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SOME CONNECTIVITY CHARACTERISTICS OF A BOOLEAN MODEL

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ABSTRACT

On the 2D simple square grid (3D simple cubic grid), the Boolean model with deterministic square (cubic) primary grains is considered. Four connectivity characteristics are studied as a function of the grain proportion: the Euler-Poincaré characteristic (EPC), the number of connected components, the relative area (volume) occupied by the biggest connected component and the connectivity function (defined as the probability that two points of the grains belong to the same connected component). Using simulations, this paper shows the influence of grain and grid size on these characteristics, for both the square and the cubic grid.

Key words: Boolean model, connectivity, discretization, Euler-Poincaré characteristic.

INTRODUCTION

During the exploration of a heterogeneous petroleum reservoir, the connectivity of the permeable phase (which is an important parameter for the exploitation) must be estimated using core samples from wells (1D) as well as seismic profiles (2D). In both cases, data are of lower dimension than the reservoir itself, and a stereological problem arises.
During the exploitation, it is a common practice to perform numerical simulations of the geometry of the reservoir, in order to estimate the production by flow computations. At this stage, it is important to know the main connectivity characteristics of the model of the random set used for these simulations. It is of special interest to know whether two wells are connected or not by a big component.
For these two reasons, we study some connectivity characteristics of a model of a random set commonly used for reservoir simulations: the discrete Boolean model with deterministic square (cubic) primary grains. As flow simulations are made on a regular simple cubic grid, the model must be defined on the same grid. The definition of this model is the following: on each site of a square (cubic) grid G of sidelength L, a square (cubic) primary grain of sidelenght l is set up, independently of all other sites, with probability p_0 (see Fig. 1). As the point process where the primary grains are implanted is a regular lattice, this model is not a Boolean model; but for $l \to \infty$, the point process converges to a Poisson process. Hence, we call this model the discrete Boolean model.

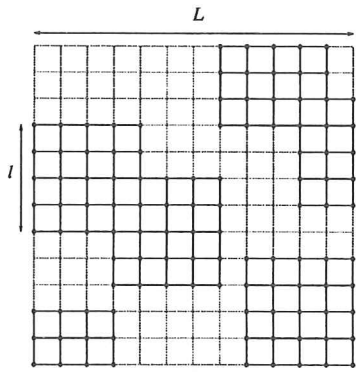

Figure 1. The discrete Boolean model with fixed square primary grains: l=5; L=13.

This model depends on three parameters: p_0, l, L (as well as the dimension of the grid, $d = 2, 3$), which are rewritten:

$p = 1 - (1 - p_0)^{l^d}$	:	the proportion, equal to 1 minus the porosity;
l	:	the discretization;
L/l	:	the scale factor.

Four connectivity characteristics are studied as a function of the proportion, as the discretization and the scale factor vary. These are the Euler-Poincaré characteristic (EPC), the number of connected components, the relative area (volume) occupied by the biggest connected component and the connectivity function, defined as the probability that two sites of the Boolean model belong to the same connected component. Except for the EPC, these characteristics are not mathematically tractable. Hence they must be estimated by simulations.

THE EULER-POINCARÉ CHARACTERISTIC

Connectivity is usually described through the EPC, χ, which is the only topological characteristic, additive in the sense of Poincaré formula. On the square grid, the specific EPC of this model is given by

$$\chi_A = l^2\, q\, \left(-1 + 2\, q^{\frac{1}{l}} + 2\, q^{\frac{2l-1}{l^2}} - 4\, q^{\frac{2}{l}} + q^{\frac{2l+1}{l^2}}\right) \qquad (1)$$

where $q = 1 - p$ is the porosity. Between $l = 1$ where $\chi_A = q(1-q)(2q-1)$ and $l = \infty$ where $\chi_A = -q \ln q(1 + \ln q)$ (which is the specific EPC of the Boolean model with unit square primary grain (Weil, 1988)), the influence of the discretization, l, is very important (Tbl. 1).

For high proportions, in the negative part of χ_A, the position of the minimum is shifted towards the right to higher proportions when l increases. The minimum itself increases by 4 between $l = 1$ ($\chi_{A,min} = -0.073$) and $l = \infty$ ($\chi_{A,min} = -0.309$). On the contrary, for low and medium proportions (where $\chi_A \geq 0$), both the position of the maximum and the transition through zero are shifted towards the left between $l = 1$ and $l = 2$, and towards the right to higher proportions when $l > 2$. The same behaviour holds for the maximum

Table 1. Typical values of the specific EPC χ_A for different grain sizes

l	$p(\chi_{A,max})$	$p(\chi_{A,min})$	$\chi_{A,max}$	$\chi_{A,min}$	$p(\chi_A = 0)$
1	0.270	0.837	0.129	-0.073	0.618
2	0.263	0.858	0.126	-0.150	0.574
3	0.270	0.874	0.131	-0.193	0.574
5	0.281	0.892	0.139	-0.236	0.586
11	0.300	0.910	0.149	-0.277	0.605
50	0.311	0.924	0.158	-0.302	0.625
∞	0.317	0.927	0.161	-0.309	0.632

value of χ_A: it decreases between $l = 1$ and $l = 2$ whereas it increases monotonically with l when $l > 2$.

On a finite square grid G, $\chi(X \cap G)$ is random. Its mean value, $E\{\chi(X \cap G)\}$ is made up of three terms: one for each dimension less than or equal to 2. Border and corner terms are positive for all proportions. Hence, when the grid size increases, i.e. when the scale factor is increased, $E\{\chi(X \cap G)\}/L^2$ decreases and the proportions corresponding to the maximum, minimum and transition through zero are shifted to lower proportions.

On the cubic grid, χ_V presents two transitions through zero and is positive for proportions close to 1. The influence of the grain size is similar: the first maximum and the first zero are shifted towards the left between $l = 1$ and $l = 3$ and towards the right for $l > 3$.

THE NUMBER OF CONNECTED COMPONENTS

Let p increase from $p = 0$ (where $N(X \cap G) = 0$) to $p = 1$ (where $N(X \cap G) = 1$). For low proportions, new connected components are created in the many empty spaces and $E\{N\}$ increases. But, little by little, different connected components come in contact, so that $E\{N\}$ decreases. These antagonistic effects lead to a balance for a particular proportion where $E\{N\}$ reaches its maximum. Beyond this value, $E\{N\}$ decreases monotonically. The number of connected components, $N(X \cap G)$, is linked to the EPC by the equation

$$N(X \cap G) = \chi(X \cap G) + \sum_{i=1}^{N} \Gamma(C_i) \qquad (2)$$

where $\Gamma(C_i)$ is the genus of the i-th connected component. For low proportions (i.e. approximately up to the maximum of χ), the genus of the connected components is 0, and $N \simeq \chi$. Hence, the curve of $\overline{N} = E\{N\}/(L/l)^d$ as a function of p behaves roughly like the positive part of $\overline{\chi} = E\{\chi\}/(L/l)^d$ for any change of the discretization and scale factor.

For a given proportion, $\overline{N}$ generally increases with l up to the limit of the continuous Boolean model. But, once again, the behaviour of the connectivity characteristic is not monotonic with the discretization, and there is a minimum in the number of connected components when $l = 2$ if $d = 2$, when $l = 3$ if $d = 3$ (Fig. 2).

If two sites of a grid G are connected, they are also connected on a grid $G' \supset G$. But, if two sites of G are connected in G', they may not be connected in G. Hence, for a given

discretization, $\overline{N}$ decreases as L increases. For the same reason, for a given discretization and a given scale factor, $\overline{N}$ decreases when d increases. Furthermore, the position of the maximum of $\overline{N}$ is shifted towards the left to the lower proportions.

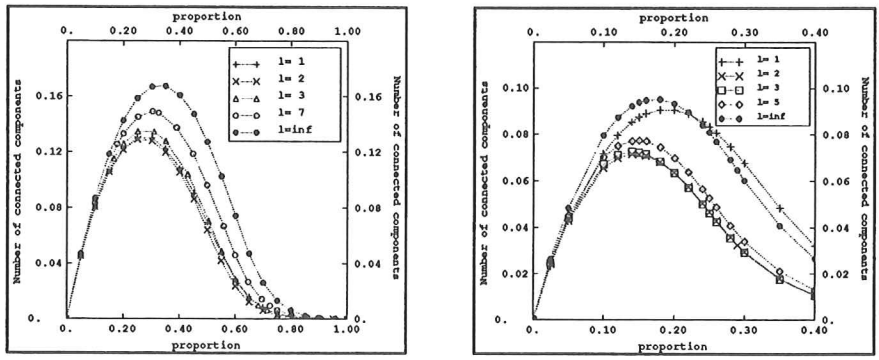

Figure 2. $\overline{N}$ estimated from 250 simulations vs. proportion. Left: simple square grid ($d = 2$), $L/l = 100$. Right: simple cubic grid ($d = 3$), $L/l = 20$.

THE BIGGEST CONNECTED COMPONENT

For a small proportion the Boolean model is constituted of many, widely scattered connected components, whereas for a large proportion almost all the connected components have merged into a huge cluster covering the entire field, except for some small holes. Between these two extremes, one of the connected components individualizes and grows quicker than any other. To quantify this qualitative description, the area (volume) occupied by the biggest connected component in the random set is considered. Fig. 3 represents the mean value $\overline{V}$ of this relative area (volume), versus the proportion for some choices of l and L. $\overline{V}$ suddenly increases around a critical proportion in a sharp transition. This transition is situated between $p = 0.55$ and $p = 0.65$ for $d = 2$, and between $p = 0.3$ and $p = 0.4$ for $d = 3$.

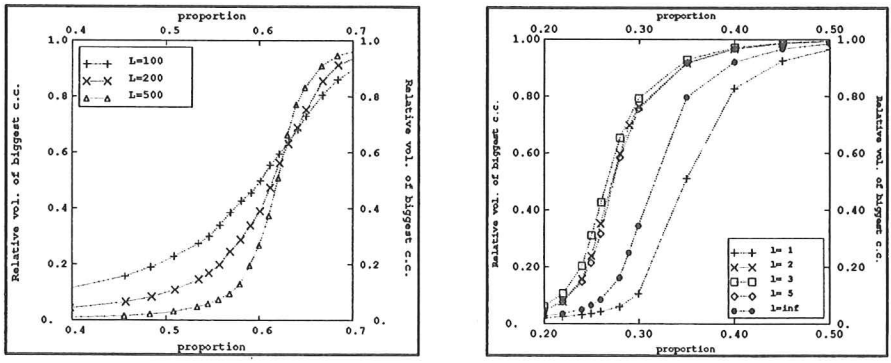

Figure 3. Mean value of the relative area (volume) of the biggest connected component estimated from 250 simulations, vs. proportion. Left: simple square grid ($d = 2$), $l = 5$. Right: simple cubic grid ($d = 3$), $L/l = 20$.

For a given grain size, the sharpness of the transition increases with the grid size, i.e. with the scale factor (Fig. 3, left). This can be explained as follows. For small proportions, all the connected components are small (even the biggest one), and they do not become larger with a larger grid; hence $\overline{V}$ decreases. On the other hand, for large proportions, the biggest connected component is very large (more than half the random set, say); in this case, increasing the grid size creates new connections and the biggest connected component becomes larger relatively to the random set. For a given scale factor L/l, the transition is shifted to lower proportions between $l = 1$ and $l = 2$ if $d = 2$, and between $l = 1$ and $l = 3$ if $d = 3$. For larger discretizations, the transition is monotonically shifted to higher proportions up to the limit of the continuous Boolean model (Fig. 3, right).

THE CONNECTIVITY FUNCTION

The connectivity function is defined as the probability that two sites of the grid belong to the same connected component, knowing that they both belong to the random set. As the Boolean model is stationary, this function is also stationary, and we write

$$\tau_p(h) = P_p(x \leftrightarrow x + h \,|\, x, x + h \in X) \tag{3}$$

where "$\leftrightarrow$" means "belong to the same connected component". It has been observed that this function is isotropic (for the Euclidien metrics) in spite of the anisotropy of the square (cubic) lattice, and of the Boolean model. The way $\tau_p(h)$ decreases with $|h|$ depends on the proportion p and is illustrated on Fig. 4, left.

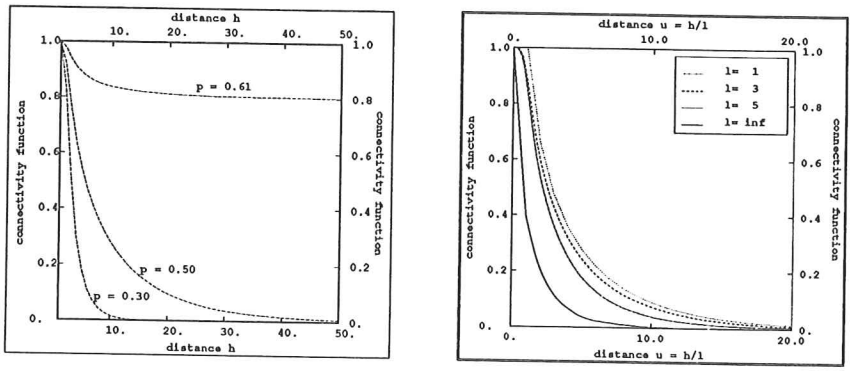

Figure 4. Connectivity function on the square grid ($d = 2$). Left: τ_p vs. physical distance $|h|$ for three proportions; from bottom to top, $p = 0.3$, $p = 0.5$ and $p = 0.61$ ($l = 5$). Right: τ_p vs. adimensional distance $|u| = |h|/l$; $p = 0.5$.

- For small proportions, $\tau_p(h)$ rapidly decreases and converges towards 0 for large $|h|$, since all connected components are small.

- For large proportions, $\tau_p(h)$ decreases until it reaches a sill greater than $\overline{V}^2(p)$. Indeed, for short distances, two connected points of the Boolean model can either belong to a small connected component or to the biggest one, C_{max}. But for long distances, two points are connected only if they both belong to C_{max}. Hence, for large $|h|$,

$$\begin{aligned} \tau_p(h) &\simeq P_p(x \in C_{max} \text{ and } x+h \in C_{max}) \\ &\geq P_p(x \in C_{max}) P_p(x+h \in C_{max}) \simeq \overline{V}^2(p) \end{aligned} \quad (4)$$

according to a standard inequality on increasing events (Grimmett, 1992).

The influence of the scale factor will not be studied here. The grid is supposed to be large enough to avoid important borders effects. In order to compare the connectivity functions for different grain sizes, an adimensional distance $u = h/l$ is considered.
As shown on Fig. 4 (right), for a larger grain size, the adimensional connectivity function decreases for a given distance and a given proportion. This is also the case for $l = 2$, although it has not been represented, because its function $\tau_p(u)$ is very similar to the $l = 1$ one. It must be noted that the connectivity function of the continuous Boolean model with unit square decreases linearly from $u = 0$ to $u = 1$.

CONCLUSIONS

The influence of the three parameters: l, L/l, and d can be summarized as follows.

1. For a given discretization, a larger grid size increases the possibility of connections. Hence, the number of connected components divided by the grid size and the connectivity function increases. The relative area (volume) of the biggest connected component decreases for small proportions whereas it increases for large proportions.

2. The influence of the discretization is complex. In general, increasing the discretization for a given scale factor L/l breaks connections. Hence, the number of connected components increases relatively to $(L/l)^d$, the relative area (volume) of the biggest connected component decreases as well as the adimensional connectivity function. As a consequence, $\overline{V}(p)$ is shifted towards the right and $\tau_p(u)$ is shifted towards the left. However, this is not the case for small grain sizes. It can be observed that between $l = 1$ and $l = 2$ on the simple square grid ($l = 1$ and $l = 3$ on the simple cubic grid), the converse holds for the the number of connected components and for $\overline{V}$. Increasing the discretization has another consequence: the number of holes increases for high proportions, as can be seen on χ_A.

3. On the simple cubic grid, each site has 6 nearest neighbours whereas it has only 4 on the simple square grid. The number and the complexity of the possible connections increase. Hence, there are fewer connected components on the cubic grid and they are larger on average. In particular, the biggest connected component merges much earlier and the transition of $\overline{V}(p)$ is shifted from $p = 0.6$ to $p = 0.3$.

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