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CHARACTERIZATION OF TOPOLOGICAL PROPERTIES OF 2D CELLULAR STRUCTURES BY IMAGE ANALYSIS

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ABSTRACT

The disordered 2D cellular structure constituted by the administrative division of Lorraine into civil parishes has been investigated by image analysis. The latter structure and polycrystal cuts have very similar topological properties. This confirms the reduced variability of such properties in homogeneous structures with comparable disorder.

Key words: administrative divisions, tessellation, topological properties.

INTRODUCTION

We consider cellular structures which are disordered bidimensional assemblies of cells which fill a given spatial domain C. The cells are arranged so that no gaps are left and that they do not overlap. The intersection of any two cells is either empty or is one side. Cells have in general a rather regular geometric shape (Fig. 1a) and have three sides at least. Cell vertices belong to three cells, except for cells whose edges are the boundaries of C. Some froths, such as the Johnson-Mehl froth (Stoyan et al.,1987), however do not fulfil the previous conditions. The distributions of metric and of topological properties such as cell areas, edge lengths, the number n of edges of cells ($n \geq 3$) etc.., have been characterized in some 2D random cellular structures. The two-cell correlation m(n) is the mean number of edges of the first neighbour cells of n-sided cells (called hereafter n-cells). It has been often reported that nm(n) varies linearly with n :

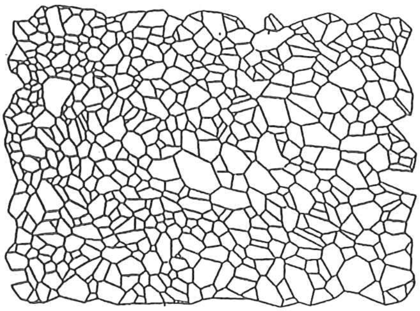
$$nm(n) = (6\text{-}a)\, n + 6a + \mu_2 \qquad (1)$$

The latter semi-empirical relation, called the Aboav-Weaire law, plays a central role in the maximum entropy theory of Rivier (1993). The variance μ_2 of the probability distribution P(n) of the number n of edges of cells (average $<n>$), $\mu_2 = <n^2> - <n>^2$, is a measure of the topological disorder. The parameter "a" is in general positive and of the order of 1 (Rivier,1993). As the structures investigated here are finite, $<n>$ differs in general from 6. A slightly modified Eq.1 yields a parameter a_w from a weighted least-squares fit with weights P(n). The latter parameter is a useful characteristic even if nm(n) departs from the Aboav-Weaire law. It is given by (Le Caër and Delannay, 1993 a,b) :

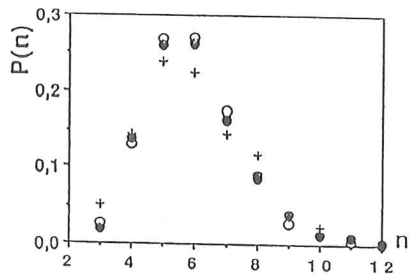
$$a_w = [< n > \{ <nm(n)> + \mu_2 \} - < n^2 m(n) >] / \mu_2 \qquad (2)$$

The correlation $M_k(n)$, which is the average number of k-sided neighbours of an n-cell, has been recently investigated (Peshkin et al., 1991, Le Caër and Delannay, 1993 a,b). The topological correlation functions A_{kn} (= $M_k(n)/P(k)$) allows comparing properties of tissues with different distributions P(n) (Le Caër and Delannay,1993 a and b).

Before assessing the validity of any theory devoted to random cellular structures, it is necessary to characterize the previous properties in various natural structures. In a recent study, Le Caër and Delannay (1993 b) have shown that the topological properties of the administrative divisions of mainland France in departments (μ_2=0.91) and in districts (μ_2=1.41) are comparable to those of biological tissues (μ_2 is typically $\lesssim$ 1.5). The magnitude of μ_2 seems to increase when the characteristic distance of the geographical network (e.g. average cell size) decreases. Indeed, μ_2 is much larger for the civil parishes of three English counties (Boots, 1980). It varies from 2.91 (Wiltshire) to 4.62 (Devon, but with one cell with n=23). Moreover, the corresponding P(n) and m(n) are quite similar to those of planar cuts of alumina (μ_2 = 2.58) (Le Caër and Delannay, 1993 b, Tbl. 1). This agrees with recent suggestions of Le Caër and Delannay (1993 a) about the restricted variability of topological correlations in structures which have comparable values of μ_2 . It is however necessary to confirm the previous observations. The topological and some metric properties of the cellular structure constituted by the civil parishes of the largest part of the region of Lorraine (East of France, Fig. 1a) have therefore been investigated by image analysis. The Lorraine administrative cellular structure will be referred to hereafter as the LACES.



(a)



(b)

Fig. 1. a) 2D cellular structure associated with the administrative division of Lorraine into civil parishes b) distributions P(n) for the LACES (open circles), for a planar cut of an alumina polycrystal (full circles, Righetti et al., unpublished) and for the civil parishes of Wiltshire (crosses, Boots, 1980).

METHODS

Maps of the territorial division of Lorraine into civil parishes are available from I.N.S.E.E. Images are collected with a video CCD camera and analyzed with an image analysis software, VISILOG 3.6, which yields a skeleton of the edges of the cellular structure. All cells which intersect the edges of the window are eliminated. The images are further analyzed with the help of the software package 2D-CELL (Righetti et al.,1992) which enables to determine the vertices, the edges and the neighbours of every cell . An additional program calculates some topological correlations (m(n), A_{kn}). Various topological and metric properties of the previous structures are thus characterized. Only cells, called interior cells, which are completely surrounded by cells have been taken into account. The correlations in the arrangements of cells , as quantified by m(n), A_{kn}, can only be safely determined for the latter cells as edge corrections have not been worked out till now for such characteristics. We have checked that a possible bias due to the present method is without significant consequences on the distribution P(n). The total number of interior cells is 1129 for the LACES (Fig.1).

RESULTS

The distribution P(n) for the LACES is shown in Fig. 1 and is compared to the distribution P(n) for a planar cut of an alumina polycrystal (4310 cells have been used here) (Righetti, Liebling, Le Caër and Mocellin, unpublished) and to P(n) for the civil parishes of Wiltshire (138 cells, Boots, 1980). Some characteristics of various cellular structures are given in Tbl. 1. Exact results are calculated for the two structures called z=5 and 4-8-8 respectively (Le

Caër and Delannay, 1993 a). They are topological models associated with planar tessellations in which every vertex is characterized by its valence z which is the number of edges emanating from that vertex. The valence is z=5 for all vertices of the tessellation by squares and triangles shown in Le Caër (1991). The tessellation by triangles named 4-8-8 shows vertices with three different valences : z=4,8,8 respectively (Le Caër, 1991). In the 4-8-8 tessellation, the three vertices of any triangle are shared by z=4,8,8 triangles respectively. The vertices of every square or of every triangle of the considered tessellations are structurally unstable as they belong to more than three polygons. The stable configurations are obtained by replacing every vertex by z-3 added sides so that only trivalent vertices exist in the final arrangement (Le Caër, 1991).

Table 1. Average number of cell sides $< n >$, variance μ_2, Aboav-Weaire parameter a_W (Eq.2), P(6) observed and calculated from Eq. 3 for various cellular structures

Structure	$<n>$	μ_2	a_W (Eq.2)	P(6) obs.	P(6) (Eq. 3)
LACES	5.92	2.25	0.79	0.267	0.257
Wiltshire	5.80	2.91	0.66	0.225	0.225
Polyc. cut	6.00	2.585	1.19	0.261	0.239
z=5	6	2.5867	1.2126	0.23893	0.2391
4-8-8	6	2.9583	0.7218	0.21935	0.2232

Fig. 1b and Fig. 2a show that the distributions of the number of cell sides P(n) and the correlations nm(n) for a planar cut of an alumina polycrystal and for the civil parishes of Wiltshire and of Lorraine are quite similar. The Aboav-Weaire law (Eq. 1) is fairly well obeyed with slopes deduced from a_W in Tbl. 1. Comparable topological correlations A_{kn} are also measured for the polycrystal and for the LACES (Fig. 2b, c for k=4,5,7,8). The parameter a_W ranges from ~0.7 to ~1.2 and tends to decrease rapidly when μ_2 increases from ~2 to ~3. The reduced variability of the topological properties of disordered structures with similar μ_2 is therefore confirmed. The variance μ_2 varies quasi-universally in random mosaics with the fraction P(6) of 6-cells (Lemaître et al., 1992). An approximate relation (Le Caër and Delannay, 1993 a) :

$$\mu_2^{0.513}\, P(6) = 0.3893 \qquad \text{for } 0.1 \lesssim P(6) \lesssim 0.7 \qquad (3)$$

yields P(6) with a precision typically better than 5 % for the structures considered by Le Caër and Delannay (1993a), better than 9 % for the structures of Tbl. 1.

A reduced area a is defined for every parish as a=A/<A>*, where A is the area of the considered parish and <A>* is the mean area calculated for the studied sample.

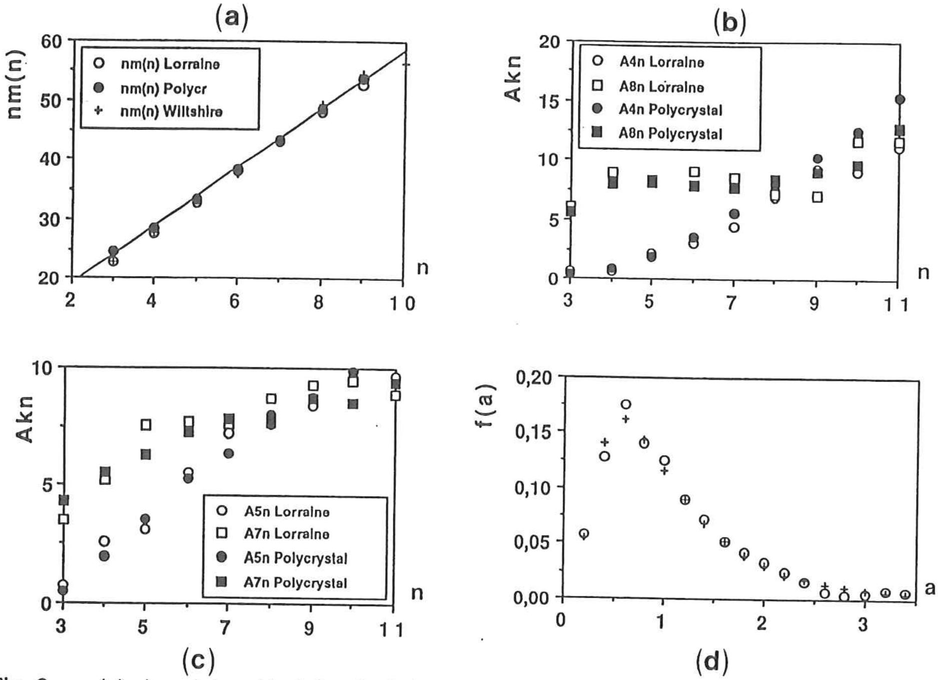


Fig. 2. nm(n) a) and A_{kn} b) c) for the LACES (open circles and squares) and for a planar cut of an alumina polycrystal (full circles and squares), nm(n) for the Wiltshire (crosses) is also shown in a). d) Distribution of the reduced areas a=A/<A>* of the LACES : observed (open circles) and calculated (crosses, Eq.4).

The distribution of the reduced area a is well fitted by a lognormal distribution (m = 0.90 and σ = 0.68) (Fig. 2d) :

$$f(a) = \exp\{ - \mathrm{Log}(a / m)^2 / 2\, \sigma^2 \} / [\, \sigma\, a\, (2\pi)^{1/2}] \quad (4)$$

The theoretical mean <a>, $\langle a\rangle = m \exp(\sigma^2/2) = 1.13$ differs from 1 as both f(a) is an approximation of the actual distribution and <A>* is estimated from the investigated sample. The mean area of n-cells varies furthermore linearly with n, and follows Lewis's law ($n \leq 9$) which is frequently observed in 2D cellular structures (Rivier, 1993):

$$a(n) = (n - n_0)/(6 - n_0) \quad (5)$$

where a(n) is the normalized area (a(6)=1). Metric properties of structures with similar μ_2 have a priori no reason to be also similar. The intercept n_0 in Eq.5 is typically less than ~3 in biological tissues and larger than ~4 in polycrystal cuts (Rivier 1993). The observed value, $n_0 = 2.78 \pm 0.06$, is surprisingly not so much less than 4 . More important, the value of n_0 in this a priori "artificial" structure is characteristic of values measured for natural structures.

CONCLUSION

The topological properties of the territorial division of Lorraine into civil parishes do not differ from the properties which are measured in natural 2D cellular structures with comparable disorder μ_2, for instance in polycrystal cuts. The disorder, quantified by μ_2, increases in the administrative divisions of France when the geographic scale decreases from department (μ_2=0.91) to district (μ_2=1.41) and from district to parish (μ_2=2.25). Even parish areas obey the usual semi-empirical laws known for cellular structures. The reduced variability of topological properties of structures with similar μ_2 is confirmed. An unpublished study shows that the topological properties of heterogeneous young soap froths differ however from those of structures with similar μ_2. Only structures with overall homogeneous cell sizes may therefore be compared. The disorder μ_2, measured till now in natural and homogeneous structures never exceeds ~4.

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