=0}^{\infty} G_n(x) \xi^n \right) \zeta,
\]

where \( F_n, G_n \in C^\infty(S^{1\|1}) \), \( \zeta = \bar{\theta} + \theta \xi \) corresponds to \( \eta \) and \( \bar{\theta} \) corresponds to \( \frac{\partial}{\partial \theta} \); with \( \bar{\theta}^2 = \zeta^2 = 0 \) and \( \zeta \cdot F \xi^m = \sigma(F) \xi^m \zeta \), \( F \in C^\infty(S^{1\|1}) \). Then, the multiplication of symbols is obvious.

We define the Poisson bracket on \( \mathcal{SP} \) by

\[
\{S, T\} = \frac{\partial}{\partial \xi}(S) \frac{\partial}{\partial x}(T) - \frac{\partial}{\partial x}(S) \frac{\partial}{\partial \xi}(T) - (-1)^{\text{deg}(S)} \left( \frac{\partial}{\partial \theta}(S) \frac{\partial}{\partial \bar{\theta}}(T) + \frac{\partial}{\partial \bar{\theta}}(S) \frac{\partial}{\partial \theta}(T) \right),
\]

where \( S, T \in \mathcal{SP} \) (cf. [7]).

3 The space of tensor densities on \( S^{1\|1} \)

Let us first recall the \( \text{Vect}(S^1) \)-module of tensor densities on \( S^1 \). Consider the one parameter action of \( \text{Vect}(S^1) \) on \( C^\infty(S^1) \) given by

\[
L^\lambda_{X(x)\eta}(f(x)) = X(x)f'(x) + \lambda X'(x)f(x),
\]

where \( f \in C^\infty(S^1) \) and \( \lambda \in \mathbb{R} \). Denote \( \mathcal{F}_\lambda \) the \( \text{Vect}(S^1) \)-module structure on \( C^\infty(S^1) \) given by (3.1). Note that the adjoint \( \text{Vect}(S^1) \)-module is isomorphic to \( \mathcal{F}_{-1} \). Geometrically, \( \mathcal{F}_\lambda \) is the space of tensor densities of degree \( \lambda \) on \( S^1 \), i.e. the set of all expressions: \( f(x)(dx)^\lambda \), where \( f \in C^\infty(S^1) \).

We have analogous definition of tensor densities in the super-case (see [9]). Let \( \alpha = dx + \theta dt \) be the contact 1-form on \( S^{1\|1} \) and let \( \mathcal{K}(1) \) be the Lie superalgebra of vector fields on \( S^{1\|1} \) preserving the 1-form \( \alpha \). The Lie Superalgebra \( \mathcal{K}(1) \) is also known as the algebra of Neveu-Schwarz without central charge or the Lie superalgebra of contact vector fields on \( S^{1\|1} \).

Every vector field in \( \mathcal{K}(1) \) has the form

\[
v_F = \frac{1}{2}(F + \sigma(F))\eta^2 + \eta(F)\eta, \quad F \in C^\infty(S^{1\|1}).
\]

We introduce a one parameter action of \( \mathcal{K}(1) \) on \( C^\infty(S^{1\|1}) \) by the rule:

\[
\text{368}
\]
\[ \mathcal{L}_\gamma^\lambda (G) = F \cdot \eta^2 (G) + \frac{(-1)^{p(F)\rho(G)+1}}{2} \eta(F) \cdot \eta(G) + \lambda \eta^2 (F) \cdot G, \]

where \( F, G \in C^\infty(S^{11}) \). We denote this \( \mathcal{K}(1) \)-module by \( \mathfrak{F}_\lambda \).

Geometrically, the space \( \mathfrak{F}_\lambda \) is no other then the space of all tensor densities on \( S^{11} \) of degree \( \lambda \):

\[ \phi = f(x, \theta) \alpha^\lambda, \quad f(x, \theta) \in C^\infty(S^{11}), \]

where the action (3.3) of \( \mathcal{K}(1) \) is the Lie derivative action on tensor densities.

**Remarks 3.1.**

1) The action (3.3) of \( \mathcal{K}(1) \) on \( \mathfrak{F}_\lambda \) is given explicitly by

\[ \mathcal{L}_\gamma^\lambda (G) = \mathcal{L}_\gamma^\lambda (g_0) + 2b g_1 + 2(\mathcal{L}_\gamma^\lambda + \frac{1}{2} h_1 + J_1 (h, g_0)) \theta \]

where \( F = a + 2b \), \( G = g_0 + 2g_1 \) and the operator \( J_1 \) is defined on \( \mathcal{F}_a \otimes \mathcal{F}_b \) by

\[ J_1 (f, g) = -\lambda f g' + \mu g f'. \]

As a \( \text{Vect}(S^1) \)-module (i.e. \( b = 0 \)) the space of tensor densities \( \mathfrak{F}_\lambda \) is isomorphic to \( \mathcal{F}_a \otimes \mathcal{F}_{a+\frac{1}{2}} \), which is the \( \mathbb{Z}_2 \)-grading of \( \mathfrak{F}_\lambda \). In particular, the Lie superalgebra \( \mathcal{K}(1) \) is isomorphic to \( \mathcal{F}_{-1} \otimes \mathcal{F}_{-\frac{1}{2}} \) as \( \text{Vect}(S^1) \)-module.

2) The adjoint \( \mathcal{K}(1) \)-module is isomorphic to the module \( \mathfrak{F}_{-1} \). This isomorphism induces a super Poisson bracket on \( C^\infty(S^{11}) \) given by:

\[ \{F, G\} = \mathcal{L}_\gamma^{-1} (G) = FG' - GF' + \frac{(-1)^{p(F)\rho(G)+1}}{2} \eta(F) \cdot \eta(G). \]

4 The structure of \( \mathcal{S}\mathcal{P} \) as a \( \mathcal{K}(1) \)-module

The natural embedding of \( \mathcal{K}(1) \) inside \( \mathcal{S}\mathcal{P}\mathcal{D}\mathcal{O} \) given by the expression (3.2) induces a \( \mathcal{K}(1) \)-module structure on \( \mathcal{S}\mathcal{P}\mathcal{D}\mathcal{O} \). Analogously, we have a \( \mathcal{K}(1) \)-module structure on \( \mathcal{S}\mathcal{P} \) given by the natural embedding of \( \mathcal{K}(1) \):

\[ \pi : v_p \mapsto \frac{1}{2} (F + \sigma(F)) \xi + \eta(F) \zeta. \]

The Poisson super-algebra \( \mathcal{S} \mathcal{P} \) is \( \mathbb{Z} \)-graded, where we give \( \sigma, \theta \) the degree zero and \( \xi, \zeta \) the degree one.

Then we have

\[ \mathcal{S} \mathcal{P} = \bigoplus_{n \in \mathbb{Z}} \mathcal{S} \mathcal{P}_n \]

where, \( \mathcal{S} \mathcal{P}_n = (\oplus_{n \geq 0}) \bigoplus_{n \geq 0} \mathcal{S} \mathcal{P}_n \) and \( \mathcal{S} \mathcal{P}_n = \{F \xi^{-n} + G \xi^{-n-1} \zeta, F, G \in C^\infty(S^{11})\} \) is the homogeneous subspace of degree \( -n \).

Each element of \( \mathcal{S}\mathcal{P}\mathcal{D}\mathcal{O} \) can be written as

\[ A = \sum_{k \in \mathbb{Z}} (F_k + G_k \eta^{-1}) \eta^{2k}, \]

369
where \( F_k, G_k \in C^\infty(S^{1|1}) \). We define the order of \( A \) by

\[
\text{ord}(A) = \sup\{k; F_k \neq 0 \text{ or } G_k \neq 0\}.
\]

This definition of order equips \( \mathcal{S} \mathcal{P} \mathcal{D} \mathcal{O} \) with a decreasing filtration as follows: let us set

\[
F_n = \{ A \in \mathcal{S}\mathcal{P}\mathcal{D}O, \text{ord}(A) \leq -n \}
\]

where \( n \in \mathbb{Z} \). So one has

\[
\ldots \subseteq F_{n+1} \subseteq F_n \subseteq \ldots \tag{4.3}
\]

This filtration is compatible with the multiplication and the super Poisson bracket, that is, for \( A \in F_n \) and \( B \in F_m \), one has \( A \circ B \in F_{n+m} \) and \( \{A, B\} \in F_{n+m-1} \), where we identify \( \mathcal{S} \mathcal{P} \) with \( \mathcal{S} \mathcal{P} \mathcal{D} \mathcal{O} \). This filtration makes \( \mathcal{S}\mathcal{P}\mathcal{D}O \) as an associative filtered superalgebra. Moreover, this filtration is compatible with the natural action of \( \mathcal{K}(1) \) on \( \mathcal{S}\mathcal{P}\mathcal{D}O \). Indeed, if \( v_F \in \mathcal{K}(1) \) and \( A \in F_n \), then

\[
v_F \cdot A = [v_F, A] \in F_n.
\]

The induced \( \mathcal{K}(1) \)-module on the quotient \( F_n/F_{n+1} \) is isomorphic to the \( \mathcal{K}(1) \)-module \( \mathcal{S} \mathcal{P}_n \). Therefore, the \( \mathcal{K}(1) \)-module on the associated graded space of the filtration (4.3), is isomorphic to the graded \( \mathcal{K}(1) \)-module \( \mathcal{S} \mathcal{P} \), that is

\[
\mathcal{S} \mathcal{P} \simeq \bigoplus_{n \in \mathbb{Z}} F_n/F_{n+1}.
\]

**Proposition 4.1.** As a \( \mathcal{K}(1) \)-module we have

\[
\mathcal{S} \mathcal{P} \simeq \bigoplus_{n \in \mathbb{Z}} (\mathfrak{g}_n \oplus \mathfrak{z}_{n+\frac{1}{2}}).
\]

**Proof.** The \( \mathcal{K}(1) \)-module \( \mathcal{S} \mathcal{P}_n \) of the grading (4.2) has the direct sum decomposition of the two \( \mathcal{K}(1) \)-modules, \( \mathcal{S} \mathcal{P}^1_n \) and \( \mathcal{S} \mathcal{P}^2_n \), defined by

\[
\mathcal{S} \mathcal{P}^1_n = \{(F + \sigma(F))\xi^{-n} + \eta(F)\xi^{-n-1}\zeta; \ F \in C^\infty(S^{1|1})\},
\]

and

\[
\mathcal{S} \mathcal{P}^2_n = \{F\xi^{-n-1}\zeta - 2\theta F\xi^{-n}; \ F \in C^\infty(S^{1|1})\}.
\]

The action of \( \mathcal{K}(1) \) on \( \mathcal{S} \mathcal{P}^1_n \) is induced by the embedding (4.1) as follows:

\[
v_F \cdot \left( \frac{1}{2} (G + \sigma(G))\xi^{-n} + \eta(G)\xi^{-n-1}\zeta \right) = \{\pi(v_F), (G + \sigma(G))\xi^{-n} + \eta(G)\xi^{-n-1}\zeta\} = (L^n(g)(F) + \sigma(L^n(g)(F)))\xi^{-n} + \eta(L^n(g)(F))\xi^{-n-1}\zeta.
\]

The natural map \( \phi_1 : \mathfrak{g}_n \rightarrow \mathcal{S} \mathcal{P}^1_n \) defined by

\[
\phi_1(F) = (F + \sigma(F))\xi^{-n} + \eta(F)\xi^{-n-1}\zeta;
\]

provides us with an isomorphism of \( \mathcal{K}(1) \)-modules.
The action of \( \mathcal{K}(1) \) on \( SP_n^2 \) is given by

\[
v_F \cdot (G \xi^{-n-1} \xi + 2G \xi^{-n}) = \{ \pi(v_F), G \xi^{-n-1} \xi - 2G \xi^{-n} \}
\]

\[
= C_{\mu_F}^{n+1}(G) \xi^{-n-1} \xi - 2G C_{\mu_F}^{n+1}(G) \xi^{-n}.
\]

The natural map \( \varphi_2 : S_n + 1 \to SP_n^2 \) defined by:

\[
\varphi_2(F) = F \xi^{-n-1} \xi - 2G F \xi^{-n},
\]

provides us with an isomorphism of \( \mathcal{K}(1) \)-modules.

5 The first cohomology space \( H^1(\mathcal{K}(1), SP) \)

In this section, we will compute the first cohomology space of \( \mathcal{K}(1) \) with coefficients in \( SP \). To do this, we first recall some fundamental concepts from cohomology theory ([6]).

Let \( g = g_0 \oplus g_1 \) be a Lie superalgebra acting on a super vector space \( V = V_0 \oplus V_1 \).

The space \( \text{Hom}(g, V) \) is \( \mathbb{Z}_2 \)-graded via

\[
\text{Hom}(g, V)_b = \oplus_{a \in \mathbb{Z}_2} \text{Hom}(g_a, V_{a+b}), \ b \in \mathbb{Z}_2.
\]

Let \( Z^1(g, V) = \{ c \in \text{Hom}(g, V) : c([g, h]) = g \cdot c(h) - (-1)^{\rho(h)} h \cdot c(g), \forall g, h \in g \} \) be the space of 1-cocycles for the Chevalley-Eilenberg differential. According to the \( \mathbb{Z}_2 \)-grading (5.1), each \( c \in Z^1(g, V) \) is broken to \( (c', c'') \in \text{Hom}(g_0, V) \oplus \text{Hom}(g_1, V) \) subject to the following three equations:

\[
\begin{align*}
(E_1) \quad & c'([g_1, g_2]) - g_1 \cdot c'(g_2) + g_2 \cdot c'(g_1) = 0, \ g_1, g_2 \in g_0 \\
(E_2) \quad & c''([g, h]) - g \cdot c''(h) + h \cdot c''(g) = 0, \ g \in g_0, h \in g_1 \\
(E_3) \quad & c'([h_1, h_2]) - h_1 \cdot c''(h_2) - h_2 \cdot c''(h_1) = 0, \ h_1, h_2 \in g_1.
\end{align*}
\]

In the sequel let us consider the Lie superalgebra \( \mathcal{K}(1) \) acting on \( F \). The first cohomology space \( H^1(\mathcal{K}(1), F) \) inherits the \( \mathbb{Z}_2 \)-grading from (5.1) and it decomposes to a odd part and a even part as follows:

\[
H^1(\mathcal{K}(1), F) = H^1(\mathcal{K}(1), \mathcal{F}_1) \oplus H^1(\mathcal{K}(1), \mathcal{F}_0).
\]

We calculate each part independently. The following proposition is the main result of this section:

**Proposition 5.1.** 1) The first cohomology space \( H^1(\mathcal{K}(1), \mathcal{F}_0) \) has the following structure:

\[
H^1(\mathcal{K}(1), \mathcal{F}_0) = \begin{cases} \mathbb{R}^2 & \text{if } \lambda = 0 \\ 0 & \text{otherwise.} \end{cases}
\]

The space \( H^1(\mathcal{K}(1), \mathcal{F}_0) \) is generated by the cohomology classes of the 1-cocycles

\[
c_0(v_F) = \frac{1}{4} (F + \sigma(F)) + \frac{1}{2} F \quad \text{and} \quad c_1(v_F) = \eta^2(F)
\]

\[
(5.3)
\]

371
2) The cohomology space

\[ H^1(\mathcal{K}(1), \mathfrak{F}_\lambda) = \begin{cases} \mathbb{R} & \text{if } \lambda = \frac{1}{3}, \frac{2}{3} \\ 0 & \text{otherwise}. \end{cases} \]

It is spanned by the non-trivial cohomology class corresponding to the 1-cocycle

\[ c_2(v_F) = \eta^3(F) \quad \text{if } \lambda = \frac{1}{2}, \quad (5.4) \]

and

\[ c_3(v_F) = \eta^3(F) \quad \text{if } \lambda = \frac{3}{2}. \quad (5.5) \]

To prove this proposition, we will need the following two results:

**Proposition 5.2.** [6] The space of first cohomology of \( \text{Vect}(S^1) \) with coefficients in the space of tensor densitives \( \mathfrak{F}_\lambda \) has the following structure:

\[ H^1(\text{Vect}(S^1), \mathfrak{F}_\lambda) = \begin{cases} \mathbb{R}^2 & \text{if } \lambda = 0 \\ \mathbb{R} & \text{if } \lambda = 1 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases} \quad (5.6) \]

It is spanned by the classes of the following non-trivial 1-cocycles:

\[ \beta_0 \left( f(x) \frac{d}{dx} \right) = f(x) \text{ and } \beta_1 \left( f(x) \frac{d}{dx} \right) = f'(x), \quad \text{if } \lambda = 0. \]

\[ \beta_2 \left( f(x) \frac{d}{dx} \right) = f''(x), \quad \text{if } \lambda = 1 \text{ and } \quad (5.7) \]

\[ \beta_3 \left( f(x) \frac{d}{dx} \right) = f'''(x), \quad \text{if } \lambda = 2. \]

Moreover, we have the following lemma

**Lemma 5.3.** Let \( C_0 = (C_{00}, C_{11}) \) be a even 1-cocycle from \( \mathcal{K}(1) \) to \( \mathfrak{F}_\lambda \), where \( C_{00} : \text{Vect}(S^1) \to \mathfrak{F}_\lambda \) and \( C_{11} : \mathfrak{F}_{\frac{3}{2}} \to \mathfrak{F}_{\frac{3}{2} + \frac{1}{2}} \) are given by the grading (5.1). Then, if \( C_{00} \) is a coboundary over \( \text{Vect}(S^1) \) then, \( C_0 \) is a coboundary over \( \mathcal{K}(1) \).

**Proof.** Recall that a 1-coboundary of \( \text{Vect}(S^1) \) with coefficients in \( \mathfrak{F}_\lambda \) has the form \( c(a(x) \frac{d}{dx}) = L^a \eta^3(f) \) for some \( f \in \mathfrak{F}_\lambda \). Now let \( C_{00}(v_F) = L^a \eta^3(f) \) for some \( f \in \mathfrak{F}_\lambda \) be a coboundary where \( F = a(x) + 2\theta b(x) \). If we apply the equations \( (E_2) \) and \( (E_3) \) from (5.2) to \( C_0 \), we will obtain \( C_{11}(F) = 2\theta J_1(b(x), f) \) and then, \( C_0(v_F) = L^a \eta^3(f) \) is a coboundary of \( \mathcal{K}(1) \).

**Remark 5.4.** We have the same Lemma for odd 1-cocycle \( C_1 = (C_{01}, C_{10}) \), where \( C_{01} : \text{Vect}(S^1) \to \mathfrak{F}_{\frac{3}{2} + \frac{1}{2}} \) and \( C_{10} : \mathfrak{F}_{\frac{3}{2}} \to \mathfrak{F}_\lambda \).
Proof of Proposition 5.1. Since the space of 1-cocycles from $\mathcal{K}(1)$ to $\mathcal{F}_\lambda$ is $\mathbb{Z}_2$-graded, we first assume that $C$ is a non-trivial 1-cocycle. According to the $\mathbb{Z}_2$-gradation (5.1) of even cocycles, $C = C' + C''$ where the linear maps $C' : \text{Vect}(S^1) \to \mathcal{F}_\lambda$ and $C'' : \mathcal{F}_{\lambda + \frac{1}{2}} \to \mathcal{F}_{\lambda + \frac{1}{2}}$ are the homogenous parts. The equation $(E_1)$ from (5.2), implies that $C'$ is a 1-cocycle of $\text{Vect}(S^1)$ and the lemma (5.3) implies that $C'$ is non-trivial. By proposition (5.2), $C'$ is cohomologous to one of the cocycles (5.7). To compute $C''$, we apply the equations $(E_2)$ and $(E_3)$ from (5.2) to the cocycle $C$. We have solutions only if $C' = \beta_0$ or $C' = \beta_1$, and we obtain that $C$ is one of the cocycles $c_0$ or $c_1$.

Next, if $C$ is odd, the same arguments show that $(E_2)$ and $(E_3)$ are compatible if and only if $C' = \beta_2$ or $C' = \beta_3$, and then we obtain $c_2$ and $c_3$. \hfill \square

The first cohomology space of $\mathcal{K}(1)$ with coefficients in the space of symbols $\mathcal{SP}$ inherits the grading (4.2) of $\mathcal{SP}$, so it suffices to compute it in each degree. Combining propositions (4.1) and (5.1), we obtain the main result of this section, that can be stated as follows:

**Theorem 5.5.** The first cohomology space of $\mathcal{K}(1)$ with coefficients in the space of symbols $\mathcal{SP}$ is four-dimensional with only even 1-cocycles. It is spanned by the classes of the following non-trivial 1-cocycles

\begin{align}
C_0(v_F) &= \frac{1}{4} (F + \sigma(F)) + \frac{1}{2} F, \\
C_1(v_F) &= \eta^2(F), \\
C_2(v_F) &= \text{ad}_\zeta^3(\pi(v_F))\xi^{-2} \xi, \text{ and} \\
C_3(v_F) &= \text{ad}_\zeta^3(\pi(v_F))\xi^{-3} \xi.
\end{align}

where $\text{ad}_\zeta(\pi(v_F)) = \{\zeta, \pi(v_F)\}$ with $\pi$ is the map (4.1) and $\zeta = \bar{\theta} - \theta \xi$ ($\xi^2 = 0$).

Proof. According to propositions (4.1) and (5.1), the cohomology space of $\mathcal{K}(1)$ with coefficients in $\mathcal{SP}_n$ has the following structure

\begin{equation}
H^1(\mathcal{K}(1), \mathcal{SP}_n) = \begin{cases} 
\mathbb{R}^3, & \text{if } n = 0 \\
\mathbb{R}, & \text{if } n = 1 \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

(5.9)

In the case $n = 0$, the cohomology space $H^1(\mathcal{K}(1), \mathcal{SP}_0)$ is generated by the non-trivial cohomology classes of the cocycles $\bar{C}_0, \bar{C}_1$ and $\bar{C}_2$ corresponding to the cocycles $c_0, c_1$ and $c_2$ of proposition (5.1) via the isomorphism in proposition (4.1). They are given by

\begin{align}
\bar{C}_0(v_F) &= \frac{1}{2} \left( F + \sigma(F) + \eta(F)\xi^{-1} \xi - \frac{1}{4} \eta(F - \sigma(F))\xi^{-1} \xi \right), \\
\bar{C}_1(v_F) &= \text{ad}_\zeta^3(\pi(v_F))\xi^{-1} \xi \text{ and} \\
\bar{C}_2(v_F) &= \text{ad}_\zeta^3(\pi(v_F))\xi^{-3} \xi.
\end{align}

(5.10)

In the case $n = 1$, the cohomology space $H^1(\mathcal{K}(1), \mathcal{SP}_1)$ is generated by the non-trivial cohomology class of the cocycle $\bar{C}_3$ corresponding to the 1-cocycle $c_3$ and it is given by

373
As a 1-cocycle of \( \mathcal{SP} \), \( \tilde{C}_0 \) is cohomologous to \( C_0 \). Indeed, \( C_0 - \tilde{C}_0 = \text{ad}_{\delta \mu^{-1} \zeta} (\pi(v_F)) \) and \( \tilde{C}_1 = C_1 + \frac{1}{2} C_2 \). This completes the proof of the theorem.

\[ \Box \]

6 The first cohomology space \( H^1(\mathcal{K}(1), \mathcal{SP}) \)

In this section, we will compute the cohomology space of \( \mathcal{K}(1) \) with coefficients in the filtered module \( \mathcal{SP} \). A straightforward but long computations, using spectral sequences associated to \( \text{Grad}(\mathcal{SP}) \) [8] and Theorem (5.5) leads to the following theorem:

**Theorem 6.1.** The first cohomology space \( H^1_0(\mathcal{K}(1), \mathcal{SP}) \) of \( \mathcal{K}(1) \) with coefficients in the space \( \mathcal{SP} \) is four-dimensional with only even 1-cocycles. It is spanned by the classes of the following non-trivial 1-cocycles

\[
\begin{align*}
\Theta_0(v_F) & = \frac{1}{4} (F + \sigma(F)) + \frac{1}{2} F, \\
\Theta_1(v_F) & = \eta^2(F), \\
\Theta_2(v_F) & = \sum_{n=1}^{\infty} (-1)^n \frac{n - 2}{n} \sigma(\eta^{2n+1}(F)) \eta^{-2n+1} + \sum_{n=1}^{\infty} (-1)^n \frac{n - 3}{n + 1} \eta^{2n+2}(F) \eta^{-2n}, \\
\Theta_3(v_F) & = \sum_{n=2}^{\infty} (-1)^n \frac{n - 1}{n} \sigma(\eta^{2n+1}(F)) \eta^{-2n+1} + \sum_{n=2}^{\infty} (-1)^n \frac{n - 1}{n + 1} \eta^{2n+2}(F) \eta^{-2n},
\end{align*}
\]

(6.1)

where \( \eta = \frac{\theta}{\delta \mu} - \theta \frac{\delta \mu}{\zeta} \).

**Proof.** Since the cohomology space \( H^1(\mathcal{K}(1), \mathcal{SP}) \) is obviously upper-bounded by \( H^1(\mathcal{K}(1), \mathcal{SP}) \), we have to find explicit expressions for the non trivial cocycles generating the former cohomology space. To construct these cocycles, we follow the lines in [10] based on the computations of successive differentials of the spectral sequences corresponding to the complex \( C^*(\mathcal{K}(1), \mathcal{SP}) \). So, we consider a cocycle with values in \( \mathcal{SP} \), but we compute its boundary as it was with values in \( \mathcal{SP} \) and keep a symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one. We iterate this procedure, we establish a recurrent formula between successive terms. The cocycles \( c_0 \) and \( c_1 \) survive in the same form, we will denote them \( \Theta_0 \) and \( \Theta_1 \) when seen as cocycles with values in \( \mathcal{SP} \). The previous procedure applied to \( c_2 \) and \( c_3 \) leads to the cocycles \( \Theta_2 \) and \( \Theta_3 \).

\( \Box \)

**Remark 6.2.** The parts of \( \Theta_2 \) and \( \Theta_3 \) which are maps from \( \text{Vect}(S^1) \) to \( \Psi D(S^1) \) in the grading (5.1) are a multiple by a coefficient of the 1-cocycles \( \theta_2 \) and \( \theta_3 \) in [10].
References


B. Agrebaoui:
Département de Mathématiques, Faculté des Sciences de Sfax, Route de Soukra,
3018 Sfax BP 802, Tunisie,
E-mail address: bagreba@fss.rnu.tn,
N. Ben Fraj:
Institut Supérieur de Sciences Appliquées et Technologie, Sousse,
E-mail address: benfraj_nizar@yahoo.fr